Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 731021, 6 pages
http://dx.doi.org/10.1155/2013/731021

Research Article

Existence of Oscillatory Solutions of Second Order Delay Differential Equations with Distributed Deviating Arguments

Youjun Liu,[1] Jianwen Zhang,[2] and Jurang Yan[3]

[1] *College of Mathematics and Computer Sciences, Shanxi Datong University, Datong, Shanxi 037009, China*
[2] *Institute of Applied Mechanics and Biomedical Engineering, Taiyuan University of Technology, Taiyuan, Shanxi 030024, China*
[3] *School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi 030006, China*

Correspondence should be addressed to Jianwen Zhang; zjw9791@126.com

Received 9 June 2013; Revised 25 October 2013; Accepted 13 November 2013

Academic Editor: Elena Litsyn

Under weaker hypothesis, we use the Schauder-Tychonoff theorem to obtain new sufficient condition for the global existence of oscillatory solutions for forced second order nonlinear delay differential equations with distributed deviating arguments.

1. Introduction

In this paper, we study the existence of oscillatory solutions for the nonlinear second order delay differential equations with the perturbed term

$$\left[r(t)\Phi\left(x'(t)\right)\right]' + \int_a^b p(t,\tau) f(x(t-\tau))\,d\tau = q(t), \quad t \geq t_0. \tag{1}$$

Under the following conditions:

(1) $r \in C^1([t_0,\infty),R^+)$, $p \in C([t_0,\infty)\times[a,b],R)$, $q \in C([t_0,\infty),R)$, $f_i \in C([t_0,\infty),R)$;

(2) $\Phi \in C^1(R,R)$, $\Phi(u)$ is increasing function for all $u \in R$, $\Phi^{-1}(u)$ satisfies the local Lipischitz condition.

During the past three decades, the investigation of oscillatory theory for delay differential equations and delay dynamic equations has attracted attention of numerous researchers due to their significance in theory and applications. We mention here the monographs of Myshkis [1], Norkin [2], Shevelo [3], and Agarwal et al. [4]. The oscillation properties of second order delay differential equations were considered also in Koplatadze et al. [5], Shmul'yan [6], and Skubachevskii [7]. Distances between adjacent zeros of oscillating solutions are estimated in [8, 9] for delay and for neutral second order equation in [10]. Distances between zero of solution and zero of its derivative were estimated in [11]. Based on oscillation properties, asymptotic properties of second order delay differential equations were studied in [12]. For related work, we refer the reader to the references [13–24]. However, to the best of our knowledge, the existence of oscillatory solutions for differential equation with distributed deviating arguments has been scarcely investigated. Thus, the research presents its significance.

As usual, a solution of (1) is a function $x(t)$ defined on $[t_0-b,\infty)$ such that $x(t)$ and $r(t)\Phi(x'(t))$ are continuously differentiable on $[t_0-b,\infty)$ and $[t_0,\infty)$ and (1) holds. Our attention will be restricted to the solution $x(t)$ of (1) which satisfy $\sup|x(t)|>0$, for $t\geq t_0-b$. Such a solution is said to be oscillatory if it has a sequence of zeros tending to infinity. Otherwise, it is said to be nonoscillatory.

The purpose of this paper is to prove a general result for (1) on the existence of oscillatory solutions. Throughout this paper, we will use the following notations. For a constant $\gamma>0$, $\theta_\gamma = \max_{|x|\leq\gamma}|f(x)|, t\geq t_0, L_\gamma$ denote the local lipischitz constants of functions $\Phi^{-1}(u)$.

2. The Main Results

Lemma 1 (see [13]). *Let X be a locally convex space, $K \subset X$ nonempty and convex, $S \subset K$, and S compact. Given a* *continuous map $F : K \to S$, there exists $\tilde{x} \in S$ such that $F(\tilde{x}) = \tilde{x}$.*

Theorem 2. *Assume that there exist $\eta, \gamma > 0$ such that $r(t) > \eta$,*

$$\frac{1}{r(t)} \int_t^\infty q(s)\,ds \text{ is integrable on } [t_0, \infty), \qquad \frac{\theta_\gamma}{r(t)} \int_t^\infty \int_a^b p(s,\tau)\,d\tau\,ds \text{ is integrable on } [t_0, \infty); \tag{2}$$

moreover, there exist two sequences $\{t_n\}$, $\{s_n\}$ with $t_n \to \infty$, $s_n \to \infty$ such that

$$\int_{t_n}^\infty \Phi^{-1}\left(\frac{1}{r(s)} \int_s^\infty \left(q(u) + \theta_\gamma \int_a^b p(u,\tau)\,d\tau\right) du\right) ds < 0,$$
$$\int_{s_n}^\infty \Phi^{-1}\left(\frac{1}{r(s)} \int_s^\infty \left(q(u) - \theta_\gamma \int_a^b p(u,\tau)\,d\tau\right) du\right) ds > 0. \tag{3}$$

Then (1), *has an oscillatory solution $x(t)$ defined on $[t_0, \infty)$ with $|x| \le \gamma$, and $\lim_{t\to\infty} x(t) = 0$.*

Proof. The proof is based on an application of the well-known Schauder-Tychonoff fixed point theorem. From (2), for any $\gamma > 0$, we have to choose a large $T_\gamma \ge T$ such that for all $t \ge T_\gamma$,

$$\int_t^\infty \Phi^{-1}\left(\frac{1}{r(s)} \int_s^\infty \left(q(u) + \theta_\gamma \int_a^b p(u,\tau)\,d\tau\right) du\right) ds \le \gamma,$$
$$\int_t^\infty \Phi^{-1}\left(\frac{1}{r(s)} \int_s^\infty \left(q(u) - \theta_\gamma \int_a^b p(u,\tau)\,d\tau\right) du\right) ds \ge -\gamma. \tag{4}$$

Let $C[t_0 - b, \infty)$ denote the locally convex space of all continuous functions with topology of uniform convergence on compact subsets of $[t_0 - b, \infty)$. Let $S = \{x \in C[t_0 - b, \infty), |x(t)| \le \gamma\}$. Clearly, S is a close convex subset of $C[t_0 - b, \infty)$.

