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Research Article

A New Super Extension of Dirac Hierarchy

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We derive a new super extension of the Dirac hierarchy associated with a 3×3 matrix super spectral problem with the help of the zero-curvature equation. Super trace identity is used to furnish the super Hamiltonian structures for the resulting nonlinear super integrable hierarchy.

1. Introduction

In 1984, Kupershmidt [1] proposed a fermionic extension of the KdV equation with a Lax pair and a local super bi-Hamiltonian structure. Then, super integrable systems have received much attention with the development of integrable systems. Many experts and scholars do research on the topic. So far, many classical integrable hierarchies have been extended to the super ones by adding fermion fields, such as the super AKNS hierarchy [2, 3], the super Kaup-Newell hierarchy [4], the super Dirac hierarchy [2, 5], and the super Kadomtsev-Petviashvili (KP) hierarchy, and so on [6–10].

The Dirac hierarchy is based on the spectral problem

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x = \begin{pmatrix} r & \lambda + s \\ -\lambda + s & -r \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{1}$$

and a super extension Dirac hierarchy can be constructed by the matrix super spectral [2]

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_x = \begin{pmatrix} r & \lambda + s & \alpha \\ -\lambda + s & -r & \beta \\ \beta & -\alpha & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \tag{2}$$

where ϕ_3, α, and β are fermion fields. It reduces to the spectral Dirac system (1) as $\alpha = \beta = 0$.

In this paper, we consider a new 3×3 matrix super spectral problem which generates a generalized super Dirac hierarchy with four Fermi variables. As we will show, this spectral problem takes the spectral Dirac system (1) and the super Dirac system (2) as special cases.

The paper is organized as follows. In the Section 2, we will construct a generalized super Dirac hierarchy related to the 3×3 matrix super spectral problem and consider some special reductions. In Section 3, we prove the localness of the whole super soliton hierarchy. In Section 4, we present the super Hamiltonian structures for the generalized super Dirac hierarchy with the help of the super trace identity. The last section contains concluding remarks.

2. Super Extension Dirac Hierarchy

In this section, we will derive a generalized super Dirac hierarchy. To this end, we take a matrix super spectral problem

$$\begin{gathered} \phi_x = U\phi = U(u, \lambda)\phi, \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \\ U = \begin{pmatrix} r & \lambda + s & \alpha_1 \\ -\lambda + s & -r & \beta_1 \\ \beta_2 & \alpha_2 & 0 \end{pmatrix}, \end{gathered} \tag{3}$$

where r, s, λ, ϕ_1, and ϕ_2 are the commuting variables, which can be indicated by the degree (mod 2) p as $p(r) = p(s) = p(\lambda) = p(\phi_1) = p(\phi_2) = 0$ and α_1, α_2, β_1, β_2, and ϕ_3 are the anticommuting variables, which can be indicated by the degree p as $p(\alpha_1) = p(\alpha_2) = p(\beta_1) = p(\beta_2) = p(\phi_3) = 1$. Here λ is assumed to be a constant spectral parameter.

We first solve the stationary zero-curvature equation

$$V_x = [U, V], \qquad V = \left(V_{ij}\right)_{3\times 3}, \tag{4}$$

where

$$\begin{gathered} p(V_{11}) = p(V_{12}) = p(V_{21}) = p(V_{22}) = p(V_{33}) = 0, \\ p(V_{13}) = p(V_{23}) = p(V_{31}) = p(V_{32}) = 1, \end{gathered} \tag{5}$$

and each entry $V_{ij} = V_{ij}(A, B, C, G, \rho, \delta, \tau, \varepsilon)$ with the function of A, B, C, G, ρ, δ, τ, ε with $p(A) = p(B) = p(C) = p(G) = 0$, $p(\rho) = p(\delta) = p(\tau) = p(\varepsilon) = 1$:

$$\begin{gathered} V_{11} = \frac{1}{2}(C + G), \qquad V_{12} = A + B, \qquad V_{13} = \rho, \\ V_{21} = A - B, \qquad V_{22} = \frac{1}{2}(-C + G), \qquad V_{23} = \delta, \\ V_{31} = \varepsilon, \qquad V_{32} = \tau, \qquad V_{33} = G. \end{gathered} \tag{6}$$

