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Research Article

LaSalle-Type Theorems for General Nonlinear Stochastic Functional Differential Equations by Multiple Lyapunov Functions

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We investigate LaSalle-type theorems for general nonlinear stochastic functional differential equations. With some preliminaries on lemmas and the derivation techniques, we establish three LaSalle-type theorems for the general nonlinear stochastic functional differential equations via multiple Lyapunov functions. For the typical special case with estimations involving $|x_t|^p$ for the derivatives of the Lyapunov functions, a theorem is established as the corollary of the main theorem. At the end of the paper, an example is given to illustrate the usage of the method proposed and show the advantage of the results obtained.

1. Introduction

As it is well known, the Lyapunov function method is the most widely used tool to establish criteria for stability or other asymptotic properties of dynamic systems governed by differential equations or difference equations. With this method, the derivatives of the Lyapunov functions or their upper bounds are often desired to be negative definite. For some complex equations, for example, the functional differential equations, this point may be somehow difficult for us at some times. Thus a spontaneous question will arise, that is, may we weaken the negative definiteness conditions for the derivatives of the Lyapunov functions or their upper bounds? In fact, some investigations have been made in the past years in this aspect. For example, LaSalle established a very important theorem named LaSalle invariance principle or LaSalle's theorem [1], which weakened the condition of the Lyapunov function method on the negative definiteness of the derivatives of the Lyapunov functions along the solutions of the equations, and it has been widely used in the theory of ordinary differential equations. In the recent years, LaSalle's theorem has been generalized directly to the stochastic differential delay equations by Mao [2–5], and a kind of LaSalle-type theorems had been established. Then the LaSalle-type theorems for stochastic differential delay equations have also been generalized to a kind of stochastic functional differential equations with distributed delays by Shen et al. [6–11]. Limited by the derivation techniques, this kind of theorems has not been generalized to the most general nonlinear stochastic functional differential equations so far.

In this paper, we generalize the investigation by Xuerong Mao, Yi Shen, and other authors to the general nonlinear stochastic functional differential equations. With some preliminaries on lemmas and the derivation techniques, we establish three LaSalle-type theorems for the general nonlinear stochastic functional differential equations via multiple Lyapunov functions. For the typical special case with estimations involving $|x_t|^p$ for the derivatives of the Lyapunov functions, a theorem is established as the corollary of the main theorem of the paper. The key point of the paper lies in the treatment of the general retarded terms in the estimations for the derivatives of the Lyapunov functions or their upper bounds. At the end of the paper, an example is given to illustrate the usage of the method proposed in the paper.

2. Preliminaries

2.1. Basic Notations. Throughout the paper, unless otherwise specified, we will employ the following notions. τ is a positive constant which stands for the upper bound for the bounded time delays involved possibly in the involved inequalities or equations, and $I_\tau = [-\tau, 0]$, $I_{2\tau} = [-2\tau, 0]$. $C(I_\tau, R^n)$ denotes the space of continuous functions ϕ from I_τ to R^n with norm $|\phi| = \sup_{\theta\in I_\tau}\|\phi(\theta)\|$, where $\|\cdot\|$ is any kind of norms for vectors. Let $t_0 \in R^+ = [0, +\infty)$, $I = [t_0 - \tau, +\infty)$. $C^b_{\mathscr{F}_0}(I_\tau, R^n)$ denotes the family of all $\mathscr{F}_0$-measurable bounded $C(I_\tau, R^n)$-valued random variables.

For a given function $x(t) \in C(I, R^n)$, the associated function $x_t \in C(I_\tau, R^n)$ is defined as $x_t(\theta) = x(t+\theta), \theta \in I_\tau$. In real applications of the results of this paper, the criteria may be used in the function space $C(I_{2\tau}, R^n)$; in this case, one can extend the norm of $\phi \in C(I_{2\tau}, R^n)$ as $|\phi|_{2\tau} = \sup_{\theta\in I_{2\tau}}\|\phi(\theta)\|$. For the general theory of functional differential equations, the readers are referred to [12–17].

In the paper, we will replace the function x_t in $C(I_\tau, R^n)$ by $x(t)$ when necessary via an operator "Fr(·)", named freezing operator. In this case, as a function in $C(I_\tau, R^n)$, $x(t)$ will be counted as a constant; that is, for any $\theta \in I_\tau$, $\mathrm{Fr}[x(t)](\theta) = x(t)$; thus we have $|\mathrm{Fr}[x(t)]| = |x(t)| = \|x(t)\|$. For example, for a functional $\mathscr{K}(t, x_t) = \int_{-\tau}^0 k(t,\theta)x_t(\theta)\mathrm{d}\theta$, if we replace x_t by $x(t)$, then it becomes

$$\begin{aligned}\mathscr{K}(t, \mathrm{Fr}[x(t)]) &= \int_{-\tau}^0 k(t,\theta)\, x(t)\,\mathrm{d}\theta \\ &= \left(\int_{-\tau}^0 k(t,\theta)\,\mathrm{d}\theta\right) x(t).\end{aligned} \tag{1}$$

In the paper, we also define $L^1(R^+, R^+) = \{\gamma(\cdot) : \int_0^{+\infty}\gamma(s)\mathrm{d}s < +\infty\}$ and $D(R^+, R^+) = \{\eta(\cdot) : \int_0^{+\infty}\eta(s)\mathrm{d}s = +\infty\}$.

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge0}, \mathscr{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\ge0}$ satisfying the usual conditions; that is, it is right continuous and $\mathscr{F}_0$ contains all P-null sets, and let $W(t) = (W_1(t), W_2(t), \dots, W_m(t))^T$ be an m-dimensional Wiener process defined on the probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge0}, \mathscr{P})$.

2.2. Equation Description and Basic Assumptions. Given the stochastic functional differential equation

$$\mathrm{d}x(t) = f(t, x_t)\,\mathrm{d}t + g(t, x_t)\,\mathrm{d}W(t), \quad t \ge t_0 \ge 0, \tag{2}$$

where the state $x \in R^n$. $f : R^+ \times C(I_\tau, R^n) \to R^n$ and $g : R^+ \times C(I_\tau, R^n) \to R^{n\times m}$, $f(t, 0) = 0$, $g(t, 0) = 0$. The given initial data is given by $\phi_0 = \{x(\theta) : -\tau \le \theta \le 0\} \in C^b_{\mathscr{F}_0}(I_\tau, R^n)$. $x_t = \{x(t+\theta) : \theta \in I_\tau\}$ is a stochastic process in $C(I_\tau, R^n)$. When the criteria are used in the function space $C(I_{2\tau}, R^n)$, the initial data $\phi_0 \in C^b_{\mathscr{F}_0}(I_\tau, R^n)$ can extend to $\phi_0 \in C^b_{\mathscr{F}_0}(I_{2\tau}, R^n)$ by $\phi_0(\theta) = \phi_0(-\tau)$ for $\theta \in [-2\tau, -\tau]$.

As a standing condition, we impose the Lipschitz condition for the coefficients.

(H_1) Both f and g satisfy the Lipschitz condition. That is, there is a constant $L > 0$ such that

$$\|f(t,\phi) - f(t,\varphi)\| \vee \|g(t,\phi) - g(t,\varphi)\| \le L|\phi - \varphi|, \tag{3}$$

for all $t > 0$ and $\phi, \varphi \in C(I_\tau, R^n)$, which may be directly useful in our estimations when necessary.

In this case, $f(t,\phi)$ and $g(t,\phi)$ also satisfy the linear growth condition. In fact, by (H_1), we have $\|f(t,\phi)\| \vee \|g(t,\phi)\| = \|f(t,\phi) - f(t,0)\| \vee \|g(t,\phi) - g(t,0)\| \le L|\phi - 0| = L|\phi|$, for $\phi \in C(I_\tau, R^n)$. Thus, under the Lipschitz condition, there exists a unique global solution, which is denoted by $x(t, t_0, \phi_0)$ in this paper, to the equation for each initial data $(t_0, \phi_0) \in R^+ \times C^b_{\mathscr{F}_0}(I_\tau, R^n)$; see [14].

It is also known that, under the Lipschitz condition (H_1), the solution $x(t)$ with initial data (t_0, ϕ_0) to (2) is continuous, satisfying $E\{\sup_{t_0-\tau\le s\le t}\|x(s,\phi_0)\|^p\} < +\infty$ for $t \ge t_0$ and arbitrary $p > 0$; see [14].

