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Research Article

The Modification of Kernel Function and Its Applications

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By virtue of the modified Riesz kernel introduced by Qiao (2012), we give the integral representations for solutions of the Neumann problems in a half space.

1. Introduction and Main Results

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the sets of all real numbers and of all positive real numbers, respectively. Let $\mathbf{R}^n$ $(n \geq 3)$ denote the n-dimensional Euclidean space with points $x = (x', x_n)$, where $x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. The boundary and closure of an open set Ω of $\mathbf{R}^n$ are denoted by $\partial\Omega$ and $\overline{\Omega}$, respectively. For $x \in \mathbf{R}^n$ and $r > 0$, let $B_n(x, r)$ denote the open ball with center at x and radius r in $\mathbf{R}^n$. Let $B_n(r) = B_n(O, r)$.

The upper half space is the set $H = \{(x', x_n) \in \mathbf{R}^n : x_n > 0\}$, whose boundary is ∂H. For a set F, $F \subset \mathbf{R}_+ \cup \{0\}$, we denote $\{x \in H; |x| \in F\}$ and $\{x \in \partial H; |x| \in F\}$ by HF and ∂HF, respectively. We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$ and $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$, writing typical points x, $y \in \mathbf{R}^n$ as $x = (x', x_n)$, $y = (y', y_n)$, where $y' = (y_1, y_2, \ldots, y_{n-1}) \in \mathbf{R}^{n-1}$. Let θ be the angle between x and $\widehat{e}_n$, that is, $x_n = |x| \cos\theta$ and $0 \leq \theta < \pi/2$, where $\widehat{e}_n$ is the ith unit coordinate vector and $\widehat{e}_n$ is normal to ∂H.

We will say that a set $E \subset H$ has a covering $\{r_j, R_j\}$ if there exists a sequence of balls $\{B_j\}$ with centers in H such that $E \subset \cup_{j=1}^{\infty} B_j$, where r_j is the radius of B_j and R_j is the distance between the origin and the center of B_j.

For positive functions g_1 and g_2, we say that $g_1 \lesssim g_2$ if $g_1 \leq M g_2$ for some positive constant M. Throughout this paper, let M denote various constants independent of the variables in question. Further, we use the standard notations $u^+ = \max\{u, 0\}$, $[d]$ is the integer part of d, and $d = [d] + \{d\}$, where d is a positive real number.

Given a continuous function f on ∂H, we say that h is a solution of the Neumann problem on H with f, if h is a harmonic function on H and

$$\lim_{x \in H,\, x \to y'} \frac{\partial}{\partial x_n} h(x) = f(y') \tag{1}$$

for every point $y' \in \partial H$.

For $x \in \mathbf{R}^n$ and $y' \in \mathbf{R}^{n-1}$, consider the kernel function

$$K_n(x, y') = -\frac{\beta_n}{|x - y'|^{n-2}}, \tag{2}$$

where $\beta_n = 2/(n-2)\sigma_n$ and σ_n is the surface area of the n-dimensional unit sphere.

The Neumann integral on H is defined by

$$N[f](x) = \int_{\partial H} K_n(x, y') f(y') \, dy', \tag{3}$$

where f is a continuous function on ∂H.

The Neumann integral $N[f](x)$ is a solution of the Neumann problem on H with f if (see [1, Theorem 1 and Remarks])

$$\int_{\partial H} \frac{f(y')}{(1 + |y'|)^{n-2}} dy' < \infty. \tag{4}$$

In this paper, we consider functions f satisfying

$$\int_{\partial H} \frac{|f(y')|^p}{(1 + |y'|)^{\gamma}} dy' < \infty \tag{5}$$

for $1 \leq p < \infty$ and $\gamma \in \mathbf{R}$.

For p and α, we define the positive measure μ on $\mathbf{R}^n$ by

$$d\mu(y') = \begin{cases} |f(y')|^p |y'|^{-\gamma} dy' & y' \in \partial H(1,+\infty), \\ 0 & Q \in \mathbf{R}^n - \partial H(1,+\infty). \end{cases} \quad (6)$$

If f is a measurable function on ∂H satisfying (5), we remark that the total mass of μ is finite.