Substituting (6) into (4) yields

$$\begin{gathered} A_x = -\lambda C + 2rB + \frac{1}{2}\alpha_1\tau + \frac{1}{2}\alpha_2\rho + \frac{1}{2}\beta_1\varepsilon + \frac{1}{2}\beta_2\delta, \\ B_x = 2rA - sC + \frac{1}{2}\alpha_1\tau + \frac{1}{2}\alpha_2\rho - \frac{1}{2}\beta_1\varepsilon - \frac{1}{2}\beta_2\delta, \\ C_x = 4\lambda A - 4sB + \alpha_1\varepsilon + \beta_2\rho - \beta_1\tau - \alpha_2\delta, \\ G_x = \alpha_1\varepsilon + \beta_2\rho + \beta_1\tau + \alpha_2\delta, \\ \rho_x = \lambda\delta + r\rho + s\delta + \frac{1}{2}\alpha_1 G - \frac{1}{2}\alpha_1 C - \beta_1 A - \beta_1 B, \\ \delta_x = -\lambda\rho + s\rho - r\delta + \frac{1}{2}\beta_1 G - \alpha_1 A + \alpha_1 B + \frac{1}{2}\beta_1 C, \\ \tau_x = -\lambda\varepsilon + \beta_2 A + \beta_2 B - \frac{1}{2}\alpha_2 C - \frac{1}{2}\alpha_2 G - s\varepsilon + r\tau, \\ \varepsilon_x = \lambda\tau + \frac{1}{2}\beta_2 C - \frac{1}{2}\beta_2 G + \alpha_2 A - \alpha_2 B - r\varepsilon - s\tau. \end{gathered} \tag{7}$$

We put A, B, C, G, ρ, δ, ε, and τ to be polynomials of λ:

$$\begin{gathered} A = \sum_{m\geq 0}^{\infty} A_m\lambda^{-m}, \qquad B = \sum_{m\geq 0}^{\infty} B_m\lambda^{-m}, \\ C = \sum_{m\geq 0}^{\infty} C_m\lambda^{-m}, \qquad G = \sum_{m\geq 0}^{\infty} G_m\lambda^{-m}, \\ \rho = \sum_{m\geq 0}^{\infty} \rho_m\lambda^{-m}, \qquad \varepsilon = \sum_{m\geq 0}^{\infty} \varepsilon_m\lambda^{-m}, \\ \delta = \sum_{m\geq 0}^{\infty} \delta_m\lambda^{-m}, \qquad \tau = \sum_{m\geq 0}^{\infty} \tau_m\lambda^{-m}. \end{gathered} \tag{8}$$

Substituting (8) into (7) and equating the coefficients of λ, we obtain

$$\begin{gathered} A_{m,x} = -C_{m+1} + 2rB_m + \frac{1}{2}\alpha_1\tau_m + \frac{1}{2}\alpha_2\rho_m \\ + \frac{1}{2}\beta_1\varepsilon_m + \frac{1}{2}\beta_2\delta_m, \\ B_{m,x} = 2rA_m - sC_m + \frac{1}{2}\alpha_1\tau_m + \frac{1}{2}\alpha_2\rho_m \\ - \frac{1}{2}\beta_1\varepsilon_m - \frac{1}{2}\beta_2\delta_m, \\ C_{m,x} = 4A_{m+1} - 4sB_m + \alpha_1\varepsilon_m + \beta_2\rho_m \\ - \beta_1\tau_m - \alpha_2\delta_m, \\ G_{m,x} = \alpha_1\varepsilon_m + \beta_2\rho_m + \beta_1\tau_m + \alpha_2\delta_m, \\ \rho_{m,x} = \delta_{m+1} + r\rho_m + s\delta_m + \frac{1}{2}\alpha_1 G_m \\ - \frac{1}{2}\alpha_1 C_m - \beta_1 A_m - \beta_1 B_m, \\ \delta_{m,x} = -\rho_{m+1} + s\rho_m - r\delta_m + \frac{1}{2}\beta_1 G_m - \alpha_1 A_m \\ + \alpha_1 B_m + \frac{1}{2}\beta_1 C_m, \\ \tau_{m,x} = -\varepsilon_{m+1} + \beta_2 A_m + \beta_2 B_m - \frac{1}{2}\alpha_2 C_m \\ - \frac{1}{2}\alpha_2 G_m - s\varepsilon_m + r\tau_m, \\ \varepsilon_{m,x} = \tau_{m+1} + \frac{1}{2}\beta_2 C_m - \frac{1}{2}\beta_2 G_m + \alpha_2 A_m \\ - \alpha_2 B_m - r\varepsilon_m - s\tau_m. \end{gathered} \tag{9}$$

