Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 893906, 7 pages
http://dx.doi.org/10.1155/2014/893906

Research Article

The Convergence of Double-Indexed Weighted Sums of Martingale Differences and Its Application

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Received 23 January 2014; Accepted 19 February 2014; Published 24 March 2014

Academic Editor: Ivan Ivanov

We investigate the complete moment convergence of double-indexed weighted sums of martingale differences. Then it is easy to obtain the Marcinkiewicz-Zygmund-type strong law of large numbers of double-indexed weighted sums of martingale differences. Moreover, the convergence of double-indexed weighted sums of martingale differences is presented in mean square. On the other hand, we give the application to study the convergence of the state observers of linear-time-invariant systems and present the convergence with probability one and in mean square.

1. Introduction

Hsu and Robbins [1] introduced the concept of complete convergence; that is, a sequence of random variables $\{X_n, n \geq 1\}$ is said to *converge completely* to a constant C if $\sum_{n=1}^{\infty} P(|X_n - C| \geq \varepsilon) < \infty$ for all $\varepsilon > 0$. By Borel-Cantelli lemma, it follows that $X_n \rightarrow C$ almost surely (a.s.). The converse is true if $\{X_n, n \geq 1\}$ is independent. But the converse cannot always be true for the dependent case. Hsu and Robbins [1] obtained that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdös [2] proved the converse. The result of Hsu-Robbins-Erdös is a fundamental theorem in probability theory, and it has been generalized and extended in several directions by many authors. Baum and Katz [3] gave the following generalization to establish a rate of convergence in the sense of Marcinkiewicz-Zygmund-type strong law of large numbers.

Theorem 1. *Let $\alpha > 1/2$, $\alpha p > 1$, and $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables. Assume that $EX_1 = 0$ if $\alpha \leq 1$. Then the following statements are equivalent:*

(i) $E|X_1|^p < \infty$;

(ii) $\sum_{n=1}^{\infty} n^{\alpha p-2} P(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i| > \varepsilon n^{\alpha}) < \infty$ *for all* $\varepsilon > 0$.

Many authors have extended Theorem 1 to the martingale differences. For example, Yu [4] obtained the complete convergence for weighted sums of martingale differences; Ghosal and Chandra [5] gave the complete convergence of martingale arrays; Stoica [6, 7] investigated the Baum-Katz-Nagaev-type results for martingale differences and the rate of convergence in the strong law of large numbers for martingale differences; Wang et al. [8] also studied the complete convergence and complete moment convergence for martingale differences, which generalized some results of Stoica [6, 7]; Yang et al. [9] obtained the complete convergence for the moving average process of martingale differences and so forth. For other works about convergence analysis, one can refer to Gut [10], Chen et al. [11], Sung [12–14], Sung and Volodin [15], Hu et al. [16], and the references therein.

In this paper, we study the moment complete convergence of double-indexed weighted sums of martingale differences. Then it is easy to obtain the Marcinkiewicz-Zygmund-type strong law of large numbers of double-indexed weighted sums of martingale differences. Moreover, the convergence of double-indexed weighted sums of martingale differences is presented in mean square. For the details, see Theorem 5, Corollary 6, and Theorem 7 in Section 2. On the other hand, we give the applications of Corollary 6 and Theorem 7 to study the convergence of the state observers of linear-time-invariant systems and present their convergence with probability one and in mean square, respectively (see Theorems 11 and 12 in Section 3).

Recall that the sequence $\{X_n, n \geq 1\}$ is stochastically dominated by a nonnegative random variable X if $\sup_{n\geq 1} P(|X_n| > t) \leq KP(X > t)$ for some positive constant K and for all $t \geq 0$.

Throughout the paper, let $\mathscr{F}_0 = \{\emptyset, \Omega\}$, $\mathbb{1}(B)$ be the indicator function of set $Bx^+ = x\mathbb{1}(x \geq 0)$, and let $K, K_1, K_2, \ldots$ denote some positive constants not depending on n, which may be different in various places.

The following lemmas are useful for the proofs of the main results.