Let $\epsilon > 0$ and $\delta \geq 0$. For each $x \in \mathbf{R}^n$, the maximal function $M(x;\mu,\delta)$ is defined by

$$M(x;\mu,\delta) = \sup_{0<\rho<|x|/2} \frac{\mu(B_n(x,\rho))}{\rho^\delta}. \quad (7)$$

The set $\{x \in \mathbf{R}^n; M(x;\mu,\delta)|x|^\delta > \epsilon\}$ is denoted by $E(\epsilon;\mu,\delta)$.

To obtain the Neumann solution for the boundary data f on H, as in [2, 3], we use the following modified Riesz kernel defined by

$$L_{n,m}(x,y') = \begin{cases} -\beta_n \sum_{k=0}^{m-1} \frac{|x|^k}{|y|^{n+k-2}} C_k^{(n-2)/2}\left(\frac{x\cdot y'}{|x||y'|}\right) & |y'| \geq 1,\ m \geq 1, \\ 0 & |y'| < 1,\ m \geq 1, \\ 0 & m = 0, \end{cases} \quad (8)$$

where m is a nonnegative integer.

For $x \in \mathbf{R}^n$ and $y' \in \mathbf{R}^{n-1}$, the generalized Neumann kernel is defined by

$$K_{n,m}(x,y') = K_n(x,y') - L_{n,m}(x,y') \quad (m \geq 0). \quad (9)$$

Put

$$N_m[f](x) = \int_{\partial H} K_{n,m}(x,y') f(y') dy', \quad (10)$$

where f is continuous function on ∂H. Here note that $N_0[f](x)$ is nothing but the Neumann integral $N[f](x)$.

The following result is due to Su (see [4]).

Theorem A. *If f is a continuous function on ∂H satisfying* (5) *with $p = 1$ and $\alpha = m$, then*

$$\lim_{|x|\to\infty, x\in H} N_m[f](x) = o(|x|^m \sec^{n-2}\theta). \quad (11)$$

Our first aim is to be concerned with the growth property of $N_m[f]$ at infinity in a half space and establish the following theorem.

Theorem 1. *Let $1 \leq p < \infty$, $0 \leq \beta \leq (n-2)p$, $\gamma > -(n-1)(p-1)$ and*

$$1 - \frac{n-\gamma-1}{p} < m < 2 - \frac{n-\gamma-1}{p} \quad \text{if } p > 1,$$
$$\gamma - n + 2 \leq m < \gamma - n + 3 \quad \text{if } p = 1. \quad (12)$$

If f is a measurable function on ∂H satisfying (5)*, then there exists a covering $\{r_j, R_j\}$ of $E(\epsilon;\mu,(n-2)p-\beta)(\subset H)$ satisfying*

$$\sum_{j=0}^{\infty} \left(\frac{r_j}{R_j}\right)^{(n-2)p-\beta} < \infty \quad (13)$$

such that

$$\lim_{|x|\to\infty, x\in H-E(\epsilon;\mu,(n-2)p-\beta)} N_m[f](x) = o(|x|^{1+((\gamma-n+1)/p)}). \quad (14)$$

Remark 2. In the case that $p = 1$, $\alpha = m$, and $\beta = n-2$, then (13) is a finite sum and the set $E(\epsilon;\mu,0)$ is a bounded set. So (14) holds in H. That is to say, (11) holds. This is just the result of Theorem A.

Corollary 3. *Let $1 < p < \infty$, $n + \alpha - 2 > -(n-1)(p-1)$ and*

$$1 - \frac{n-\gamma-1}{p} < m < 2 - \frac{n-\gamma-1}{p}. \quad (15)$$

If f is a measurable function on ∂H satisfying (5)*, then*

$$\lim_{|x|\to\infty, x\in H} N_m[f](x) = o(|x|^{1+((\gamma-n+1)/p)}). \quad (16)$$

As an application of Theorem 1, we now show the solution of the Neumann problem with continuous data on H. About the solutions of the Dirichlet problem with respect to the Schrödinger operator in a half space, we refer readers to the paper by Su (see [5]).