In the paper, some coefficients and functions will be involved. Assume that, for each $i = 1, 2, \dots, k$, functions $\gamma_i(\cdot) \in L^1(R^+, R^+)$, $\widehat{k}_i(\cdot) \in C([-\tau, 0], R^+)$, $\eta_i(\cdot) \in D(R^+, R^+)$, $\rho_i(\cdot) \in C(R^+ \times R^+ \times \cdots \times R^+, R^+)$, $\rho_i^* \in C(R^n, R^+)$, $w_{1i}(t,x)$, $w_{3i}(t,x) \in C(R^+ \times R^n, R^+)$, $w_{2i}(t,\phi) \in C(R^+ \times C(I_\tau, R^n), R^+)$.

For $w_{2i}(t, x_t)$, we further assume the following.

(H_2) Along the solution of the equation, for each $i = 1, 2, \dots, k$, we have estimation

$$\begin{aligned} & w_{2i}(t, x_t) - w_{2i}(t, \mathrm{Fr}[x(t)]) \\ & \le \int_{t-\tau}^t \Big(\widehat{\mathscr{C}}_{1i}(t)\, w_{1i}(s, x(s)) + \widehat{\mathscr{C}}_{2i}(t)\, w_{2i}(s, x_s) \\ & \qquad + \widehat{\mathscr{C}}_{3i}(t)\, w_{3i}(s, x(s))\Big)\,\mathrm{d}s \\ & \qquad + \gamma_i^*(t) + \zeta_i(t, x_t, W(\cdot)), \end{aligned} \tag{4}$$

where $\widehat{\mathscr{C}}_{ji}(t)$, $j = 1, 2, 3$ are nonnegative continuous functions on R^+ and

$$\overline{\mathscr{C}}_{ji}(t) \le \mathscr{C}_{ji}(t), \quad j = 1, 2, 3 \tag{5}$$

with $0 \le \mathscr{C}_{2i}(t) = \mathscr{C}_{2i} < 1$. $\gamma_i^*(\cdot) \in L^1(R^+, R^+)$, $E\int_{t_0}^t \zeta_i(s, x_s, W(\cdot))\mathrm{d}s = 0$, and, especially,

$$w_i(t,x) \ge 0, \quad i = 1, 2, \dots, k, \tag{6}$$

where

$$\begin{aligned} \overline{\mathscr{C}}_{ji}(t) &= \int_t^{t+\tau}\widehat{\mathscr{C}}_{ji}(s)\,\mathrm{d}s, \\ w_i(t,x) &= \left(1 - \frac{\mathscr{C}_{1i}(t)}{1-\mathscr{C}_{2i}}\right) w_{1i}(t,x) - \frac{1}{1-\mathscr{C}_{2i}} w_{2i}(t, \mathrm{Fr}[x]) \\ & \quad - \left(k_i + \frac{\mathscr{C}_{3i}(t)}{1-\mathscr{C}_{2i}}\right) w_{3i}(t,x), \\ & \quad j = 1, 2, 3, \ i = 1, 2, \dots, k, \end{aligned} \tag{7}$$

where k_i will be defined in Lemma 4.

Remark 1. The assumption on $w_{2i}(t,x_t)-w_{2i}(t,\mathrm{Fr}[x(t)])$ can also be described in the following form theoretically if $w_{2i}(t,\cdot)$ is differentiable. Denote

$$\begin{aligned} M_i(\mu) &= M_i(\mu,t,x_t) = w_{2i}(t,\mu x_t + (1-\mu)\,\mathrm{Fr}[x(t)]), \\ \overline{M}_i(\mu) &= \overline{M}_i(\mu,t,x_t) = \frac{\mathrm{d}M_i(\mu)}{\mathrm{d}\mu}, \\ \widehat{M}_i(t,x_t) &= \int_0^1 \overline{M}_i(\mu)\,\mathrm{d}\mu. \end{aligned} \tag{8}$$

Assume that

$$\begin{aligned} \widehat{M}_i(t,x_t) \le \int_{t-\tau}^{t} &\left(\widehat{\mathscr{C}}_{1i}(t)\,w_{1i}(s,x(s)) + \widehat{\mathscr{C}}_{2i}(t)\,w_{2i}(s,x_s)\right. \\ &\left. + \widehat{\mathscr{C}}_{3i}(t)\,w_{3i}(s,x(s))\right)\mathrm{d}s \\ &+ \gamma_i^*(t) + \zeta_i(t,x_t,W(\cdot)). \end{aligned} \tag{9}$$

Let $V(t,x) \in C^{1,2}(R^+ \times R^n, R^+)$; that is, $V(t,x)$ is once differentiable in t and twice continuously differentiable in x. Define a differential operator $\mathscr{L}$, associated with (2), acting on $V(t,x)$ by

$$\begin{aligned} \mathscr{L}V(t,\phi) = V_t(t,\phi(0)) &+ V_x(t,\phi(0))\,f(t,\phi) \\ &+ \frac{1}{2}\mathrm{Tr}\left(g^T(t,\phi)\,V_{xx}(t,\phi(0))\,g(t,\phi)\right), \end{aligned} \tag{10}$$

where

$$\begin{aligned} V_t(t,x) &= \frac{\partial V(t,x)}{\partial t}, \\ V_x(t,x) &= \left(\frac{\partial V(t,x)}{\partial x_1},\ldots,\frac{\partial V(t,x)}{\partial x_n}\right), \\ V_{xx} &= \left(\frac{\partial^2 V(t,x)}{\partial x_i \partial x_j}\right)_{n\times n}, \quad i,j = 1,\ldots,n. \end{aligned} \tag{11}$$

The assumptions for the involved Lyapunov functions will, respectively, be as follows:

(H_3)

$$\begin{aligned} &\mathscr{L}V_i(t,\phi) \\ &\quad \le \gamma_i(t) - w_{1i}(t,\phi(0)) \\ &\qquad + w_{2i}(t,\phi) + \int_{-\tau}^{0}\widehat{k}_i(\theta)\,w_{3i}(t,\phi(\theta))\,\mathrm{d}\theta, \\ &\gamma_i(t) - \mathscr{L}V_i(t,\phi) - w_{1i}(t,\phi(0)) \\ &\qquad + w_{2i}(t,\phi) + \int_{-\tau}^{0}\widehat{k}_i(\theta)\,w_{3i}(t,\phi(\theta))\,\mathrm{d}\theta \\ &\qquad + \left\|V_{ix}(t,\phi(0))\,g(t,\phi)\right\|^2 \\ &\quad \ge \eta_i(t)\,\rho_i\left(V_1(t,\phi(0)),V_2(t,\phi(0)),\right. \\ &\qquad\qquad \left.\ldots,V_k(t,\phi(0))\right); \end{aligned} \tag{12}$$

(H_4)

$$\begin{aligned} &\mathscr{L}V_i(t,\phi) \\ &\quad \le \gamma_i(t) - w_{1i}(t,\phi(0)) \\ &\qquad + w_{2i}(t,\phi) + \int_{-\tau}^{0}\widehat{k}_i(\theta)\,w_{3i}(t,\phi(\theta))\,\mathrm{d}\theta, \\ &\gamma_i(t) - \mathscr{L}V_i(t,\phi) - w_{1i}(t,\phi(0)) \\ &\qquad + w_{2i}(t,\phi) + \int_{-\tau}^{0}\widehat{k}_i(\theta)\,w_{3i}(t,\phi(\theta))\,\mathrm{d}\theta \\ &\qquad + \left\|V_{ix}(t,\phi(0))\,g(t,\phi)\right\|^2 \\ &\quad \ge \eta_i(t)\,\rho_i^*(\phi(0)). \end{aligned} \tag{13}$$

2.3. Lemmas. By the nonnegative semimartingale convergence theorem [18], with a simple variable substitution for time t, we directly have the following.