Upon choosing the initial data

$$\begin{gathered} A_0 = C_0 = \rho_0 = \tau_0 = \delta_0 = \varepsilon_0 = 0, \\ B_0 = 1, \qquad G_0 = -g_0 = \text{constant}, \end{gathered} \tag{10}$$

then the recursion relations in (9) uniquely define a series of sets of differential polynomial functions in u with respect to x. The first two sets are as follows:

$$\begin{gathered} A_1 = s, \qquad C_1 = 2r, \qquad \delta_1 = \beta_1 + \frac{1}{2}g_0\alpha_1, \\ \rho_1 = \alpha_1 - \frac{1}{2}g_0\beta_1, \qquad \tau_1 = \alpha_2 - \frac{1}{2}g_0\beta_2, \\ \varepsilon_1 = \beta_2 + \frac{1}{2}g_0\alpha_2, \qquad B_1 = 0, \qquad G_1 = 0, \\ A_2 = \frac{1}{2}r_x - \frac{1}{4}g_0\left(\alpha_1\alpha_2 + \beta_1\beta_2\right), \\ C_2 = -s_x - \frac{1}{2}g_0\left(\alpha_1\beta_2 + \alpha_2\beta_1\right), \end{gathered}$$

$$\delta_2 = \alpha_{1,x} + \frac{1}{2}g_0\left(r\beta_1 - s\alpha_1 - \beta_{1,x}\right),$$
$$\rho_2 = -\beta_{1,x} - \frac{1}{2}g_0\left(\beta_1 s + \alpha_1 r + \alpha_{1,x}\right),$$
$$\tau_2 = \beta_{2,x} + \frac{1}{2}g_0\left(r\alpha_2 - s\beta_2 + \alpha_{2,x}\right),$$
$$\varepsilon_2 = -\alpha_{2,x} - \frac{1}{2}g_0\left(s\alpha_2 + r\beta_2 - \beta_{2,x}\right),$$
$$B_2 = \frac{1}{2}\left(r^2 + s^2\right) + \frac{1}{2}\alpha_1\beta_2 + \frac{1}{2}\beta_1\alpha_2 + \frac{1}{4}g_0\left(\alpha_1\alpha_2 - \beta_1\beta_2\right),$$
$$G_2 = \alpha_2\alpha_1 + \beta_1\beta_2 + \frac{1}{2}g_0\left(\alpha_1\beta_2 - \alpha_2\beta_1\right). \tag{11}$$

From the recursion relations in (9), we can obtain the hereditary recursion operator L which satisfies that

$$\begin{aligned}&\left(C_{m+1}, 2A_{m+1}, -\varepsilon_{m+1}, -\tau_{m+1}, \delta_{m+1}, \rho_{m+1}\right)^T\\ &\quad = L\left(C_m, 2A_m, -\varepsilon_m, -\tau_m, \delta_m, \rho_m\right)^T,\end{aligned} \tag{12}$$

where

$$L = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix},$$

$$L_{11} = \begin{pmatrix} -2r\partial^{-1}s & 2r\partial^{-1}r - \frac{1}{2}\partial \\ -2s\partial^{-1}s + \frac{1}{2}\partial & 2s\partial^{-1}r \end{pmatrix},$$