Lemma 2 (cf. Hall and Heyde [17, Theorem 2.11]). *If $\{X_i, \mathscr{F}_i, 1 \leq i \leq n\}$ are martingale differences and $p > 0$, then there exists a constant K depending only on p such that*

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leq K\left\{E\left(\sum_{i=1}^{n} E\left(X_i^2 \mid \mathscr{F}_{i-1}\right)\right)^{p/2} + E\left(\max_{1\leq i\leq n}|X_i|^p\right)\right\}, \quad n \geq 1. \tag{1}$$

Lemma 3 (cf. Sung [12, Lemma 2.4]). *Let $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be sequences of random variables. Then for any $n \geq 1$, $q > 1$, $\varepsilon > 0$, and $a > 0$, one has*

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(X_i + Y_i)\right| - \varepsilon a\right)^+ \leq \left(\frac{1}{\varepsilon^q} + \frac{1}{q-1}\right)\frac{1}{a^{q-1}} E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^q\right) + E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} Y_i\right|\right). \tag{2}$$

Lemma 4 (cf. Wang et al. [8, Lemma 2.2]). *Let $\{X_n, n \geq 1\}$ be a sequence of random variables stochastically dominated by a nonnegative random variable X. Then for any $n \geq 1$, $a > 0$, and $b > 0$, the following two statements hold:*

$$\begin{aligned} E\left[|X_n|^a \mathbb{1}\left(|X_n| \leq b\right)\right] &\leq K_1\left\{E\left[X^a \mathbb{1}(X \leq b)\right] + b^a P(X > b)\right\}, \\ E\left[|X_n|^a \mathbb{1}\left(|X_n| > b\right)\right] &\leq K_2 E\left[X^a \mathbb{1}(X > b)\right]. \end{aligned} \tag{3}$$

Consequently, $E|X_n|^a \leq K_3 EX^a$. Here K_1, K_2, and K_3 are positive constants.

2. The Convergence of Double-Indexed Weighted Sums of Martingale Differences

First, we give the complete moment convergence of double-indexed weighted sums of martingale differences.

Theorem 5. *Let $\alpha > 1/2$, $p \geq 2$, and $\{X_n, \mathscr{F}_n, n \geq 1\}$ be martingale differences stochastically dominated by a nonnegative random variable X with $EX^p < \infty$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of real numbers. For some $q > 2(\alpha p-1)/(2\alpha-1)$, we assume that $E[\sup_{n\geq 1} E(X_n^2 \mid \mathscr{F}_{n-1})]^{q/2} < \infty$ and*

$$\sum_{i=1}^{n} |a_{ni}|^q = O(n). \tag{4}$$

Then for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^{\alpha}\right)^+ < \infty. \tag{5}$$

Taking $p = 2l$ and $\alpha = 2/p$ for $1 \leq l < 2$ in Theorem 5, we have the following result.

Corollary 6. *Let $1 \leq l < 2$, $\{X_n, \mathscr{F}_n, n \geq 1\}$ be martingale differences stochastically dominated by a nonnegative random variable X with $EX^{2l} < \infty$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of real numbers. For some $q > 2l/(2-l)$, one assumes that $E[\sup_{n\geq 1} E(X_n^2 \mid \mathscr{F}_{n-1})]^{q/2} < \infty$ and (4) holds true. Then for every $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{-1/l} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^{1/l}\right)^+ < \infty. \tag{6}$$

In particular, one has

$$\lim_{n\to\infty} \frac{1}{n^{1/l}} \sum_{i=1}^{n} a_{ni}X_i = 0, \quad a.s. \tag{7}$$

Next, we investigate the convergence in mean square.

Theorem 7. *Let $r > 1/2$ and $\{X_n, \mathscr{F}_n, n \geq 1\}$ be martingale differences stochastically dominated by a nonnegative random variable X with $EX^2 < \infty$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of real numbers and*

$$\sum_{i=1}^{n} a_{ni}^2 = O(n). \tag{8}$$

Then, one has

$$n^{2r-1} E\left(\frac{1}{n^r}\sum_{i=1}^{n} a_{ni}X_i\right)^2 \leq K, \quad n \geq 1, \tag{9}$$

where K is a positive constant.