Lemma 2. *Let $A(t)$ and $U(t)$ be two continuous adapted increasing processes on $t \ge t_0 \ge 0$ with $A(t_0) = U(t_0) = 0$, a.s., $M(t) = M(t,\omega)$ a real-valued continuous local martingale with $M(t_0,\omega) = 0$, a.s., and ξ a nonnegative $\mathscr{F}_0$-measurable random variable such that $E\xi < \infty$. Define*

$$X(t) = \xi + A(t) - U(t) + M(t), \quad \text{for } t \ge t_0. \tag{14}$$

If $X(t)$ is nonnegative, then

$$\begin{aligned} \left\{\lim_{t\to+\infty} A(t) < +\infty\right\} \subset &\left\{\lim_{t\to+\infty} X(t) < +\infty\right\} \\ &\cap \left\{\lim_{t\to+\infty} U(t) < +\infty\right\}, \quad \text{a.s.}, \end{aligned} \tag{15}$$

where $B \subset D$, a.s. means $P(B \cap D^C) = 0$. In particular, if $\lim_{t\to+\infty} A(t) < +\infty$, a.s., then for almost all $\omega \in \Omega$,

$$\begin{aligned} \lim_{t\to+\infty} X(t) < +\infty, \qquad \lim_{t\to+\infty} U(t) < +\infty, \\ -\infty < \lim_{t\to+\infty} M(t) < +\infty. \end{aligned} \tag{16}$$

That is, both $X(t)$ and $U(t)$ a.s. converge to finite random variables as $t \to +\infty$.

Lemma 3. *Under the assumption (H_2), along the solution of (2), one has, for each $i = 1,2,\ldots,k$,*

$$\begin{aligned} &\int_{t_0}^{t} w_{2i}(s,x_s)\,\mathrm{d}s \\ &\quad \le \int_{t_0}^{t} w_{2i}(s,Fr[x(s)])\,\mathrm{d}s \end{aligned}$$

$$+\int_{t_0}^{t}\left(\mathscr{C}_{1i}(s)\,w_{1i}(s,x(s))\right.$$
$$\left.+\mathscr{C}_{2i}w_{2i}(s,x_s)+\mathscr{C}_{3i}(s)\,w_{3i}(s,x(s))\right)\mathrm{d}s$$
$$+\int_{t_0}^{t}\gamma_i^*(s)\,ds+\int_{t_0}^{t}\zeta_i(s,x_s,W(\cdot))\,\mathrm{d}s+U_{1i}(t_0,\phi_0),\tag{17}$$

where

$$U_{1i}(t_0,\phi_0)=\int_{t_0-\tau}^{t_0}\left(\mathscr{C}_{1i}(s)\,w_{1i}(s,x(s))+\mathscr{C}_{2i}w_{2i}(s,x_s)\right.$$
$$\left.+\mathscr{C}_{3i}(s)\,w_{3i}(s,x(s))\right)\mathrm{d}s\geq 0.\tag{18}$$

Proof. Firstly, we directly have

$$w_{2i}(t,x_t)$$
$$=w_{2i}(t,\mathrm{Fr}[x(t)])+\left(w_{2i}(t,x_t)-w_{2i}(t,\mathrm{Fr}[x(t)])\right),\tag{19}$$

and thus we have

$$\int_{t_0}^{t}w_{2i}(s,x_s)\,\mathrm{d}s=\int_{t_0}^{t}w_{2i}(s,\mathrm{Fr}[x(s)])\,\mathrm{d}s$$
$$+\int_{t_0}^{t}\left(w_{2i}(s,x_s)-w_{2i}(s,\mathrm{Fr}[x(s)])\right)\mathrm{d}s.\tag{20}$$

Combining with the assumption (H_2), this yields

$$\int_{t_0}^{t}w_{2i}(s,x_s)\,\mathrm{d}s$$
$$\leq\int_{t_0}^{t}w_{2i}(s,\mathrm{Fr}[x(s)])\,\mathrm{d}s$$
$$+\int_{t_0}^{t}\int_{s-\tau}^{s}\left(\widehat{\mathscr{C}}_{1i}(s)\,w_{1i}(u,x(u))+\widehat{\mathscr{C}}_{2i}(s)\,w_{2i}(u,x_u)\right.$$
$$\left.+\widehat{\mathscr{C}}_{3i}(s)\,w_{3i}(u,x(u))\right)\mathrm{d}u\,\mathrm{d}s$$
$$+\int_{t_0}^{t}\gamma_i^*(s)\,\mathrm{d}s+\int_{t_0}^{t}\zeta_i(s,x_s,W(\cdot))\,\mathrm{d}s$$
$$=\int_{t_0}^{t}w_{2i}(s,\mathrm{Fr}[x(s)])\,\mathrm{d}s$$
$$+\int_{t_0-\tau}^{t}\int_{t_0\vee u}^{t\wedge(u+\tau)}\left(\widehat{\mathscr{C}}_{1i}(s)\,w_{1i}(u,x(u))\right.$$
$$+\widehat{\mathscr{C}}_{2i}(s)\,w_{2i}(u,x_u)$$
$$\left.+\widehat{\mathscr{C}}_{3i}(s)\,w_{3i}(u,x(u))\right)\mathrm{d}s\,\mathrm{d}u$$
$$+\int_{t_0}^{t}\gamma_i^*(s)\,\mathrm{d}s+\int_{t_0}^{t}\zeta_i(s,x_s,W(\cdot))\,\mathrm{d}s$$
$$\leq\int_{t_0}^{t}w_{2i}(s,\mathrm{Fr}[x(s)])\,\mathrm{d}s$$
$$+\int_{t_0-\tau}^{t}\int_{u}^{u+\tau}\left(\widehat{\mathscr{C}}_{1i}(s)\,w_{1i}(u,x(u))\right.$$
$$+\widehat{\mathscr{C}}_{2i}(s)\,w_{2i}(u,x_u)$$
$$\left.+\widehat{\mathscr{C}}_{3i}(s)\,w_{3i}(u,x(u))\right)\mathrm{d}s\,\mathrm{d}u$$
$$+\int_{t_0}^{t}\gamma_i^*(s)\,\mathrm{d}s+\int_{t_0}^{t}\zeta_i(s,x_s,W(\cdot))\,\mathrm{d}s$$
$$=\int_{t_0}^{t}w_{2i}(s,\mathrm{Fr}[x(s)])\,\mathrm{d}s$$
$$+\int_{t_0-\tau}^{t}\left(\overline{\mathscr{C}}_{1i}(u)\,w_{1i}(u,x(u))\right.$$
$$+\overline{\mathscr{C}}_{2i}(u)\,w_{2i}(u,x_u)$$
$$\left.+\overline{\mathscr{C}}_{3i}(u)\,w_{3i}(u,x(u))\right)\mathrm{d}u$$
$$+\int_{t_0}^{t}\gamma_i^*(s)\,\mathrm{d}s+\int_{t_0}^{t}\zeta_i(s,x_s,W(\cdot))\,\mathrm{d}s$$
$$\leq\int_{t_0}^{t}w_{2i}(s,\mathrm{Fr}[x(s)])\,\mathrm{d}s$$
$$+\int_{t_0-\tau}^{t}\left(\mathscr{C}_{1i}(s)\,w_{1i}(s,x(s))+\mathscr{C}_{2i}w_{2i}(s,x_s)\right.$$
$$\left.+\mathscr{C}_{3i}(s)\,w_{3i}(s,x(s))\right)\mathrm{d}s$$
$$+\int_{t_0}^{t}\gamma_i^*(s)\,\mathrm{d}s+\int_{t_0}^{t}\zeta_i(s,x_s,W(\cdot))\,\mathrm{d}s$$
$$=\int_{t_0}^{t}w_{2i}(s,\mathrm{Fr}[x(s)])\,\mathrm{d}s$$
$$+\int_{t_0}^{t}\left(\mathscr{C}_{1i}(s)\,w_{1i}(s,x(s))+\mathscr{C}_{2i}w_{2i}(s,x_s)\right.$$
$$\left.+\mathscr{C}_{3i}(s)\,w_{3i}(s,x(s))\right)\mathrm{d}s$$
$$+\int_{t_0}^{t}\gamma_i^*(s)\,\mathrm{d}s+\int_{t_0}^{t}\zeta_i(s,x_s,W(\cdot))\,\mathrm{d}s$$
$$+U_{1i}(t_0,\phi_0).\tag{21}$$

The proof is complete. □

Lemma 4. *One has the following estimation:*

$$\int_{t_0}^{t}\int_{-\tau}^{0}\widehat{k}_i(\theta)\,w_{3i}(s,x(s+\theta))\,\mathrm{d}\theta\,\mathrm{d}s$$
$$\leq k_i\int_{t_0}^{t}w_{3i}(s,x(s))\,\mathrm{d}s+U_{2i}(t_0,\phi_0),\tag{22}$$

where $k_i = \int_{-\tau}^{0} \widehat{k}_i(\theta)\mathrm{d}\theta$ *and*

$$U_{2i}(t_0,\phi_0) = k_i \int_{t_0-\tau}^{t_0} w_{3i}(s,x(s))\,\mathrm{d}s, \quad i = 1,2,\ldots,k. \quad (23)$$