$$L_{12} = \begin{pmatrix} r\partial^{-1}\beta_1 - \frac{1}{2}\beta_1 & -r\partial^{-1}\alpha_1 - \frac{1}{2}\alpha_1 \\ s\partial^{-1}\beta_1 + \frac{1}{2}\alpha_1 & -\frac{1}{2}\beta_1 - s\partial^{-1}\alpha_1 \end{pmatrix},$$

$$L_{13} = \begin{pmatrix} -r\partial^{-1}\beta_2 + \frac{1}{2}\beta_2 & r\partial^{-1}\alpha_2 + \frac{1}{2}\alpha_2 \\ -s\partial^{-1}\beta_2 + \frac{1}{2}\alpha_2 & s\partial^{-1}\alpha_2 - \frac{1}{2}\beta_2 \end{pmatrix},$$

$$L_{21} = \begin{pmatrix} \beta_2\partial^{-1}s + \frac{1}{2}\alpha_2 & -\beta_2\partial^{-1}r - \frac{1}{2}\beta_2 \\ \alpha_2\partial^{-1}s + \frac{1}{2}\beta_2 & -\alpha_2\partial^{-1}r + \frac{1}{2}\alpha_2 \end{pmatrix},$$

$$L_{22} = \begin{pmatrix} -\frac{1}{2}\beta_2\partial^{-1}\beta_1 - \frac{1}{2}\alpha_2\partial^{-1}\alpha_1 - s & \frac{1}{2}\beta_2\partial^{-1}\alpha_1 - \frac{1}{2}\alpha_2\partial^{-1}\beta_1 + r - \partial \\ \frac{1}{2}\beta_2\partial^{-1}\alpha_1 - \frac{1}{2}\alpha_2\partial^{-1}\beta_1 + r + \partial & \frac{1}{2}\beta_2\partial^{-1}\beta_1 + \frac{1}{2}\alpha_2\partial^{-1}\alpha_1 + s \end{pmatrix},$$

$$L_{23} = \begin{pmatrix} \frac{1}{2}\beta_2\partial^{-1}\beta_2 + \frac{1}{2}\alpha_2\partial^{-1}\alpha_2 & -\frac{1}{2}\beta_2\partial^{-1}\alpha_2 + \frac{1}{2}\alpha_2\partial^{-1}\beta_2 \\ -\frac{1}{2}\beta_2\partial^{-1}\alpha_2 + \frac{1}{2}\alpha_2\partial^{-1}\beta_2 & -\frac{1}{2}\beta_2\partial^{-1}\beta_2 - \frac{1}{2}\alpha_2\partial^{-1}\alpha_2 \end{pmatrix},$$

$$L_{31} = \begin{pmatrix} -\beta_1\partial^{-1}s + \frac{1}{2}\alpha_1 & \beta_1\partial^{-1}r + \frac{1}{2}\beta_1 \\ -\alpha_1\partial^{-1}s + \frac{1}{2}\beta_1 & \alpha_1\partial^{-1}r - \frac{1}{2}\alpha_1 \end{pmatrix},$$

$$L_{32} = \begin{pmatrix} \frac{1}{2}\alpha_1\partial^{-1}\alpha_1 + \frac{1}{2}\beta_1\partial^{-1}\beta_1 & \frac{1}{2}\alpha_1\partial^{-1}\beta_1 - \frac{1}{2}\beta_1\partial^{-1}\alpha_1 \\ \frac{1}{2}\alpha_1\partial^{-1}\beta_1 - \frac{1}{2}\beta_1\partial^{-1}\alpha_1 & -\frac{1}{2}\alpha_1\partial^{-1}\alpha_1 - \frac{1}{2}\beta_1\partial^{-1}\beta_1 \end{pmatrix},$$

$$L_{33} = \begin{pmatrix} -\frac{1}{2}\alpha_1\partial^{-1}\alpha_2 - \frac{1}{2}\beta_1\partial^{-1}\beta_2 - s & -\frac{1}{2}\alpha_1\partial^{-1}\beta_2 + \frac{1}{2}\beta_1\partial^{-1}\alpha_2 + \partial - r \\ \frac{1}{2}\beta_1\partial^{-1}\alpha_2 - \frac{1}{2}\alpha_1\partial^{-1}\beta_2 - r - \partial & \frac{1}{2}\beta_1\partial^{-1}\beta_2 + \frac{1}{2}\alpha_1\partial^{-1}\alpha_2 + s \end{pmatrix}. \tag{13}$$