Remark 8. Wang et al. [8] obtained the complete convergence and complete moment convergence for nonweighted martingale differences, which generalized some results of Stoica [6, 7]. In this paper, we study the complete moment convergence of double-indexed weighted sums of martingale differences. So we extend the results of Wang et al. [8] and Stoica [6, 7] to the case of double-indexed weighted sums of martingale differences. On the other hand, we give the applications of

Corollary 6 and Theorem 7 to study the convergence of the state observers of linear-time-invariant systems and present the convergence with probability one and in mean square, respectively (see Theorems 11 and 12 in Section 3).

Proof of Theorem 5. Let $X_{ni} = X_i \mathbb{1}(|X_i| \le n^\alpha)$, $1 \le i \le n$. It can be found that $a_{ni}X_i = a_{ni}X_i\mathbb{1}(|X_i| > n^\alpha) + [a_{ni}X_{ni} - a_{ni}E(X_{ni} \mid \mathscr{F}_{i-1})] + a_{ni}E(X_{ni} \mid \mathscr{F}_{i-1})$, $1 \le i \le n$.

By Lemma 3 with $a = n^\alpha$, for any $q > 1$, we obtain that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^\alpha\right)^+ \\
&\le K_1 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} \\
&\quad \times E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\left[a_{ni}X_{ni} - a_{ni}E\left(X_{ni} \mid \mathscr{F}_{i-1}\right)\right]\right|^q\right) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\mathbb{1}\left(\left|X_i\right| > n^\alpha\right)\right|\right) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}E\left(X_{ni} \mid \mathscr{F}_{i-1}\right)\right|\right) \\
&:= H_1 + H_2 + H_3.
\end{aligned} \tag{10}$$

For $p \ge 2$, it is easy to see that $q > 2(\alpha p - 1)/(2\alpha - 1) \ge 2$. Consequently, for any $1 \le s \le 2$, we get by Hölder's inequality and (4) that

$$\sum_{i=1}^{n}\left|a_{ni}\right|^s \le \left(\sum_{i=1}^{n}\left|a_{ni}\right|^q\right)^{s/q}\left(\sum_{i=1}^{n}1\right)^{1-s/q} = O(n). \tag{11}$$

So, it can be checked by Markov's inequality, Lemma 4, (11), and $EX^p < \infty$ ($p \ge 2$) that

$$\begin{aligned}
H_2 &\le \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\sum_{i=1}^{n}\left|a_{ni}\right| E\left[\left|X_i\right|\mathbb{1}\left(\left|X_i\right| > n^\alpha\right)\right] \\
&\le K_1 \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E\left[X\mathbb{1}\left(X > n^\alpha\right)\right] \\
&= K_1 \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha}\sum_{m=n}^{\infty} E\left[X\mathbb{1}\left(m^\alpha < X \le (m+1)^\alpha\right)\right] \\
&= K_1 \sum_{m=1}^{\infty} E\left[X\mathbb{1}\left(m^\alpha < X \le (m+1)^\alpha\right)\right]\sum_{n=1}^{m} n^{\alpha p-1-\alpha} \\
&\le K_2 \sum_{m=1}^{\infty} m^{\alpha p-\alpha} E\left[X\mathbb{1}\left(m^\alpha < X \le (m+1)^\alpha\right)\right] \\
&\le K_2 EX^p < \infty.
\end{aligned} \tag{12}$$

Since $\{X_i, \mathscr{F}_i, 1 \le i \le n\}$ are martingale differences, by the martingale property and the proof of (12), one has that

$$\begin{aligned}
H_3 &= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}E\left[X_i\mathbb{1}\left(\left|X_i\right| \le n^\alpha\right) \mid \mathscr{F}_{i-1}\right]\right|\right) \\
&= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}E\left[X_i\mathbb{1}\left(\left|X_i\right| > n^\alpha\right) \mid \mathscr{F}_{i-1}\right]\right|\right) \\
&\le K_1 \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha}\sum_{i=1}^{n}\left|a_{ni}\right| E\left[\left|X_i\right|\mathbb{1}\left(\left|X_i\right| > n^\alpha\right)\right] \\
&\le K_2 \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E\left[X\mathbb{1}\left(X > n^\alpha\right)\right] \le K_3 EX^p < \infty.
\end{aligned} \tag{13}$$