Theorem 4. *Let p, β, α, and m be defined as in Theorem 1. If f is a continuous function on ∂H satisfying* (5)*, then the function $N_m[f]$ is a solution of the Neumann problem on H with f and* (14) *holds, where the exceptional set $E(\epsilon;\mu,(n-2)p-\beta)(\subset H)$ has a covering $\{r_j, R_j\}$ satisfying* (13).

Finally we have the following result.

Theorem 5. *Let $1 \leq p < \infty$, $\alpha > 1 - p$, l be a positive integer and*

$$1 - \frac{n-\gamma-1}{p} < m < 2 - \frac{n-\gamma-1}{p} \quad \text{if } p > 1,$$
$$\alpha \leq m < \alpha + 1 \quad \text{if } p = 1. \quad (17)$$

If f is a continuous function on ∂H satisfying (5) *and h is a solution of the Neumann problem on H with f such that*

$$\lim_{|x|\to\infty, x\in H} h^+(x) = o(|x|^{l+[1+((\gamma-n+1)/p)]}), \quad (18)$$

then

$$h(x) = N_m[f](x) + \Pi(x') + \sum_{j=1}^{[l+[1+((\gamma-n+1)/p)]/2]} \frac{(-1)^j}{(2j)!} x_n^{2j} \Delta^j \Pi(x') \quad (19)$$

for any $x = (x', x_n) \in H$, *where*

$$\Delta^j = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2}\right) \quad (j = 1, 2, \ldots) \tag{20}$$

and $\Pi(x')$ *is a polynomial of* $x' \in \mathbf{R}^{n-1}$ *of degree less than* $l + [1 + ((\gamma - n + 1)/p)]$.

2. Lemmas

In our discussions, the following estimates for the kernel function $K_{n,m}(x, y')$ are fundamental (see [6, Lemma 4.2] and [3, Lemmas 2.1 and 2.4]).

Lemma 6. *(1) If* $1 \le |y'| \le |x|/2$, *then* $|K_{n,m}(x, y')| \lesssim |x|^{m-1}|y'|^{-n-m+3}$.

(2) If $|x|/2 < |y'| \le (3/2)|x|$, *then* $|K_{n,m}(x, y')| \lesssim |x - y'|^{2-n}$.

(3) If $(3/2)|x| < |y'| \le 2|x|$, *then* $|K_{n,m}(x, y')| \lesssim x_n^{2-n}$.

(4) If $|y'| \ge 2|x|$ *and* $|y'| \ge 1$, *then* $|K_{n,m}(x, y')| \lesssim |x|^m|y'|^{2-n-m}$.

The following Lemma is due to Qiao (see [3]).

Lemma 7. *If* $\epsilon > 0$, $\eta \ge 0$, *and* λ *is a positive measure in* $\mathbf{R}^n$ *satisfying* $\lambda(\mathbf{R}^n) < \infty$, *then* $E(\epsilon; \lambda, \eta)$ *has a covering* $\{r_j, R_j\}$ $(j = 1, 2, \ldots)$ *such that*

$$\sum_{j=1}^{\infty}\left(\frac{r_j}{R_j}\right)^{\eta} < \infty. \tag{21}$$

Lemma 8. *Let* p, β, α, *and* m *be defined as in Theorem 1. If* f *is a local integral and upper semicontinuous function on* ∂H *satisfying* (5), *then*

$$\limsup_{x\in H,\, x\to y'} \frac{\partial}{\partial x_n} N_m[f](x) \le f(y'), \tag{22}$$

for any fixed point $y' \in \partial H$.

Proof. Let y^* be any fixed point an ∂H and let ϵ be any positive number. Take a positive number δ, $\delta < 1$, such that

$$f(y) < f(y^*) + \epsilon, \tag{23}$$

for any $y \in B_{n-1}(y^*, \delta)$.

By Lemma 6(4) and (5), we can choose a number R^*, $R^* > 2(|y^*| + 1)$, such that

$$\int_{\partial H\backslash B_{n-1}(R^*)} \left|\frac{\partial}{\partial x_n} K_{n,m}(x, y')\right| \left|f(y')\right| dy' < \epsilon, \tag{24}$$

for any $x \in \partial H \cap B_{n-1}(y^*, \delta)$.