Let ϕ satisfy the spectral problem (3) and an auxiliary problem

$$\phi_{t_n} = V^{(n)}\phi, \qquad n \geq 0, \tag{14}$$

where

$$V^{(n)} = (\lambda^n V)_+ = \sum_{j=0}^{n} \begin{pmatrix} \frac{C_j + G_j}{2} & A_j + B_j & \rho_j \\ A_j - B_j & \frac{-C_j + G_j}{2} & \delta_j \\ \varepsilon_j & \tau_j & G_j \end{pmatrix} \lambda^{n-j}, \tag{15}$$

where $(\lambda^n V)_+$ denotes the polynomial part of $\lambda^n V$.

Then, the compatibility condition of (3) and (14) yields the zero-curvature equation

$$U_t - V_x^{(n)} + \left[U, V^{(n)}\right] = 0, \tag{16}$$

which is equivalent to a hierarchy of generalized super Dirac equations

$$u_{t_n} = \begin{pmatrix} q \\ r \\ \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \end{pmatrix}_{t_n} = K_n(u) = \begin{pmatrix} 2A_{n+1} \\ -C_{n+1} \\ \delta_{n+1} \\ -\rho_{n+1} \\ -\varepsilon_{n+1} \\ \tau_{n+1} \end{pmatrix}, \qquad n \geq 0. \tag{17}$$

The first nontrivial member in the hierarchy (17) is as follows:

$$\begin{aligned} r_{t_1} &= r_x - \frac{1}{2}g_0(\alpha_1\alpha_2 + \beta_1\beta_2), \\ s_{t_1} &= -\beta_{2,x} - \frac{1}{2}(\alpha_{2,x} + r\alpha_2 - s\beta_2), \\ \alpha_{1,t_1} &= \alpha_{1,x} - \frac{1}{2}g_0(\beta_{1,x} - r\beta_1 + s\alpha_1), \\ \beta_{1,t_1} &= \beta_{1,x} + \frac{1}{2}g_0(\alpha_{1,x} + r\alpha_1 + s\beta_1), \\ \alpha_{2,t_1} &= \alpha_{2,x} - \frac{1}{2}g_0(\beta_{2,x} - r\beta_2 - s\alpha_2), \\ \beta_{2,t_1} &= \beta_{2,x} + \frac{1}{2}g_0(\alpha_{2,x} + r\alpha_2 - s\beta_2). \end{aligned} \tag{18}$$

When $n = 2$ and $g_0 = 0$ in (17), we can obtain the second-order nonlinear super integrable equations

$$\begin{aligned} r_{t_2} &= -\frac{1}{2}s_{xx} - \frac{1}{2}\alpha_{2,x}\alpha_1 - \frac{1}{2}\alpha_{1,x}\alpha_2 - \frac{1}{2}\beta_{2,x}\beta_1 \\ &\quad - \frac{1}{2}\beta_{1,x}\beta_2 + sr^2 + s^3 - s\alpha_2\beta_1 + s\alpha_1\beta_2, \\ s_{t_2} &= \alpha_{2,xx} + \frac{1}{2}r_x\alpha_2 + r\alpha_{2,x} - \frac{1}{2}s_x\beta_2 - s\beta_{2,x} \\ &\quad + \beta_2\alpha_1\alpha_2 - \frac{1}{2}\alpha_2 r^2 - \frac{1}{2}\alpha_2 s^2, \\ \alpha_{1,t_2} &= -\beta_{1,xx} + \frac{1}{2}r_x\beta_1 + r\beta_{1,x} - \frac{1}{2}s_x\alpha_1 \\ &\quad - s\alpha_{1,x} - \alpha_1\beta_1\beta_2 + \frac{1}{2}\beta_1 r^2 + \frac{1}{2}\beta_1 s^2, \\ \beta_{1,t_2} &= \alpha_{1,xx} + \beta_{1,x}s + \frac{1}{2}s_x\beta_1 + \alpha_{1,x}r + \frac{1}{2}r_x\alpha_1 \\ &\quad - \frac{1}{2}\alpha_1 r^2 - \frac{1}{2}\alpha_1 s^2 + \alpha_1\alpha_2\beta_1, \\ \alpha_{2,t_2} &= \beta_{2,xx} - \alpha_{2,x}s - \frac{1}{2}\alpha_2 s_x - \beta_{2,x}r - \frac{1}{2}\beta_2 r_x \\ &\quad - \frac{1}{2}\beta_2 r^2 - \frac{1}{2}\beta_2 s^2 + \alpha_2\beta_1\beta_2, \\ \beta_{2,t_2} &= -\alpha_{2,xx} - \frac{1}{2}r_x\alpha_2 - r\alpha_{2,x} + \frac{1}{2}s_x\beta_2 + s\beta_{2,x} \\ &\quad + \frac{1}{2}\alpha_2 r^2 + \frac{1}{2}\alpha_2 s^2 - \beta_2\alpha_1\alpha_2, \end{aligned} \tag{19}$$