Next, we turn to prove $H_1 < \infty$ under conditions of Theorem 5. It can be seen that

$$\left\{\left[a_{ni}X_{ni} - a_{ni}E\left(X_{ni} \mid \mathscr{F}_{i-1}\right)\right], \mathscr{F}_i, 1 \le i \le n\right\} \tag{14}$$

are also martingale differences. So, by Markov's inequality, (10), and Lemma 2 with $p = q$, it can be found that

$$\begin{aligned}
H_1 &= K_1 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}\left[a_{ni}X_{ni} - a_{ni}E\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. \times \left(X_{ni} \mid \mathscr{F}_{i-1}\right)\right]\right|^q\right) \\
&\le K_2 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} E\left(\sum_{i=1}^{n} E\left\{\left[a_{ni}X_{ni} - a_{ni}E\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. \times \left(X_{ni} \mid \mathscr{F}_{i-1}\right)\right]^2 \mid \mathscr{F}_{i-1}\right\}\right)^{q/2} \\
&\quad + K_3 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha}\sum_{i=1}^{n} E\left|a_{ni}X_{ni} - a_{ni}E\left(X_{ni} \mid \mathscr{F}_{i-1}\right)\right|^q \\
&=: K_2 H_{11} + K_3 H_{12}.
\end{aligned} \tag{15}$$

Obviously, it follows that

$$\begin{aligned}
&E\left\{\left[a_{ni}X_{ni} - E\left(a_{ni}X_{ni} \mid \mathscr{F}_{i-1}\right)\right]^2 \mid \mathscr{F}_{i-1}\right\} \\
&\quad = E\left[a_{ni}^2 X_i^2 \mathbb{1}\left(\left|X_i\right| \le n^\alpha\right) \mid \mathscr{F}_{i-1}\right] \\
&\qquad - \left[E\left(a_{ni}X_i\mathbb{1}\left(\left|X_i\right| \le n^\alpha\right) \mid \mathscr{F}_{i-1}\right)\right]^2 \\
&\quad \le a_{ni}^2 E\left[X_i^2\mathbb{1}\left(\left|X_i\right| \le n^\alpha\right) \mid \mathscr{F}_{i-1}\right] \\
&\quad \le a_{ni}^2 E\left(X_i^2 \mid \mathscr{F}_{i-1}\right), \quad \text{a.s.}, 1 \le i \le n.
\end{aligned} \tag{16}$$

Combining (11) with $E[\sup_{i\geq1}E(X_i^2 \mid \mathcal{F}_{i-1})]^{q/2} < \infty$, we obtain that

$$H_{11} \leq \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha}\left(\sum_{i=1}^{n} a_{ni}^2\right)^{q/2} E\left(\sup_{i\geq1} E\left(X_i^2 \mid \mathcal{F}_{i-1}\right)\right)^{q/2}$$
$$\leq K_4 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha+q/2} < \infty, \quad (17)$$

following from the fact that $q > 2(\alpha p-1)/(2\alpha-1)$. Meanwhile, by C_r inequality, Lemma 4, and (4),

$$H_{12} \leq K_5 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} \sum_{i=1}^{n} |a_{ni}|^q E\left[|X_i|^q \mathbb{1}\left(|X_i| \leq n^\alpha\right)\right]$$
$$\leq K_6 \sum_{n=1}^{\infty} n^{\alpha p-1-q\alpha} E\left[X^q \mathbb{1}\left(X \leq n^\alpha\right)\right] + K_7 \sum_{n=1}^{\infty} n^{\alpha p-1} P\left(X > n^\alpha\right)$$
$$\leq K_6 \sum_{n=1}^{\infty} n^{\alpha p-1-q\alpha} E\left[X^q \mathbb{1}\left(X \leq n^\alpha\right)\right] + K_7 \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E\left[X \mathbb{1}\left(X > n^\alpha\right)\right]$$
$$=: K_6 H_{11}^* + K_7 H_{12}^*. \quad (18)$$