Proof. We directly have

$$\begin{aligned}
&\int_{t_0}^{t}\int_{-\tau}^{0} \widehat{k}_i(\theta)\,\omega_{3i}(s,x(s+\theta))\,\mathrm{d}\theta\,\mathrm{d}s \\
&= \int_{t_0}^{t}\int_{s-\tau}^{s} \widehat{k}_i(u-s)\,\omega_{3i}(u,x(u))\,\mathrm{d}u\,\mathrm{d}s \\
&= \int_{t_0-\tau}^{t}\int_{u\vee t_0}^{(u+\tau)\wedge t} \widehat{k}_i(u-s)\,\mathrm{d}s\,w_{3i}(u,x(u))\,\mathrm{d}u \\
&\le \int_{t_0-\tau}^{t}\int_{u}^{u+\tau} \widehat{k}_i(u-s)\,\mathrm{d}s\,w_{3i}(u,x(u))\,\mathrm{d}u \\
&= k_i \int_{t_0-\tau}^{t} w_{3i}(u,x(u))\,\mathrm{d}u = k_i \int_{t_0-\tau}^{t} w_{3i}(s,x(s))\,\mathrm{d}s \\
&= k_i \int_{t_0}^{t} w_{3i}(s,x(s))\,\mathrm{d}s + U_{2i}(t_0,\phi_0),
\end{aligned} \quad (24)$$

and this completes the proof. □

3. Main Results

Theorem 5. *Assume that* (H_2) *holds. If there are k Lyapunov functions* $V_1, V_2, \ldots, V_k \in C^{1,2}(R^+ \times R^n, R^+)$ *satisfying* (H_3), *and for each* $i = 1,2,\ldots,k$, $\ker \rho_i = \{u \in R^+ \times R^+ \times \cdots \times R^+ : \rho_i(u) = 0\} \neq \emptyset$, *then for all* $\phi_0 \in C^b_{\mathcal{F}_0}(I_\tau, R^n)$, *the solution* $x(t,t_0,\phi_0)$ *of* (2) *satisfies*

$$\begin{aligned}
&\limsup_{t\to+\infty} (V_1(t,x(t,t_0,\phi_0)), V_2(t,x(t,t_0,\phi_0)), \\
&\qquad \ldots, V_k(t,x(t,t_0,\phi_0))) \\
&\quad \in \ker\rho_i \quad \text{a.s.}, \quad i = 1,2,\ldots,k.
\end{aligned} \quad (25)$$

Proof. Denote $U_i(t_0,\phi_0) = (1/(1-\mathscr{C}_{2i}))U_{1i}(t_0,\phi_0) + U_{2i}(t_0,\phi_0)$. Firstly, by Lemma 3 we have

$$\begin{aligned}
&\int_{t_0}^{t} w_{2i}(s,x_s)\,\mathrm{d}s \\
&\le \frac{1}{1-\mathscr{C}_{2i}}\Big(\Big(\int_{t_0}^{t} w_{2i}(s,\mathrm{Fr}[x(s)])\,\mathrm{d}s \\
&\qquad + \int_{t_0}^{t} (\mathscr{C}_{1i}(s)\,w_{1i}(s,x(s)) \\
&\qquad\qquad + \mathscr{C}_{3i}(s)\,w_{3i}(s,x(s)))\,\mathrm{d}s \\
&\qquad + \int_{t_0}^{t} \gamma_i^*(s)\,\mathrm{d}s + \int_{t_0}^{t} \zeta_i(s,x_s,W(\cdot))\,\mathrm{d}s \\
&\qquad + U_{1i}(t_0,\phi_0)\Big),
\end{aligned} \quad (26)$$

and, combining with Lemma 4, we have

$$\begin{aligned}
&\int_{t_0}^{t}\Big(-w_{1i}(s,x(s)) + w_{2i}(s,x_s) \\
&\qquad + \int_{-\tau}^{0} \widehat{k}_i(\theta)\,w_{3i}(s,x(s+\theta))\,\mathrm{d}\theta\Big)\mathrm{d}s \\
&\le -\int_{t_0}^{t} w_i(s,x(s))\,\mathrm{d}s + \frac{1}{1-\mathscr{C}_{2i}}\int_{t_0}^{t} \gamma_i^*(s)\,\mathrm{d}s \\
&\qquad + \frac{1}{1-\mathscr{C}_{2i}}\int_{t_0}^{t} \zeta_i(s,x_s,W(\cdot))\,\mathrm{d}s + U_i(t_0,\phi_0).
\end{aligned} \quad (27)$$

Secondly, by the first assumption of the theorem, we have

$$\begin{aligned}
&\int_{t_0}^{t} \mathscr{L}V_i(s,x_s)\,\mathrm{d}s \\
&\le \int_{t_0}^{t}\Big(\gamma_i(s) - w_{1i}(s,x(s)) + w_{2i}(s,x_s) \\
&\qquad + \int_{-\tau}^{0} \widehat{k}_i(\theta)\,w_{3i}(s,x(s+\theta))\,\mathrm{d}\theta\Big)\mathrm{d}s \\
&\le \int_{t_0}^{t}\Big(\gamma_i(s) + \frac{1}{1-\mathscr{C}_{2i}}\gamma_i^*(s)\Big)\mathrm{d}s \\
&\qquad + \frac{1}{1-\mathscr{C}_{2i}}\int_{t_0}^{t} \zeta_i(s,x_s,W(\cdot))\,\mathrm{d}s \\
&\qquad + U_i(t_0,\phi_0) - \int_{t_0}^{t} w_i(s,x(s))\,\mathrm{d}s,
\end{aligned} \quad (28)$$

and thus we have

$$\begin{aligned}
&EV_i(t,x(t)) - EV_i(t_0,x(t_0)) \\
&= \int_{t_0}^{t} E\mathscr{L}V_i(s,x_s)\,\mathrm{d}s \\
&\le \int_{t_0}^{t}\Big(\gamma_i(s) + \frac{1}{1-\mathscr{C}_{2i}}\gamma_i^*(s)\Big)\mathrm{d}s + EU_i(t_0,\phi_0) \\
&\qquad - \int_{t_0}^{t} Ew_i(s,x(s))\,\mathrm{d}s,
\end{aligned} \quad (29)$$

due to $E\int_{t_0}^{t} \zeta_i(s,x_s,W(\cdot))\mathrm{d}s = 0$. With this we know that

$$\begin{aligned}
&\int_{t_0}^{t} Ew_i(s,x(s))\,\mathrm{d}s \\
&\le \int_{t_0}^{t} Ew_i(s,x(s))\,\mathrm{d}s + EV_i(t,x(t)) \\
&\le EV_i(t_0,x(t_0)) \\
&\qquad + \int_{t_0}^{t}\Big(\gamma_i(s) + \frac{1}{1-\mathscr{C}_{2i}}\gamma_i^*(s)\Big)\mathrm{d}s + EU_i(t_0,\phi_0),
\end{aligned} \quad (30)$$

and this implies that $\int_{t_0}^{+\infty} Ew_i(s,x(s))\mathrm{d}s$ converges, so is $\int_{t_0}^{+\infty} w_i(s,x(s))\mathrm{d}s$, a.s.

Third, denote

$$M_i(t)=\int_{t_0}^{t}V_{ix}(s,x(s))\,g(s,x_s)\,\mathrm{d}W(s). \quad (31)$$

For every integer $j\geq 1$, define a stopping time

$$\tau_{i,j}=\inf\{t\geq 0:\|M_i(t)\|\geq j\}. \quad (32)$$

Define sets $\mathcal{S}_{i,j}=\{t\geq 0:\|M_i(t)\|\geq j\}$; then one easily verifies that $\mathcal{S}_{i,j+1}\subset\mathcal{S}_{i,j}$; thus we have $\tau_{i,j+1}\geq\tau_{i,j}$; that is, the sequence $\tau_{i,j}$ is increasing for j.