Put

$$\begin{aligned} \Lambda_1(x) &= \int_{B_{n-1}(R^*)} \frac{\partial}{\partial x_n} K_{n,0}(x, y') f(y') dy', \\ \Lambda_2(x) &= -\int_{B_{n-1}(R^*)} \frac{\partial}{\partial x_n} L_{n,m}(x, y') f(y') dy'. \end{aligned} \tag{25}$$

Since

$$\frac{\partial}{\partial x_n} K_{n,0}(x, y') = \frac{2x_n}{\sigma_n} \frac{1}{|x - y'|^n}, \tag{26}$$

for any $x = (x', x_n) \in H$ and $y' \in \partial H$, we have

$$\begin{aligned} &\left|\int_{B_{n-1}(R^*)\backslash B_{n-1}(y^*,\delta)} \frac{\partial}{\partial x_n} K_{n,0}(x, y') f(y') dy'\right| \\ &\quad \lesssim x_n \left(\frac{\delta}{2}\right)^{-n} \int_{B_{n-1}(R^*)\backslash B_{n-1}(y^*,\delta)} f(y') dy' \end{aligned} \tag{27}$$

for any $x \in H \cap B_n(y^*, \delta/2)$.

Since

$$\begin{aligned} 1 &- \int_{B_{n-1}(y^*,\delta)} \frac{\partial}{\partial x_n} K_{n,0}(x, y') dy' \\ &= \int_{\partial H\backslash B_{n-1}(y^*,\delta)} \frac{\partial}{\partial x_n} K_{n,0}(x, y') dy' \\ &= \frac{2x_n}{\sigma_n} \int_{\partial H\backslash B_{n-1}(y^*,\delta)} \frac{1}{|x - y'|^n} dy', \end{aligned} \tag{28}$$

for any $x \in H$, we observe that

$$\limsup_{x\in H,\, x\to y^*} \int_{B_{n-1}(y^*,\delta)} \frac{\partial}{\partial x_n} K_{n,0}(x, y') dy' = 1. \tag{29}$$

Finally (23), (27), and (29) yield

$$\lim_{x\in H,\, x\to y^*} \Lambda_1(x) \le f(y^*) + \epsilon. \tag{30}$$

From Lemma 6(4) we obtain

$$|\Lambda_2(x)| \lesssim \int_{B_{N-1}(R^*)} x_n |f(y')| dy' \lesssim x_n \tag{31}$$

for any $x \in H \cap B_{n-1}(y^*, \delta)$.

These and (24) yield

$$\begin{aligned} &\limsup_{x\in H,\, x\to y^*} \frac{\partial}{\partial x_n} N_m[f](x) \\ &\quad = \limsup_{x\in H,\, x\to y^*} \int_{\partial H} \frac{\partial}{\partial x_n} K_{n,m}(x, y') f(y') dy' \\ &\quad = \limsup_{x\in H,\, x\to y^*} \Big(\Lambda_1(x) + \Lambda_2(x) \\ &\qquad + \int_{\partial H\backslash B_{n-1}(R^*)} \frac{\partial}{\partial x_n} K_{n,m}(x, y') f(y') dy'\Big) \\ &\quad \le f(y^*) + 2\epsilon. \end{aligned} \tag{32}$$

□

Now the conclusion immediately follows.

Lemma 9 (see [1, Lemma 1]). *If $h(x)$ is a harmonic polynomial of $x=(x',x_n)\in H$ of degree m and $\partial h/\partial x_n$ vanishes on ∂H, then there exists a polynomial $\Pi(x')$ of degree m such that*

$$h(x)=\begin{cases}\Pi(x')+\sum_{j=1}^{[m/2]}\frac{(-1)^j}{(2j)!}x_n^{2j}\Delta^j\Pi(x') & \text{if } m\ge 2,\\ \Pi(x') & \text{if } m=0,1.\end{cases} \tag{33}$$

3. Proof of Theorem 1

We prove only the case $p>1$; the proof of the case $p=1$ is similar.