which are, respectively, reduced to the super Dirac equations

$$\begin{aligned} r_{t_2} &= -\frac{1}{2}s_{xx} - \alpha\alpha_x + \beta\beta_x + r^2 s + s^3 + 2s\alpha\beta, \\ s_{t_2} &= \frac{1}{2}r_{xx} - \alpha\beta_x + \alpha_x\beta - rs^2 - r^3 - 2r\alpha\beta, \\ \alpha_{t_2} &= -\beta_{xx} + r\beta_x - s\alpha_x - \frac{1}{2}s_x\alpha + \frac{1}{2}r_x\beta + \frac{1}{2}r^2\beta + \frac{1}{2}s^2\beta, \\ \beta_{t_2} &= \alpha_{xx} + r\alpha_x + s\beta_x + \frac{1}{2}r_x\alpha + \frac{1}{2}s_x\beta - \frac{1}{2}r^2\alpha - \frac{1}{2}s^2\alpha, \end{aligned} \tag{20}$$

or the famous Dirac equations

$$r_{t_2} = -\frac{1}{2}s_{xx} + r^2 s + s^3, \qquad s_{t_2} = \frac{1}{2}r_{xx} - rs^2 - r^3, \tag{21}$$

when $\alpha_1 = -\alpha_2 = \alpha$, $\beta_2 = \beta_1 = \beta$, or $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$.

3. Localness

We note that the recursion operator L is an integrodifferential operator, but the generalized super Dirac system (17) are pure differential equations.

We put

$$\begin{gathered} \overline{B} = 1 + \frac{\overline{B}_1}{\lambda} + \frac{\overline{B}_2}{\lambda^2} + \cdots, \\ \overline{G} = -g_0 + \frac{\overline{G}_1}{\lambda} + \frac{\overline{G}_2}{\lambda^2} + \cdots, \\ \overline{V} = \frac{\overline{v}_1}{\lambda} + \frac{\overline{v}_2}{\lambda^2} + \frac{\overline{v}_3}{\lambda^3} + \cdots, \\ \overline{V}^T = \left(\overline{A}, \overline{C}, \overline{\delta}, \overline{\rho}, \overline{\epsilon}, \overline{\tau}\right), \\ \overline{v}_j^T = \left(\overline{A}_j, \overline{C}_j, \overline{\delta}_j, \overline{\rho}_j, \overline{\epsilon}_j, \overline{\tau}_j\right), \end{gathered} \tag{22}$$

then, $\overline{B}$, $\overline{C}$, and $\overline{V}$ satisfy the following equations:

$$\begin{gathered} -\overline{G}_x + \alpha_1\overline{\varepsilon} + \beta_2\overline{\rho} + \beta_1\overline{\tau} + \alpha_2\overline{\delta} = 0, \\ -\overline{B}_x + 2r\overline{A} - s\overline{C} + \frac{1}{2}\alpha_1\overline{\tau} + \frac{1}{2}\alpha_2\overline{\rho} - \frac{1}{2}\beta_1\overline{\varepsilon} - \frac{1}{2}\beta_2\overline{\delta} = 0, \\ -\overline{C}_x - 4\lambda\overline{A} - 4s\overline{B} + \alpha_1\overline{\varepsilon} + \beta_2\overline{\rho} - \beta_1\overline{\tau} - \alpha_2\overline{\delta} = 0, \\ -\overline{A}_x - \lambda\overline{C} + 2r\overline{B} + \frac{1}{2}\alpha_1\overline{\tau} + \frac{1}{2}\alpha_2\overline{\rho} + \frac{1}{2}\beta_1\overline{\varepsilon} + \frac{1}{2}\beta_2\overline{\delta} = 0, \\ -\overline{\rho}_x + \lambda\overline{\delta} + r\overline{\rho} + s\overline{\delta} + \frac{1}{2}\alpha_1\overline{G} - \frac{1}{2}\alpha_1\overline{C} - \beta_1\overline{A} - \beta_1\overline{B} = 0, \\ -\overline{\delta}_x - \lambda\overline{\rho} + s\overline{\rho} - r\overline{\delta} + \frac{1}{2}\beta_1\overline{G} - \alpha_1\overline{A} + \alpha_1\overline{B} + \frac{1}{2}\beta_1\overline{C} = 0, \\ -\overline{\tau}_x - \lambda\overline{\varepsilon} + \beta_2\overline{A} + \beta_2\overline{B} - \frac{1}{2}\alpha_2\overline{C} - \frac{1}{2}\alpha_2\overline{G} - s\overline{\varepsilon} + r\overline{\tau} = 0, \\ -\overline{\varepsilon}_x + \lambda\overline{\tau} + \frac{1}{2}\beta_2\overline{C} - \frac{1}{2}\beta_2\overline{G} + \alpha_2\overline{A} - \alpha_2\overline{B} - r\overline{\varepsilon} - s\overline{\tau} = 0. \end{gathered} \tag{23}$$

From (23), we obtain

$$\left(-\frac{\overline{G}^2}{4} - \overline{B}^2 + \frac{\overline{C}^2}{4} + \overline{A}^2 - \overline{\epsilon}\overline{\rho} + \overline{\delta}\overline{\tau}\right)_x = 0. \tag{24}$$

Integrating (24) with respect to x, we get

$$-\frac{\overline{G}^2}{4} - \overline{B}^2 + \frac{\overline{C}^2}{4} + \overline{A}^2 - \overline{\epsilon}\overline{\rho} + \overline{\delta}\overline{\tau} = -1 - \frac{g_0^2}{4}. \tag{25}$$

Substituting (22) into (25) and equating the coefficients of λ^{-j} on both sides of the equation, we have

$$\overline{B}_1 = \overline{G}_1 = 0, \tag{26}$$

$$\begin{aligned} \overline{B}_j - \frac{1}{4}g_0\overline{G}_j = {} & \frac{1}{2}\sum_{l=1}^{j-1}\overline{A}_l\overline{A}_{j-l} + \frac{1}{4}\sum_{l=1}^{j-1}\overline{C}_l\overline{C}_{j-l} \\ & + \frac{1}{2}\sum_{l=1}^{j-1}\overline{\rho}_l\overline{\epsilon}_{j-l} - \frac{1}{2}\sum_{l=1}^{j-1}\overline{\tau}_l\overline{\delta}_{j-l}. \end{aligned} \tag{27}$$

We point that the left side of (27) involves an arbitrary constant g_0. Based on (27) and (9), it is easy to find that $\overline{B}_i$, $\overline{G}_i$, $\overline{A}_i$, $\overline{C}_i$, $\overline{\epsilon}_i$, $\overline{\rho}_i$, $\overline{\delta}_i$, and $\overline{\tau}_i$ are all differential polynomials in six variables r, s, α_1, α_2, β_1, and β_2. So the hierarchy of the super integrable equations (17) is pure differential equations.