Denote $A(t)=\int_{t_0}^{t}\mathscr{L}V_i(s,x_s)\mathrm{d}s+\int_{t_0}^{t}w_i(s,x(s))\mathrm{d}s$; then we have

$$\int_{t_0}^{t}\mathscr{L}V_i(s,x_s)\,\mathrm{d}s=A(t)-\int_{t_0}^{t}w_i(s,x(s))\,\mathrm{d}s. \quad (33)$$

By the above derivations, we have

$$\begin{aligned}A(t)&\leq\int_{t_0}^{t}\left(\gamma_i(s)+\frac{1}{1-\mathscr{C}_{2i}}\gamma_i^*(s)\right)\mathrm{d}s\\&\quad+\frac{1}{1-\mathscr{C}_{2i}}\int_{t_0}^{t}\zeta_i(s,x_s,W(\cdot))\,\mathrm{d}s+U_i(t_0,\phi_0),\\ EA(t)&\leq\int_{t_0}^{t}\left(\gamma_i(s)+\frac{1}{1-\mathscr{C}_{2i}}\gamma_i^*(s)\right)\mathrm{d}s+EU_i(t_0,\phi_0),\end{aligned} \quad (34)$$

and thus we have $\lim_{t\to+\infty}EA(t)<+\infty$, and then $\lim_{t\to+\infty}A(t)<+\infty$, a.s.

By the Itô's rule, we have

$$\begin{aligned}&V_i(t,x(t))\\&\quad=V_i(t_0,x(t_0))+\int_{t_0}^{t}\mathscr{L}V_i(s,x_s)\,\mathrm{d}s+M_i(t)\\&\quad=V_i(t_0,x(t_0))+A(t)-\int_{t_0}^{t}w_i(s,x(s))\,\mathrm{d}s+M_i(t).\end{aligned} \quad (35)$$

By Lemma 2, we obtain $-\infty<\lim_{t\to+\infty}M_i(t)<+\infty$, a.s., and $\lim_{t\to+\infty}V_i(t,x(t))<\infty$, a.s. Thus for sufficient large j, the probability for $\|M_i(t)\|\geq j$ is obviously 0; thus there is a subset Ω_1 of Ω with $P(\Omega_1)=1$ such that for every $\omega\in\Omega_1$ there is an $j(\omega)$ such that

$$\tau_{i,j}=+\infty\quad\forall j\geq j(\omega). \quad (36)$$

On the other hand, we have, for any $t>t_0$,

$$\begin{aligned}&E\int_{t_0}^{t\wedge\tau_{i,j}}\left\|V_{ix}(s,x(s))\,g(s,x_s)\right\|^2\,\mathrm{d}s\\&\quad=E\left\|M_i(t\wedge\tau_{i,j})\right\|^2\leq j^2.\end{aligned} \quad (37)$$

Letting $t\to+\infty$ yields

$$E\int_{t_0}^{\tau_{i,j}}\left\|V_{ix}(s,x(s))\,g(s,x_s)\right\|^2\,\mathrm{d}s\leq j^2, \quad (38)$$

which implies that

$$\int_{t_0}^{\tau_{i,j}}\left\|V_{ix}(s,x(s))\,g(s,x_s)\right\|^2\,\mathrm{d}s<+\infty, \quad (39)$$

with probability 1. Hence there is another subset Ω_2 of Ω with $P(\Omega_2)=1$ such that, if $\omega\in\Omega_2$, $\int_{t_0}^{\tau_{i,j}}\|V_{ix}(s,x(s))g(s,x_s)\|^2\,\mathrm{d}s<+\infty$ holds for every $j\geq 1$. Therefore, for any $\omega\in\Omega_1\cup\Omega_2$, we have

$$\begin{aligned}&\int_{t_0}^{+\infty}\left\|V_{ix}(s,x(s))\,g(s,x_s)\right\|^2\,\mathrm{d}s\\&\quad=\int_{t_0}^{\tau_{i,j(\omega)}(\omega)}\left\|V_{ix}(s,x(s))\,g(s,x_s)\right\|^2\,\mathrm{d}s<+\infty.\end{aligned} \quad (40)$$

Since $P(\Omega_1\cup\Omega_2)=1$, we must have

$$\int_{t_0}^{+\infty}\left\|V_{ix}(s,x(s))\,g(s,x_s)\right\|^2\,\mathrm{d}s<+\infty\quad\text{a.s.} \quad (41)$$

By Itô's rule again, together with the above derivation, we get

$$\begin{aligned}&-V_i(t,x(t))+V_i(t_0,x(t_0))\\&\quad+\frac{1}{1-\mathscr{C}_{2i}}\int_{t_0}^{t}\zeta_i(s,x_s,W(\cdot))\,\mathrm{d}s+U_i(t_0,\phi_0)\\&\quad+M_i(t)+\int_{t_0}^{t}\left(\gamma_i(s)+\frac{1}{1-\mathscr{C}_{2i}}\gamma_i^*(s)\right)\mathrm{d}s\\&\quad-\int_{t_0}^{t}w_i(s,x(s))\,\mathrm{d}s+\int_{t_0}^{t}\left\|V_{ix}(s,x(s))\,g(s,x_s)\right\|^2\mathrm{d}s\\&\geq\int_{t_0}^{t}\Big(-\mathscr{L}V_i(s,x_s)+\gamma_i(s)-w_{1i}(s,x(s))\\&\qquad+w_{2i}(s,x_s)+\int_{-\tau}^{0}k_i(\theta)\,w_{3i}(s,x(s+\theta))\,\mathrm{d}\theta\\&\qquad+\left\|V_{ix}(s,x(s))g(s,x_s)\right\|^2\Big)\,\mathrm{d}s\\&\geq\int_{t_0}^{t}\eta_i(s)\,\rho_i(V_1(s,x(s)),V_2(s,x(s)),\\&\qquad\ldots,V_k(s,x(s)))\,\mathrm{d}s,\\&\qquad i=1,2,\ldots,k.\end{aligned} \quad (42)$$

Notice that all the terms in the left side are a.s. bounded; then we have

$$\begin{aligned}&\int_{t_0}^{+\infty}\eta_i(s)\,\rho_i(V_1(s,x(s)),V_2(s,x(s)),\ldots,V_k(s,x(s)))\,\mathrm{d}s\\&\quad\leq\lim_{t\to\infty}\Big\{-V_i(t,x(t))+V_i(t_0,x(t_0))\\&\qquad+\frac{1}{1-\mathscr{C}_{2i}}\int_{t_0}^{t}\zeta_i(s,x_s,W(\cdot))\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+ U_i(t_0, \phi_0) + M_i(t) \\
&+ \int_{t_0}^{t} \left(\gamma_i(s) + \frac{1}{1 - \mathscr{C}_{2i}} \gamma_i^*(s) \right) \mathrm{d}s \\
&- \int_{t_0}^{t} w_i(s, x(s))\, \mathrm{d}s \\
&+ \int_{t_0}^{t} \left\| V_{ix}(s, x(s))\, g(s, x_s) \right\|^2 \mathrm{d}s \Big\} < +\infty \quad \text{a.s.}
\end{aligned}
\tag{43}
$$

It follows that

$$
\begin{aligned}
0 &\le \rho_i \left(\limsup_{t \to +\infty} V_1(t, x(t)), \ldots, \limsup_{t \to +\infty} V_k(t, x(t)) \right) \\
&= \limsup_{t \to +\infty} \rho_i \left(V_1(t, x(t)), \ldots, V_k(t, x(t)) \right) < +\infty \quad \text{a.s.}
\end{aligned}
\tag{44}
$$

since $\rho_i \in C(R^+ \times R^+ \times \cdots \times R^+, R^+)$. In fact, we have

$$
\begin{aligned}
&\rho_i \left(\limsup_{t \to +\infty} V_1(t, x(t)), \ldots, \limsup_{t \to +\infty} V_k(t, x(t)) \right) \\
&\quad = \limsup_{t \to +\infty} \rho_i \left(V_1(t, x(t)), \ldots, V_k(t, x(t)) \right) \\
&\quad = 0 \quad \text{a.s.}, \quad i = 1, 2, \ldots, k.
\end{aligned}
\tag{45}
$$

If not, there must be some $\overline{\Omega} \subset \Omega$ with $P(\overline{\Omega}) > 0$ such that, for any $\omega \in \overline{\Omega}$,

$$
\limsup_{t \to +\infty} \rho_i \left(V_1(t, x(t, w)), \ldots, V_k(t, x(t, w)) \right) > 0. \tag{46}
$$

Hence, for any $\omega \in \overline{\Omega}$, one can find a pair of $\varepsilon(\omega) > 0$ and $T(\omega) > t_0$ such that

$$
\begin{aligned}
&\rho_i \left(V_1(t, x(t, \omega)), \ldots, V_k(t, x(t, \omega)) \right) \ge \varepsilon(\omega) \\
&\qquad\qquad \text{whenever} \quad t \ge T(\omega).
\end{aligned}
\tag{47}
$$