By the conditions $p \geq 2$ and $\alpha > 1/2$, we have that $2(\alpha p - 1)/(2\alpha - 1) - p \geq 0$, which implies $q > p$. So, we obtain by $EX^p < \infty$ that

$$H_{11}^* = \sum_{n=1}^{\infty} n^{\alpha p-1-q\alpha} \sum_{i=1}^{n} E\left[X^q \mathbb{1}\left((i-1)^\alpha < X \leq i^\alpha\right)\right]$$
$$= \sum_{i=1}^{\infty} E\left[X^q \mathbb{1}\left((i-1)^\alpha < X \leq i^\alpha\right)\right] \sum_{n=i}^{\infty} n^{\alpha p-1-q\alpha}$$
$$\leq K_8 \sum_{i=1}^{\infty} E\left[X^p X^{q-p} \mathbb{1}\left((i-1)^\alpha < X \leq i^\alpha\right)\right] i^{\alpha p-q\alpha}$$
$$\leq K_8 EX^p < \infty. \quad (19)$$

By the proof of (12), one has that

$$H_{12}^* = \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E\left[X \mathbb{1}\left(X > n^\alpha\right)\right] \leq K_9 EX^p < \infty. \quad (20)$$

Thus, by (15)–(20), we have that $H_1 < \infty$. So, it completes the proof of (5). □

Proof of Corollary 6. If $p = 2l$ and $\alpha = 2/p$, then one has $\alpha p = 2$. So as an application of Theorem 5, one gets (6) immediately. On the other hand, it can be seen that

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^\alpha\right)^+$$
$$= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_0^\infty P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^\alpha > t\right) dt$$
$$\geq \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \int_0^{\varepsilon n^\alpha} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^\alpha > t\right) dt$$
$$\geq \varepsilon \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > 2\varepsilon n^\alpha\right). \quad (21)$$

So by (5) and (21) with $\alpha p = 2$, we have for every $\varepsilon > 0$ that

$$\sum_{n=1}^{\infty} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon n^{1/l}\right) < \infty. \quad (22)$$

It follows from Borel-Cantelli lemma that

$$\lim_{n\to\infty} \frac{1}{n^{1/l}} \sum_{i=1}^{n} a_{ni}X_i = 0, \quad \text{a.s.} \quad (23)$$

So, (7) holds. □

Proof of Theorem 7. Since $\{a_{ni}X_i, \mathcal{F}_i, 1 \leq i \leq n\}$ are martingale differences, it can be found by Lemmas 2 and 4 and (8) that

$$E\left(\frac{1}{n^r}\sum_{i=1}^{n} a_{ni}X_i\right)^2 = \frac{1}{n^{2r}} E\left(\sum_{i=1}^{n} a_{ni}X_i\right)^2 \leq \frac{K_1}{n^{2r}} \sum_{i=1}^{n} a_{ni}^2 EX_i^2$$
$$\leq \frac{K_2}{n^{2r}} EX^2 \sum_{i=1}^{n} a_{ni}^2 \leq \frac{K_3}{n^{2r-1}}, \quad n \geq 1. \quad (24)$$

Consequently, (9) holds true. □

3. Applications to the Convergence of the State Observers of Linear-Time-Invariant Systems

In this section, we give the applications of Corollary 6 and Theorem 7 to study the convergence of the state observers of linear-time-invariant systems.

For $t \geq 0$, consider an MISO (multi-input-single-output) linear-time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t), \quad (25)$$

where $A \in R^{m_0\times m_0}$, $B \in R^{m_0\times m_1}$, and $C \in R^{1\times m_0}$ are known system matrices, and for $t \geq 0$, $u(t) \in R^{m_1}$ is the control input,

$x(t) \in R^{m_0}$ is the state, and $y(t) \in R$ is the system output. The initial state $x(0)$ is unknown. We are interested in estimation of $x(t)$, from some limited observations on $y(t)$.