For any $\epsilon>0$, there exists $R_\epsilon>1$ such that

$$\int_{\partial H(R_\epsilon,\infty)}\frac{|f(y')|^p}{(1+|y'|)^{n+\alpha-2}}dy'<\epsilon. \tag{34}$$

Take any point $x\in H(R_\epsilon,\infty)-E(\epsilon;\mu,(n-2)p-\beta)$ such that $|x|>2R_\epsilon$ and write

$$\begin{aligned}&N_m[f](x)\\&=\left(\int_{G_1}+\int_{G_2}+\int_{G_3}+\int_{G_4}+\int_{G_5}\right)K_{n,m}(x,y')f(y')dy'\\&=U_1(x)+U_2(x)+U_3(x)+U_4(x)+U_5(x),\end{aligned} \tag{35}$$

where

$$\begin{aligned}G_1&=\{y'\in\partial H:|y'|\le 1\},\\G_2&=\left\{y'\in\partial H:1<|y'|\le\frac{|x|}{2}\right\},\\G_3&=\left\{y'\in\partial H:\frac{|x|}{2}<|y'|\le\frac{3}{2}|x|\right\},\\G_4&=\left\{y'\in\partial H:\frac{3}{2}|x|<|y'|\le 2|x|\right\},\\G_5&=\{y'\in\partial H:|y'|\ge 2|x|\}.\end{aligned} \tag{36}$$

First note that

$$|U_1(x)|\lesssim\int_{G_1}\frac{|f(y')|}{|x-y'|^{n-2}}dy'\lesssim|x|^{2-n}\int_{G_1}|f(y')|dy', \tag{37}$$

so that

$$\lim_{|x|\to\infty,x\in H}|x|^{-1+((n-\gamma-1)/p)}U_1(x)=0. \tag{38}$$

If $m<2-((n-\gamma-1)/p)$ and $1/p+1/q=1$, then $(3-n-m+((n+\alpha-2)/p))q+n-1>0$. By Lemma 6(1), (34), and Hölder inequality, we have

$$\begin{aligned}|U_2(x)|&\lesssim|x|^{m-1}\int_{G_2}|y'|^{-n-m+3}|f(y')|dy'\\&\lesssim|x|^{m-1}\left(\int_{G_2}\frac{|f(y')|^p}{|y'|^{n+\alpha-2}}dy'\right)^{1/p}\\&\quad\times\left(\int_{G_2}|y'|^{(-n-m+3+((n+\alpha-2)/p))q}dy'\right)^{1/q}\\&\lesssim|x|^{1-((n-\gamma-1)/p)}\left(\int_{G_2}\frac{|f(y')|^p}{|y'|^{n+\alpha-2}}dy'\right)^{1/p}.\end{aligned} \tag{39}$$

Put

$$U_2(x)=U_{21}(x)+U_{22}(x), \tag{40}$$

where

$$\begin{aligned}U_{21}(x)&=\int_{G_2\cap B_{n-1}(R_\epsilon)}K_{n,m}(x,y')f(y')dy',\\U_{22}(x)&=\int_{G_2\setminus B_{n-1}(R_\epsilon)}K_{n,m}(x,y')f(y')dy'.\end{aligned} \tag{41}$$

If $|x|\ge 2R_\epsilon$, then

$$|U_{21}(x)|\lesssim R_\epsilon^{2-m-((n-\gamma-1)/p)}|x|^{m-1}. \tag{42}$$

Moreover, by (34) and (39) we get

$$|U_{22}(x)|\lesssim\epsilon|x|^{1-((n-\gamma-1)/p)}. \tag{43}$$

That is,

$$|U_2(x)|\lesssim\epsilon|x|^{1-((n-\gamma-1)/p)}. \tag{44}$$

By Lemma 6(3), (34), and Hölder inequality, we have

$$|U_4(x)|\lesssim\epsilon x_n^{2-n}|x|^{n-1-((n-\gamma-1)/p)}. \tag{45}$$

If $m>1-((n-\gamma-1)/p)$, then $(2-n-m+((n+\alpha-2)/p))q+n-1<0$. We obtain Lemma 6(4), (34), and Hölder inequality:

$$\begin{aligned}|U_5(x)|&\lesssim|x|^m\int_{G_5}|y'|^{-n-m+2}|f(y')|dy'\\&\lesssim|x|^m\left(\int_{G_5}\frac{|f(y')|^p}{|y'|^{n+\alpha-2}}dy'\right)^{1/p}\\&\quad\times\left(\int_{G_5}|y'|^{(-n-m+2+((n+\alpha-2)/p))q}dy'\right)^{1/q}\\&\lesssim\epsilon|x|^{1-((n-\gamma-1)/p)}.\end{aligned} \tag{46}$$

Finally, we will estimate $U_3(x)$. Take a sufficiently small positive number b such that $\partial H[|x|/2, (3/2)|x|] \subset B(x, |x|/2)$ for any $x \in \Pi(b)$, where

$$\Pi(b) = \left\{ x \in H; \inf_{y' \in \partial H} \left| \frac{x}{|x|} - \frac{y'}{|y'|} \right| < b \right\}, \quad (47)$$

and divide H into two sets $\Pi(b)$ and $H - \Pi(b)$.

If $x \in H - \Pi(b)$, then there exists a positive number b' such that $|x - y'| \geq b'|x|$ for any $y' \in \partial H$, and hence

$$\begin{aligned} |U_3(x)| &\lesssim \int_{G_3} |y'|^{2-n} |f(y')| \, dy' \\ &\lesssim |x|^m \int_{G_3} |y'|^{2-n-m} |f(y')| \, dy' \\ &\lesssim \epsilon |x|^{1-((n-\gamma-1)/p)}, \end{aligned} \quad (48)$$

which is similar to the estimate of $U_5(x)$.

We will consider the case $x \in \Pi(b)$. Now put

$$\begin{aligned} H_i(x) = \Big\{ y' \in \partial H \left[\frac{|x|}{2}, \frac{3}{2}|x| \right]; 2^{i-1}\delta(x) \\ \leq |x - y'| < 2^i \delta(x) \Big\}, \end{aligned} \quad (49)$$

where $\delta(x) = \inf_{y' \in H} |x - y'|$.

Since $\partial H \cap \{y' \in \mathbf{R}^{n-1} : |x - y'| < \delta(x)\} = \emptyset$, we have

$$U_3(x) = \sum_{i=1}^{i(x)} \int_{H_i(x)} \frac{|g(y')|}{|x - y'|^{n-2}} dy', \quad (50)$$

where $i(x)$ is a positive integer satisfying $2^{i(x)-1}\delta(x) \leq |x|/2 < 2^{i(x)}\delta(x)$.

Similar to the estimate of $U_5(x)$ we obtain

$$\begin{aligned} &\int_{H_i(x)} \frac{|g(y')|}{|x - y'|^{n-2}} dy' \\ &\lesssim \int_{H_i(x)} \frac{|g(y')|}{\{2^{i-1}\delta(x)\}^{n-2}} dy' \\ &\lesssim \delta(x)^{(\beta-(n-2)p)/p} \int_{H_i(x)} \delta(x)^{(((n-2)p-\beta)/p)-n+2} |g(y')| \, dy' \\ &\lesssim \cos^{-\beta/p}\theta \delta(x)^{(\beta-(n-2)p)/p} \int_{H_i(x)} |x|^{-\beta/p} |g(y')| \, dy' \\ &\lesssim |x|^{n-2-(\beta/p)} \cos^{-\beta/p}\theta \delta(x)^{(\beta-(n-2)p)/p} \\ &\quad \times \int_{H_i(x)} |y'|^{2-n} |g(y')| \, dy' \\ &\lesssim |x|^{n-1+((\alpha-\beta-1)/p)} \left(\frac{\mu(H_i(x))}{2^i \delta(x)^{(n-2)p-\beta}} \right)^{1/p} \end{aligned} \quad (51)$$

for $i = 0, 1, 2, \ldots, i(x)$.