Consequently

$$
\begin{aligned}
&\int_{t_0}^{+\infty} \eta_i(s)\, \rho_i \left(V_1(s, x(s, \omega)), \ldots, V_k(s, x(s, \omega)) \right) \mathrm{d}s \\
&\quad \ge \varepsilon(\omega) \int_{T(\omega)}^{+\infty} \eta_i(s)\, \mathrm{d}s = +\infty.
\end{aligned}
\tag{48}
$$

This is a contradiction, so (45) holds. It now follows from (45) that

$$
\begin{aligned}
&\limsup_{t \to +\infty} \left(V_1(t, x(t)), \ldots, V_k(t, x(t)) \right) \in \ker \rho_i, \\
&\qquad\qquad i = 1, 2, \ldots, k;
\end{aligned}
\tag{49}
$$

that is, we have the conclusion of the theorem. The proof is complete. □

Theorem 6. *Assume that (H_2) holds. If there are k Lyapunov functions $V_1, V_2, \ldots, V_k \in C^{1,2}(R^+ \times R^n, R^+)$ satisfying (H_3), $\ker \rho_i = \{u \in R^+ \times R^+ \times \cdots \times R^+ : \rho_i(u) = 0\} = \{0\}$, and for some $i_0 \in \{1, 2, \ldots, k\}$, $\rho_{i_0}(u) = 0$ implies $u = 0$; $V_i(t, x) = 0$, $i = 1, 2, \ldots, k$, imply $x = 0$; then $\forall \phi_0 \in C^b_{\mathscr{F}_0}(I_\tau, R^n)$; the solution $x(t, t_0, \phi_0)$ of (2) satisfies*

$$
\lim_{t \to +\infty} x(t, t_0, \phi_0) = 0 \quad \text{a.s.} \tag{50}
$$

Similarly, we have the following theorem.

Theorem 7. *Assume that (H_2) holds. If there are k Lyapunov functions $V_1, V_2, \ldots, V_k \in C^{1,2}(R^+ \times R^n, R^+)$ satisfying (H_4) and $\ker \rho_i^* = \{u \in R^n : \rho_i^*(u) = 0\} \neq \emptyset$, then $\forall \phi_0 \in C^b_{\mathscr{F}_0}(I_\tau, R^n)$; the solution $x(t, t_0, \phi_0)$ of (2) satisfies*

$$
\lim_{t \to +\infty} \rho_i^*(x(t)) = 0, \quad \text{a.s.}, \quad i = 1, 2, \ldots, k. \tag{51}
$$

4. Special Case: $w_{2i}(t, x_t) = C_{2i}^*(t) |x_t|^p$

Consider the special case with $w_{2i}(t, x_t) = C_{2i}^*(t) |x_t|^p$, where $C_{2i}^*(t)$ is nonincreasing, and $p > 0$, $C_{2i}^*(t) \ge 0$, $i = 1, 2, \ldots, k$.

Theorem 8. *Assume that (H_1) and (H_2) hold, one of (H_3) and (H_4) holds, and $w_i(t, x) \ge 0$ for $i = 1, 2, \ldots, k$, with parameters $\mathscr{C}_{1i} = \widehat{\mathscr{C}}_{1i}(t) = \mathscr{C}_{3i} = \widehat{\mathscr{C}}_{3i}(t) = 0$, $0 \le \mathscr{C}_{2i} < 1$, $\widehat{\mathscr{C}}_{2i} = (1/2)(2 + mL + m|p - 2|)pL$. If there are k Lyapunov functions $V_1, V_2, \ldots, V_k \in C^{1,2}(R^+ \times R^n, R^+)$ satisfying (H_3) or (H_4) and $\ker \rho_i = \{u \in R^+ \times R^+ \times \cdots \times R^+ : \rho_i(u) = 0\} \neq \emptyset$, then $\forall \phi_0 \in C^b_{\mathscr{F}_0}(I_\tau, R^n)$; the solution $x(t, t_0, \phi_0)$ of (2) satisfies*

$$
\begin{aligned}
&\limsup_{t \to +\infty} \left(V_1(t, x(t, t_0, \phi_0)), V_2(t, x(t, t_0, \phi_0)), \right. \\
&\qquad \left. \ldots, V_k(t, x(t, t_0, \phi_0)) \right) \\
&\quad \in \ker \rho_i, \quad \text{a.s.}, \quad i = 1, 2, \ldots, k,
\end{aligned}
\tag{52}
$$

or

$$
\lim_{t \to +\infty} \rho_i^*(x(t)) = 0, \quad \text{a.s.}, \quad i = 1, 2, \ldots, k, \tag{53}
$$

respectively.

Proof. We prove the first case, that is, the case with assumption (H_3).

Define $\mathscr{U}(x) = \|x\|^p$, by the definition of $|x_t|^p$; there exists a $\theta(t) = \theta(t, \omega) \in I_\tau$ such that $|x_t|^p = \sup_{\theta \in I_\tau} \|x(t + \theta)\|^p = \|x(t + \theta(t))\|^p$. By the definition of freezing operator $\mathrm{Fr}(\cdot)$, we have $|\mathrm{Fr}[x(t)]|^p = \sup_{\theta \in I_\tau} \|\mathrm{Fr}[x(t)](\theta)\|^p = \sup_{\theta \in I_\tau} \|x(t)\|^p = \|x(t)\|^p$; thus

$$
\begin{aligned}
&w_{2i}(t, x_t) - w_{2i}(t, \mathrm{Fr}[x(t)]) \\
&\quad = C_{2i}^*(t) \left(\|x(t + \theta(t))\|^p - \|x(t)\|^p \right) \\
&\quad = C_{2i}^*(t) \left(\mathscr{U}(x(t + \theta(t))) - \mathscr{U}(x(t)) \right) \\
&\quad = C_{2i}^*(t) \int_t^{t + \theta(t)} \mathrm{d}\mathscr{U}(x(s)).
\end{aligned}
\tag{54}
$$

Denote the jth column of $g(t,x_t)$ by $g_j(t,x_t)$; then we can rewrite (2) as

$$\mathrm{d}x(t) = f(t,x_t)\,\mathrm{d}t + \sum_{j=1}^{m} g_j(t,x_t)\,\mathrm{d}W_j(t). \tag{55}$$

Then by Itô's rule, we have

$$\mathrm{d}\mathscr{U} = \hat{f}(t,x_t)\,\mathrm{d}t + \sum_{j=1}^{m} \hat{g}_j(t,x_t)\,\mathrm{d}W_j(t), \tag{56}$$

where

$$\begin{aligned} \hat{f}(t,x_t) &= \mathscr{L}\mathscr{U} \\ &= p\|x\|^{p-2}\left(x^T f(t,x_t) + \frac{1}{2}\sum_{j=1}^{m} g_j^T(t,x_t)\,g_j(t,x_t)\right) \\ &\quad + p\left(\frac{p}{2}-1\right)\|x\|^{p-4}\left(\sum_{j=1}^{m} x^T g_j(t,x_t)\right)^2, \\ \hat{g}_j(t,x_t) &= p\|x\|^{p-2}x^T g_j(t,x_t). \end{aligned} \tag{57}$$

Thus we have

$$\begin{aligned} & w_{2i}(t,x_t) - w_{2i}(t,\mathrm{Fr}[x(t)]) \\ & \le C_{2i}^*(t)\left(\int_{t-\tau}^{t} \left\|\hat{f}(s,x_s)\right\|\mathrm{d}s + \int_{t}^{t+\theta(t)} \sum_{j=1}^{m} \hat{g}_j(s,x_s)\,\mathrm{d}W_j(s)\right). \end{aligned} \tag{58}$$

By the linear growth condition, we have

$$\left\|\hat{f}(t,x_t)\right\| \le \frac{1}{2}\left(2 + mL + m\left|p-2\right|\right)pL|x_t|^p. \tag{59}$$