In our setup, the output $y(t)$ is only measured at a sequence of sampling time instants $\{t_i\}$ with measured values $\gamma(t_i)$, and noise d_i

$$\gamma(t_i) = y(t_i) - d_i. \quad (26)$$

We would like to estimate the state $x(t)$ from information on $u(t)$, $\{t_i\}$, and $\{\gamma(t_i)\}$. In practical systems, the irregular sampling sequences $\{\gamma(t_i)\}$ can be generated by different means such as randomized sampling, event-triggered sampling, and signal quantization.

It is obvious that state estimation will not be possible if the system is not observable. Also, in this paper, d_k is assumed to be martingale difference. We give the following assumption.

Assumption 9. The system (25) is observable; that is, the observability matrix

$$W_o' = \left[C', (CA)', \ldots, \left(CA^{m_0-1}\right)'\right] \quad (27)$$

has full rank.

For both $t > t_0$ and $t < t_0$, the solution to system (25) can be expressed as

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau. \quad (28)$$

Suppose that $\{t_i, 1 \le i \le n\}$ is a sequence of sampling times. For $t_i \le t_n$, we have

$$\gamma(t_i) + d_i = y(t_i) = Ce^{A(t_i-t_n)}x(t_n) + C\int_{t_n}^{t_i} e^{A(t_i-\tau)}Bu(\tau)\,d\tau. \quad (29)$$

Since the second term is known, it will be denoted by $v(t_i, t_n) = C\int_{t_n}^{t_i} e^{A(t_i-\tau)}Bu(\tau)d\tau$. This leads to the observations

$$Ce^{A(t_i-t_n)}x(t_n) = \gamma(t_i) - v(t_i, t_n) + d_i, \quad 1 \le i \le n. \quad (30)$$

Define

$$\Phi_n = \begin{bmatrix} Ce^{A(t_1-t_n)} \\ \vdots \\ Ce^{A(t_{n-1}-t_n)} \\ C \end{bmatrix}, \qquad \Gamma_n = \begin{bmatrix} \gamma(t_1) \\ \vdots \\ \gamma(t_{n-1}) \\ \gamma(t_n) \end{bmatrix},$$
$$V_n = \begin{bmatrix} v(t_1, t_n) \\ \vdots \\ v(t_{n-1}, t_n) \\ 0 \end{bmatrix}, \qquad D_n = \begin{bmatrix} d_1 \\ \vdots \\ d_{n-1} \\ d_n \end{bmatrix}. \quad (31)$$

Then, (30) can be written as

$$\Phi_n x(t_n) = \Gamma_n - V_n + D_n. \quad (32)$$

Suppose that Φ_n is full rank, which will be established later. Then, a least-squares estimate of $x(t_n)$ is given by

$$\hat{x}(t_n) = \left(\Phi_n'\Phi_n\right)^{-1}\Phi_n'(\Gamma_n - V_n). \quad (33)$$

Here, G' denotes the transpose of G. From (32) and (33), the estimation error for $x(t_n)$ at sampling time t_n is

$$\begin{aligned} e(t_n) &= \hat{x}(t_n) - x(t_n) = \left(\Phi_n'\Phi_n\right)^{-1}\Phi_n'D_n \\ &= \left(\frac{1}{n^r}\Phi_n'\Phi_n\right)^{-1}\frac{1}{n^r}\Phi_n'D_n \end{aligned} \quad (34)$$

for some $1/2 < r < 1$. For convergence analysis, one must consider a typical entry in $(1/n^r)\Phi_n'D_n$. By the Cayley Hamilton theorem (see Ogata [18]), the matrix exponential can be expressed by a polynomial function of A of order at most $m_0 - 1$,

$$e^{At} = \alpha_1(t)I + \cdots + \alpha_{m_0}(t)A^{m_0-1}, \quad (35)$$

where the time functions $\alpha_i(t)$ can be derived by the Lagrange-Hermite interpolation method (see Ogata [18]). This implies that

$$\begin{aligned} Ce^{A(t_i-t_n)} &= \left[\alpha_1(t_i-t_n), \ldots, \alpha_{m_0}(t_i-t_n)\right]\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m_0-1} \end{bmatrix} \\ &= \varphi'(t_i-t_n)W_o, \end{aligned} \quad (36)$$

where $\varphi'(t_i - t_n) = [\alpha_1(t_i - t_n), \ldots, \alpha_{m_0}(t_i - t_n)]$ and W_o is the observability matrix.