Since $x \notin E(\epsilon; \mu, (n-2)p - \beta)$, we have

$$\begin{aligned} \frac{\mu(H_i(x))}{\{2^i \delta(x)\}^{(n-2)p-\beta}} &\lesssim \frac{\mu(B_{n-1}(x, 2^i \delta(x)))}{\{2^i \delta(x)\}^{(n-2)p-\beta}} \\ &\lesssim M(x; \mu, (n-2)p - \beta) \\ &\lesssim \epsilon |x|^{\beta-(n-2)p} \end{aligned} \quad (52)$$

for $i = 0, 1, 2, \ldots, i(x) - 1$ and

$$\frac{\mu(H_{i(x)}(x))}{\{2^i \delta(x)\}^{(n-2)p-\beta}} \lesssim \frac{\mu(B_{n-1}(x, |x|/2))}{(|x|/2)^{(n-2)p-\beta}} \lesssim \epsilon |x|^{\beta-(n-2)p}. \quad (53)$$

So

$$|U_3(x)| \lesssim \epsilon |x|^{1+((\gamma-n+1)/p)}. \quad (54)$$

Combining (38) and (44)–(54), we obtain that if R_ϵ is sufficiently large and ϵ is a sufficiently small number, then $N_m[f](x) = o(|x|^{1+((\gamma-n+1)/p)})$ as $|x| \to \infty$, where $x \in H(R_\epsilon, +\infty) - E(\epsilon; \mu, (n-2)p - \beta)$. Finally, there exists an additional finite ball B_0 covering $H(0, R_\epsilon]$, which, together with Lemma 7, gives the conclusion of Theorem 1.

4. Proof of Theorem 4

For any fixed $x \in H$, take a number R satisfying $R > \max\{1, 2|x|\}$. If $m > (n - \gamma - 1)/p$, then $(2 - n - m + ((n + \alpha - 2)/p))q + n - 1 < 0$. By (5), Lemma 6(4), and Hölder inequality, we have

$$\begin{aligned} &\int_{\partial H(R,\infty)} |K_{n,m}(x, y')| |f(y')| \, dy' \\ &\lesssim |x|^m \int_{\partial H(R,\infty)} |y'|^{2-n-m} |f(y')| \, dy' \\ &\lesssim |x|^m \left(\int_{\partial H(R,\infty)} \frac{|f(y')|^p}{|y'|^{n+\alpha-2}} dy' \right)^{1/p} \\ &\quad \times \left(\int_{\partial H(R,\infty)} |y'|^{(-n-m+2+((n+\alpha-2)/p))q} dy' \right)^{1/q} \\ &< \infty. \end{aligned} \quad (55)$$

Hence, $N_m[f](x)$ is absolutely convergent and finite for any $x \in H$. Thus, $N_m[f](x)$ is harmonic on H.

To prove

$$\lim_{x \to y', x \in H} \frac{\partial}{\partial x_n} N_m[f](x) = f(y') \quad (56)$$

for any point $y' \in \partial H$, we only need to apply Lemma 8 to $f(y)$ and $-f(y)$.

We complete the proof of Theorem 4.

5. Proof of Theorem 5

Consider the function $h'(x) = h(x) - N_m[f](x)$. Then it follows from Theorems 4 and 5 that $h'(x)$ is a solution of the Neumann problem on H with f and it is an even function of x_n (see [1, page 92]).

Since

$$0 \le \{h - N_m[f]\}^+(x) \le h^+(x) + \{N_m[f]\}^-(x) \tag{57}$$

for any $x \in H$, we have

$$\lim_{|x| \to \infty,\, x \in H} N_m[f](x) = o\left(|x|^{1+((\gamma-n+1)/p)}\right) \tag{58}$$

from Theorem 4.

Moreover, (18) gives that

$$\lim_{|x| \to \infty,\, x \in H} (h - N_m[f])(x) = o\left(|x|^{l+[1+((\gamma-n+1)/p)]}\right). \tag{59}$$

This implies that $h'(x)$ is a polynomial of degree less than $l + [1 + ((\gamma - n + 1)/p)]$ (see [7, Appendix]), which gives the conclusion of Theorem 5 from Lemma 9.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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