With these, we then have

$$\begin{aligned} & w_{2i}(t,x_t) - w_{2i}(t,\mathrm{Fr}[x(t)]) \\ & \le \frac{1}{2}C_{2i}^*(t)\int_{t-\tau}^{t}\left(2 + mL + m\left|p-2\right|\right)pL|x_s|^p\mathrm{d}s \\ & \quad + \int_{t}^{t+\theta(t)} \sum_{j=1}^{m} C_{2i}^*(t)\,\hat{g}_j(s,x_s)\,\mathrm{d}W_j(s) \\ & \le \int_{t-\tau}^{t} \frac{1}{2}\left(2 + mL + m\left|p-2\right|\right)pL\,C_{2i}^*(s)\left|x_s\right|^p\mathrm{d}s \\ & \quad + \int_{t}^{t+\theta(t)} \sum_{j=1}^{m} C_{2i}^*(t)\,\hat{g}_j(s,x_s)\,\mathrm{d}W_j(s) \\ & = \int_{t-\tau}^{t} \mathscr{C}_{2i} w_{2i}(s,x_s)\,\mathrm{d}s \\ & \quad + \int_{t}^{t+\theta(t)} \sum_{j=1}^{m} C_{2i}^*(t)\,\hat{g}_j(s,x_s)\,\mathrm{d}W_j(s). \end{aligned} \tag{60}$$

Denote

$$\zeta_i(t,W(\cdot)) = \int_{t}^{t+\theta(t)} \sum_{j=1}^{m} C_{2i}^*(t)\,\hat{g}_j(s,x_s)\,\mathrm{d}W_j(s) \tag{61}$$

and, for each t, denote the σ-algebra generalized by $\theta(t)$ by $\Theta(t)$; then we have

$$\begin{aligned} & E\zeta_i(t,W(\cdot)) \\ & = E\left(C_{2i}^*(t)\left(E\int_{t}^{t+\theta(t)} \sum_{j=1}^{m} \hat{g}_j(s,x_s)\,\mathrm{d}W_j(s) \mid \Theta(t)\right)\right) \\ & = E\left(C_{2i}^*(t)\left(E\int_{t}^{t+\theta(t)} \sum_{j=1}^{m} \hat{g}_j(s,x_s)\,\mathrm{d}W_j(s) \mid \theta(t) = \theta\right)\right) \\ & = E\left(C_{2i}^*(t)\left(E\int_{t}^{t+\theta} \sum_{j=1}^{m} \hat{g}_j(s,x_s)\,\mathrm{d}W_j(s)\right)\right) \\ & = E(0) = 0. \end{aligned} \tag{62}$$

By Theorem 5, we get the conclusion of the theorem. □

Remark 9. Theorem 8 is considered for the case $w_{2i}(t,x_t) = C_{2i}^*(t)|x_t|^p$. The homogeneous case for $w_{1i}(t,x(t))$, $w_{2i}(t,x_t)$, and $w_{3i}(t,x(t))$ is $w_{ji}(t,x(t)) = C_{ji}^*(t)\|x(t)\|^p$, $j = 1,3$.

5. Remarks for Special Coefficients

Remark 10. For the case $w_{ji}(t,x) = C_{ji}^*(t)w_i^*(t,x) \ge 0$, $j = 1,2,3$, we have

$$w_i(t,x) = C_i(t)\,w_i^*(t,x), \tag{63}$$

where

$$\begin{aligned} C_i(t) &= \left(1 - \frac{\mathscr{C}_{1i}(t)}{1-\mathscr{C}_{2i}}\right)C_{1i}^*(t) - \frac{1}{1-\mathscr{C}_{2i}}C_{2i}^*(t) \\ &\quad - \left(k_i + \frac{\mathscr{C}_{3i}(t)}{1-\mathscr{C}_{2i}}\right)C_{3i}^*(t), \quad i = 1,2,\ldots,k; \end{aligned} \tag{64}$$

thus the condition $w_i(t,x) \ge 0$ becomes $C_i(t) \ge 0$, $i = 1,2,\ldots,k$.

Remark 11. If $w_{ji}(t,x) = C_{ji}^*(t)w_i^*(t,x) \ge 0$, $j = 1,2,3$, and $\mathscr{C}_{1i}(t) = \mathscr{C}_{3i}(t) = 0$, then we have

$$C_i(t) = C_{1i}^*(t) - \frac{1}{1-\mathscr{C}_{2i}}C_{2i}^*(t) - k_iC_{3i}^*(t), \quad i = 1,2,\ldots,k. \tag{65}$$

Remark 12. If $w_{ji}(t,x) = C_{ji}^*(t)w_i^*(t,x) \geq 0$, $j = 1,2,3$, $\mathscr{C}_{1i}(t) = \mathscr{C}_{3i}(t) = 0$, and

$$C_{ji}^*(t) = \text{constant}, \qquad \widehat{\mathscr{C}}_{2i}(t) = \text{constant}, \qquad k_i \leq 1, \tag{66}$$

for $j = 1,2,3$, $i = 1,2,\ldots,k$, then $w_i(t,x) = C_i w_i^*(t,x)$, where

$$C_i = C_{1i}^* - \frac{1}{1-\tau\mathscr{C}_{2i}} C_{2i}^* - C_{3i}^*, \quad i = 1,2,\ldots,k \tag{67}$$

due to $\overline{\mathscr{C}}_{2i} = \int_t^{t+\tau} \widehat{\mathscr{C}}_{2i} \mathrm{d}s = \tau\widehat{\mathscr{C}}_{2i}$, $i = 1,2,\ldots,k$. In this case, the condition for $w_i(t,x) \geq 0$ is $C_i \geq 0$. A sufficient condition for this is

$$\begin{gathered} C_i^* = C_{1i}^* - C_{2i}^* - C_{3i}^* > 0, \quad i = 1,2,\ldots,k, \\ 0 \leq \tau \leq \tau_M = \min\left\{ \frac{C_i^*}{\widehat{\mathscr{C}}_{2i}\left(C_{1i}^* - C_{3i}^*\right)}, \ i = 1,2,\ldots,k \right\}, \end{gathered} \tag{68}$$

which simplifies the condition of the theorem greatly.

6. Example

To illustrate the usage of our results, we construct an example.

Consider a 2-dimensional stochastic functional differential equation with time-varying delay

$$\begin{aligned} \mathrm{d}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} -3x_1 \\ 2x_1 - 2x_2 + 0.5x_2\left(t - 0.2\sin^2 t\right) \end{bmatrix} \mathrm{d}t \\ &\quad + \begin{bmatrix} e^{-t}\sin x_1 \\ e^{-t}\sin x_2 \end{bmatrix} \mathrm{d}W(t), \end{aligned} \tag{69}$$

where $W(t)$ is a scalar standard Wiener process defined on the complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathscr{P})$.

For this equation, the Lipschitz condition is satisfied and $[x_1, x_2]^T = [0,0]^T$ is the trivial solution. Besides, we have time-varying delay $\tau(t) = 0.2\sin^2 t$ with $0 \leq \tau(t) \leq \tau = 0.2$.

It is obvious that the results in the previous literature are not adapted to this kind of equations with time-varying delay. We now give a conclusion for the asymptotic property of the solution of the equation by our theorems.

Define two Lyapunov functions $V_1(t,x) = x_1^2 + x_2^2$, $V_2(t,x) = x_1^2$, and denote the states $x = [x_1, x_2]^T$, $y = [y_1, y_2]^T = [x_1(t - 0.2\sin^2 t)$, and $x_2(t - 0.2\sin^2 t)]^T$, which is the retarded state determined by x_t. Define $\gamma_1(t) = 2e^{-2t}$, $\gamma_2(t) = e^{-2t}$, $w_{11}(x) = (3/2)x_2^2$, $w_{21}(y) = (1/2)y_2^2$, $w_{31}(x) = 0$, $w_{12}(x) = 5x_1^2$, $w_{22}(y) = 0$, $w_{32}(x) = 0$, and $\widehat{k}_1(\cdot) = \widehat{k}_2(\cdot) = 0$. By a simple computation, we have $V_{1x}(t,x) = 2x$, $V_{2x}(t,x) = 2[x_1, 0]^T$, and

$$\begin{aligned} \mathscr{L}V_1(t,x,y) &= 2x_1(-3x_1) + 2x_2(2x_1 - 2x_2 + 0.5y_2) \\ &\quad + e^{-2t}\left(\sin^2 x_1 + \sin^2 x_2\right) \\ &\leq 2e^{-2t} - \left(4x_1^2 + \frac{3}{2}x_2^2\right) + \frac{1}{2}y_2^2 \\ &\leq 2e^{-2t} - \frac{3}{2}x_2^2 + \frac{1}{2}y_2^2 \\ &= \gamma_1(t) - w_{11}(x) + w_{21}(y) \\ &\quad + \int_{-\tau}^0 \widehat{k}_1(\theta) w_{31}(t, \phi(\theta))\, \mathrm{d}\theta, \\ \mathscr{L}V_2(t,x,y) &= 2x_1(-3x_1) + e^{-2t}\sin^2 x_1 \\ &\leq e^{-2t} - 5x_1^2 \\ &= \gamma_2(t) - w_{12}(x) + w_{22}(y) \\ &\quad + \int_{-\tau}^0 \widehat{k}_2(\theta) w_{32}(t, \phi(\theta))\, \mathrm{d}\theta, \end{aligned} \tag{70}$$