Denote

$$\Psi_n = \begin{bmatrix} \varphi'(t_1-t_n) \\ \vdots \\ \varphi'(0) \end{bmatrix}. \quad (37)$$

Then

$$\Phi_n = \Psi_n W_o, \quad (38)$$

which implies that

$$\begin{aligned} \frac{1}{n^r}\Phi_n'\Phi_n &= W_o'\frac{1}{n^r}\Psi_n'\Psi_n W_o, \\ \frac{1}{n^r}\Phi_n'D_n &= \frac{1}{n^r}W_o'\Psi_n'D_n. \end{aligned} \quad (39)$$

As a result, for any $r > 0$, one has

$$e(t_n) = \left(\frac{1}{n^r}\Phi_n'\Phi_n\right)^{-1}\frac{1}{n^r}\Phi_n'D_n = W_o^{-1}\left(\frac{1}{n^r}\Psi_n'\Psi_n\right)^{-1}\frac{1}{n^r}\Psi_n'D_n. \quad (40)$$

Under Assumption 9, W_0^{-1} exists. Convergence results will be established by the following two sufficient conditions: $(1/n^r)\Psi_n'D_n \to 0$ and $(1/n^r)\Psi_n'\Psi_n \ge \beta I$, for some $\beta > 0$. So we need the following persistent excitation (PE) condition, which was used by Wang et al. [19] and Thanh et al. [20].

Assumption 10. For some $1/2 < r < 1$,

$$\beta = \inf_{n\geq 1} \sigma_{\min}\left(\frac{1}{n^r}\Psi_n'\Psi_n\right) > 0, \tag{41}$$

where $\sigma_{\min}(H)$ is the small eigenvalue of H for a suitable symmetric H.

We can investigate the convergence of double-indexed summations of random variables form

$$\frac{1}{n^r}\sum_{i=1}^{n} a_{ni}d_i \tag{42}$$

for some $1/2 < r < 1$. Here, $\{a_{ni}\}$ is a triangular array of real numbers and $\{d_i\}$ is a sequence of martingale differences. It can be seen that (42) is a special case of (7) in Corollary 6. The jth component of $(1/n^r)\Psi_n'D_n$ takes the form

$$\frac{1}{n^r}\sum_{i=1}^{n} \alpha_j\left(t_i - t_n\right) d_i, \tag{43}$$

where $\{\alpha_j(t_i - t_n)\}$ is a triangular array of real numbers. The convergence analysis of (43) for $e(t_n)$ is a special case of (42) or (7) in Corollary 6.

Recently, Wang et al. [19] investigated the convergence analysis of the state observers of linear-time-invariant systems under ρ^*-mixing sampling. Thanh et al. [20] studied the convergence analysis of double-indexed and randomly weighted sums of ρ^*-mixing sequence and gave its application to state observers. For more related works, one can refer to [18–23] and the references therein.

As an application of Corollary 6 to the observers and state estimation, we obtain the following theorem.

Theorem 11. *Let Assumptions 9 and 10 hold. Let $1/2 < r < 1$ and $\{d_n, \mathcal{F}_n, n \geq 1\}$ be martingale differences stochastically dominated by a nonnegative random variable d with $Ed^{2/r} < \infty$. Suppose that for any $q > 2/(2r-1)$, one has $E[\sup_{n\geq 1} E(d_n^2 \mid \mathcal{F}_{n-1})]^{q/2} < \infty$ and*

$$\sum_{i=1}^{n} \left|\alpha_j(t_i - t_n)\right|^q = O(n), \tag{44}$$

where $1 \leq j \leq m_0$. Then

$$\frac{1}{n^r}\left\|\Psi_n'D_n\right\| \longrightarrow 0, \quad a.s. \tag{45}$$

Consequently,

$$e\left(t_n\right) \longrightarrow 0, \quad a.s. \tag{46}$$

As an application to Theorem 7, we get the following result.