4. The Super Hamiltonian Structures

In this section, we will establish the super Hamiltonian structure of the generalized super Dirac hierarchy (18) by super trace identity [2, 11]:

$$\frac{\delta}{\delta u}\int \operatorname{Str}\left(V\frac{\partial U}{\partial \lambda}\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\operatorname{Str}\left(V\frac{\partial U}{\partial u}\right), \tag{28}$$

where $u = (r, s, \alpha_1, \beta_1, \alpha_2, \beta_2)^T$ and the constant γ is determined by

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln\left|\operatorname{Str}(VV)\right|. \tag{29}$$

Through direct calculations, we have

$$\begin{gathered} \operatorname{Str}\left(V\frac{\partial U}{\partial r}\right) = C, \qquad \operatorname{Str}\left(V\frac{\partial U}{\partial \alpha_1}\right) = -\varepsilon, \\ \operatorname{Str}\left(V\frac{\partial U}{\partial s}\right) = 2A, \qquad \operatorname{Str}\left(V\frac{\partial U}{\partial \alpha_2}\right) = \delta, \\ \operatorname{Str}\left(V\frac{\partial U}{\partial \lambda}\right) = -2B, \qquad \operatorname{Str}\left(V\frac{\partial U}{\partial \beta_1}\right) = -\tau, \\ \operatorname{Str}\left(V\frac{\partial U}{\partial \beta_2}\right) = \rho. \end{gathered} \tag{30}$$

Substituting the above results into the super trace identity (28) yields that

$$\frac{\delta}{\delta u}\int -2B\,dx = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^{\gamma}\left(C, 2A, -\varepsilon, -\tau, \delta, \rho\right)^T. \tag{31}$$

Comparing the coefficients of λ^{-m-2} on both sides of (31) gives rise to

$$\frac{\delta}{\delta u}\int -2B_{m+2}\,dx = (\gamma - m - 1)\begin{pmatrix} C_{m+1} \\ 2A_{m+1} \\ -\varepsilon_{m+1} \\ -\tau_{m+1} \\ \delta_{m+1} \\ \rho_{m+1} \end{pmatrix}, \quad m \geq 0. \tag{32}$$

By employing the computing formula (29) on the constant γ, we obtain $\gamma = 0$. Thus, we have

$$\frac{\delta \mathcal{H}_m}{\delta u} = \begin{pmatrix} C_{m+1} \\ 2A_{m+1} \\ -\varepsilon_{m+1} \\ -\tau_{m+1} \\ \delta_{m+1} \\ \rho_{m+1} \end{pmatrix}, \qquad \mathcal{H}_m = \int \frac{2}{m+1} B_{m+1} dx, \qquad m \geq 0. \tag{33}$$

It then follows that the super integrable hierarchy (17) possesses the following super Hamiltonian form:

$$u_{t_n} = K_n(u) = J \frac{\delta \mathcal{H}_n}{\delta u}, \quad n \geq 0, \tag{34}$$

where the super Hamiltonian operator J is given by

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}. \tag{35}$$

5. Concluding Remarks

In this paper, based on a 3×3 matrix super spectral problem, we considered the related four Fermi component super Dirac-type systems and also proved the localness of the whole super soliton hierarchy. We obtained the super Hamiltonian structure and different reductions for the super integrable equations. Our computation reflects how to choose appropriate Lie algebras to generate soliton hierarchies [12], and the generating procedure can be applied to the other super soliton hierarchies. Let us notice that the super integrable hierarchy (17) allows for an arbitrary constant g_0 and such system is interesting since different specifications of g_0 lead to different super Dirac type equations. The nonlinearization approach for integrable systems is a powerful tool to generate finite-dimensional integrable Hamiltonian systems. The super integrable system (17) may admit nonlinearization. Moreover, the super Dirac system (17) may inherit various other integrable characteristics, such as first-degree time-dependent symmetries [13] and Bäcklund transformation [14]. In particular, it is of interest to study multi-integrable couplings and the corresponding super Hamiltonian structures of the super Dirac system (17) by the super variational identity [15]. These related issues may be considered in further publication.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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