as well as

$$\begin{aligned} &\gamma_1(t) - \mathscr{L}V_1(t,x,y) - w_{11}(x) + w_{21}(y) \\ &\quad + \int_{-\tau}^0 \widehat{k}_1(\theta) w_{31}(t,\phi(\theta))\, \mathrm{d}\theta \\ &\quad + \left\| V_{1x}(t,x) g(t,x,y) \right\|^2 \\ &= 2e^{-2t} - 2x_1(-3x_1) \\ &\quad - 2x_2(2x_1 - 2x_2 + 0.5y_2) \\ &\quad - e^{-2t}\left(\sin^2 x_1 + \sin^2 x_2\right) - \frac{3}{2}x_2^2 \\ &\quad + \frac{1}{2}y_2^2 + \left(2e^{-t}\left(x_1\sin x_1 + x_2\sin x_2\right)\right)^2 \\ &\geq \frac{5}{2}x_1^2 + 2(x_1 - x_2)^2 + \frac{1}{2}(x_2 - y_2)^2 \\ &\geq \rho_1(x) = \frac{5}{2}x_1^2 + 2(x_1 - x_2)^2, \\ &\gamma_2(t) - \mathscr{L}V_2(t,x,y) - w_{12}(x) + w_{22}(y) \\ &\quad + \int_{-\tau}^0 \widehat{k}_2(\theta) w_{32}(t,\phi(\theta))\, \mathrm{d}\theta \\ &\quad + \left\| V_{2x}(t,x) g(t,x,y) \right\|^2 \\ &= e^{-2t} - 2x_1(-3x_1) - e^{-2t}\sin^2 x_1 \\ &\quad - 5x_1^2 + \left(2e^{-t}x_1\sin x_1\right)^2 \\ &= e^{-2t}\left(1 - \sin^2 x_1\right) + x_1^2 + 4e^{-2t}x_1^2\sin^2 x_1 \\ &\geq \rho_2(x) = x_1^2. \end{aligned} \tag{71}$$

Computations show that

$$\begin{aligned} &w_{21}(t,y) - w_{21}(t,x) \\ &= \frac{1}{2}\left(x_2^2(t - \tau(t)) - x_2^2(t)\right) \\ &= \frac{1}{2}\int_t^{t-\tau(t)} \left(\mathscr{L}x_2^2(s)\,\mathrm{d}s \right. \\ &\qquad \left. + 2e^{-s}x_2(s)\sin x_2(s)\,\mathrm{d}W(s)\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{t}^{t-\tau(t)} \Big(2x_2(s) \\
&\qquad \times \left(2x_1(s) - 2x_2(s) + 0.5x_2(s-\tau(s))\right) \\
&\qquad + e^{-2s}\sin^2 x_2(s)\Big)\,\mathrm{d}s \\
&\quad + \int_{t}^{t-\tau(t)} e^{-s} x_2(s)\sin x_2(s)\,\mathrm{d}W(s) \\
&\le \int_{t-\tau}^{t} \Big(x_1^2(s) + \frac{13}{4}x_2^2(s) \\
&\qquad + \frac{1}{4}y_2^2(s) + e^{-2s}\Big)\,\mathrm{d}s \\
&\quad + \int_{t}^{t-\tau(t)} e^{-s} x_2(s)\sin x_2(s)\,\mathrm{d}W(s) \\
&\le \int_{t-\tau}^{t} \left(\frac{13}{6}w_{11}(x(s)) + \frac{1}{2}w_{21}(y(s))\right)\mathrm{d}s + \gamma_1^*(t) \\
&\quad + \int_{t}^{t-\tau(t)} e^{-s} x_2(s)\sin x_2(s)\,\mathrm{d}W(s),
\end{aligned}
\tag{72}
$$

where $\gamma_1^*(t) = \int_{t-\tau}^{t} e^{-2s}\mathrm{d}s = (1/2)(1-e^{-2\tau})e^{-2(t-\tau)} \in L^1(R^+, R^+)$.

By the above computations, we have $\widehat{\mathscr{C}}_{11} = 13/6$, $\widehat{\mathscr{C}}_{21} = 1/2$, $\widehat{\mathscr{C}}_{31} = 0$, $\widehat{\mathscr{C}}_{12} = 0$, $\widehat{\mathscr{C}}_{22} = 0$, and $\widehat{\mathscr{C}}_{32} = 0$, and then $\mathscr{C}_{11} = 13/30$, $\mathscr{C}_{21} = 1/10$, $\mathscr{C}_{31} = 0$, $\mathscr{C}_{12} = 0$, $\mathscr{C}_{22} = 0$, $\mathscr{C}_{32} = 0$, $C_{11}^* = 3/2$, $C_{21}^* = 1/2$, $C_{31}^* = 0$, $C_{12}^* = 5$, $C_{22}^* = 0$, $C_{32}^* = 0$, $C_1 = 13/18 > 0$, and $C_2 = 5 - 0 - 0 = 5 > 0$; thus, by Theorem 8 and Remark 10, we have $\lim_{t\to\infty}\rho_i(x(t)) = 0$, a.s. Because $\rho_1(x) = (5/2)x_1^2 + 2(x_1 - x_2)^2$, $\rho_2(x) = x_1^2$, then it follows that $\lim_{t\to\infty} x(t, t_0, \phi_0) = 0$, a.s.

Taking initial data $x(-0.2) = [1,3]^T$, $x(0) = [1,1]^T$, Figure 1 verifies our result about the asymptotic property of the solution of the equation.

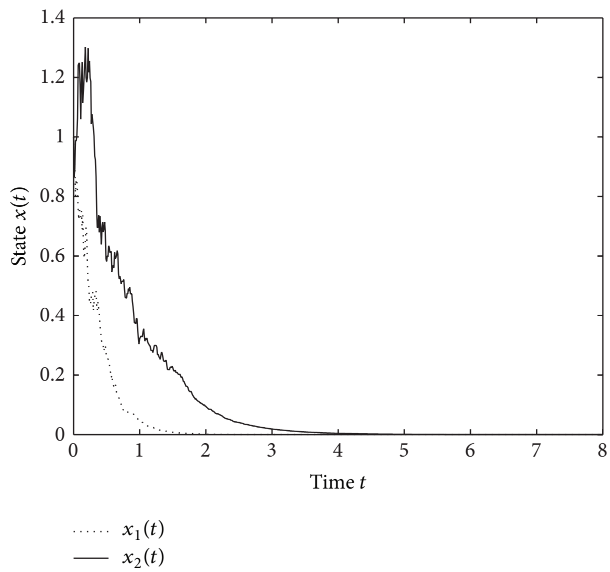

Figure 1: Solution behavior of stochastic system.

7. Conclusion

In this paper, we have generalized a kind of LaSalle-type theorems established by Xuerong Mao, Yi Shen, and other authors to the general nonlinear stochastic functional differential equations. With some preliminaries on lemmas and the derivation techniques, we establish three LaSalle-type theorems for the general nonlinear stochastic functional differential equations by the multiple Lyapunov functions. It should be pointed out that the models investigated in this paper are not desired in any kind of special form. The key point of the paper lies in the treatment of the general retarded terms in the estimations for the derivatives of the Lyapunov functions or their upper bounds. Of course, for the typical special case with estimation involving $|x_t|^p$ for the derivatives of the Lyapunov functions, a theorem is established as the corollary of the main theorem. At the end of the paper, an example is given to illustrate the usage of the method proposed in the paper. In fact, the asymptotic properties of the solutions of this kind of equations cannot be judged by the results in the previous literature mentioned above.

It is obvious that the results of this paper can be rewritten in another form by mathematical expectation with only minor changes for the assumptions on the stochastic derivatives of the Lyapunov functions.

At the end, we point out that one may determine the asymptotic behavior of the solutions of the equations by virtue of the integrability of the terms as $\int_{t_0}^{t} w_i(s, x(s))\mathrm{d}s$ directly if suitable assumptions are added to the related coefficients.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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