Theorem 12. *Let $1/2 < r < 1$ and Assumptions 9 and 10 hold. Assume that $\{d_n, \mathcal{F}_n, n \geq 1\}$ are martingales differences stochastically dominated by a nonnegative random variable d with $Ed^2 < \infty$. For $1 \leq j \leq m_0$, it is supposed that*

$$\sum_{i=1}^{n} \alpha_j^2\left(t_i - t_n\right) = O(n). \tag{47}$$

Then

$$\zeta = \sup_{n\geq 1} n^{2r-1} Ee'\left(t_n\right) e\left(t_n\right) < \infty. \tag{48}$$

Remark 13. If we assume that, for each $1 \leq i \leq n$, $\{\varphi(t_i - t_n)\}$ is uniformly bounded, then we can find that condition (44) holds for any q. On the other hand, similar to Theorems 11 and 12, Wang et al. [19] also obtained the convergence of the state observers with probability one and in mean square under ρ^*-mixing sampling (see Theorems 4 and 5 of Wang et al. [19]). So Theorems 11 and 12 generalize the results of Wang et al. [19] to the case of martingale differences.

Proof of Theorem 11. It can be seen that

$$\frac{1}{n^r}\Psi_n'D_n = \begin{bmatrix} \dfrac{1}{n^r}\displaystyle\sum_{i=1}^{n}\alpha_1\left(t_i - t_n\right) d_i \\ \vdots \\ \dfrac{1}{n^r}\displaystyle\sum_{i=1}^{n}\alpha_{m_0}\left(t_i - t_n\right) d_i \end{bmatrix}. \tag{49}$$

To prove (45), it suffices to look at the jth component

$$\frac{1}{n^r}\sum_{i=1}^{n}\alpha_j\left(t_i - t_n\right) d_i \tag{50}$$

of

$$\frac{1}{n^r}\Psi_n'D_n. \tag{51}$$

For any $q > 2/(2r-1)$, by $E[\sup_{n\geq 1} E(d_n^2 \mid \mathcal{F}_{n-1})]^{q/2} < \infty$ and (44), we can obtain (45) from Corollary 6 with $l = 1/r$, $a_{ni} = \alpha_j(t_i - t_n)$ in (43), and $X_n = d_n$.

On the other hand, by Assumption 9, W_0^{-1} exists, and by (41) in Assumption 10, $((1/n^r)\Psi_n'\Psi_n)^{-1}$ exists and

$$\sigma_{\max}\left(\left(\frac{1}{n^r}\Psi_n'\Psi_n\right)^{-1}\right) \leq \frac{1}{\beta}, \tag{52}$$

where $\sigma_{\max}(\cdot)$ is the largest eigenvalue. Together with

$$e\left(t_n\right) = W_o^{-1}\left(\frac{1}{n^r}\Psi_n'\Psi_n\right)^{-1}\frac{1}{n^r}\Psi_n'D_n \tag{53}$$

and (45), it follows (46). □

Proof of Theorem 12. For $1 \leq j \leq m_0$, by (47), (8) holds. Applying Theorem 7 with $a_{ni} = \alpha_j(t_i - t_n)$, $X_n = d_n$, and $1/2 < r < 1$, we obtain that for a typical term

$$\frac{1}{n^r}\sum_{i=1}^{n}\alpha_j\left(t_i - t_n\right) d_i \tag{54}$$

in (49),

$$n^{2r-1}E\left(\frac{1}{n^r}\sum_{i=1}^{n}\alpha_j(t_i - t_n)d_i\right)^2 \leq K_1, \quad n \geq 1. \tag{55}$$

Together with (49), (53), and (55), we obtain that

$$n^{2r-1}Ee'\left(t_n\right)e\left(t_n\right) \leq m_0K_2 < \infty, \tag{56}$$

where K_2 is a positive constant. Lastly, by (56), (48) holds true. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by the NNSF of China (11171001, 11201001, and 11326172), Natural Science Foundation of Anhui Province (1208085QA03 and 1408085QA02), Higher Education Talent Revitalization Project of Anhui Province (2013SQRL005ZD), Academic and Technology Leaders to Introduction Projects of Anhui University, and Doctoral Research Start-up Funds Projects of Anhui University.

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