Introduce an operator F by

$$(Fx)(t) = \begin{cases} \displaystyle\int_t^\infty \Phi^{-1}\left(\frac{1}{r(s)} \times \int_s^\infty \left(q(u) - \int_a^b p(u,\tau) \times f(x(t-\tau))\,d\tau \frac{1}{r(s)}\right) du\right) ds, & t > T_\gamma, \\ (Fx)(T_\gamma), & t_0 - b \le t \le T_\gamma. \end{cases} \tag{5}$$

It is easy to see that, for any $x \in S$, $(Fx)(t)$ is well defined on $[t_0 - b, \infty)$ continuously.

From (4), we obtain

$$(Fx)(t) \le \int_t^\infty \Phi^{-1}\left(\frac{1}{r(s)} \int_s^\infty \left(q(u) + \theta_\gamma \int_a^b p(u,\tau)\,d\tau\right) du\right) ds \le \gamma, \quad t \ge t_0 - b,$$
$$(Fx)(t) \ge \int_t^\infty \Phi^{-1}\left(\frac{1}{r(s)} \int_s^\infty \left(q(u) - \theta_\gamma \times \int_a^b p(u,\tau)\,d\tau\right) du\right) ds \ge -\gamma, \quad t \ge t_0 - b. \tag{6}$$

Hence, $|(Fx)(t)| \le \gamma$. Thus, we have $FS \subset S$ and Fx is uniformly bounded on S.

Let $\{x_n\}_{n=1}^\infty \in S$ be any sequence and $x_0 \in S$ with $\lim_{n\to\infty} x_n = x_0$. Let T_1 be large constant with $T_1 > T$, for any $\epsilon > 0$ so that

$$\int_{T_1}^\infty \frac{1}{r(s)} \int_s^\infty \left(\int_a^b p(s,\tau)\,d\tau\right) du\,ds < \frac{\epsilon}{3\theta_\gamma L_\gamma}. \tag{7}$$

From the compactness of the domain of f_i, there exists a large $N(\epsilon) > 0$ and a constant $\delta(\epsilon) > 0$; let $t \in [t_0 - b, T_1]$ and $n \ge N$ when $|x_n - x_0| < \delta(\epsilon)$,

$$\max_{t_0 - b \le t \le T_1} \{|f(x_n(t-\tau)) - f(x_0(t-\tau))|\} \le \frac{\epsilon}{3L_\gamma M}, \tag{8}$$

where $M = \int_{t_0-b}^{T_1} (s - t_0 + b/r(s)) \int_a^b p(s,\tau)\,d\tau\,ds$. By virtue of (1)–(8), we have that for any $t \ge t_0 - b$ and $|x_n - x_0| < \delta$,

$$\begin{aligned} &|(Fx_n)(t) - (Fx_0)(t)| \\ &= \left| \int_t^\infty \Phi^{-1}\left(\frac{1}{r(s)} \times \int_s^\infty \left(q(u) - \int_a^b p(u,\tau) \times f(x_n(t-\tau))\,d\tau\right) du\right) ds \right. \\ &\quad \left. - \int_t^\infty \Phi^{-1}\left(\frac{1}{r(s)} \times \int_s^\infty \left(q(u) - \int_a^b p(u,\tau) \times f(x_0(t-\tau))\,d\tau\right) du\right) ds \right| \end{aligned}$$

$$
\begin{aligned}
&\leq \int_t^{\infty} \Bigg| \Phi^{-1}\Bigg(\frac{1}{r(s)} \times \int_s^{\infty}\Bigg(q(u) - \int_a^b p(u,\tau) \times f(x_n(t-\tau))\, d\tau\Bigg) du\Bigg) ds \\
&\qquad - \Phi^{-1}\Bigg(\frac{1}{r(s)} \times \int_s^{\infty}\Bigg(q(u) - \int_a^b p(u,\tau) \times f(x_0(t-\tau))\, d\tau\Bigg) du\Bigg) ds\Bigg| ds \\
&\leq L_\gamma \int_t^{\infty} \Bigg(\frac{1}{r(s)} \times \int_s^{\infty}\Bigg(\Bigg|\int_a^b p(u,\tau)\Bigg| \times \big|f(x_n(u-\tau)) - f(x_0(u-\tau))\big|\, d\tau\Bigg) du\Bigg) ds \\
&\leq L_\gamma \int_{t_0-b}^{T_1} \Bigg(\frac{1}{r(s)} \times \int_s^{\infty}\Bigg(\Bigg|\int_a^b p(u,\tau)\Bigg| \times \big|f(x_n(u-\tau)) - f(x_0(u-\tau))\big|\, d\tau\Bigg) du\Bigg) ds \\
&\quad + L_\gamma \int_{T_1}^{\infty} \Bigg(\frac{1}{r(s)} \times \int_s^{\infty}\Bigg(\Bigg|\int_a^b p(u,\tau)\Bigg| \times \big|f(x_n(u-\tau)) - f(x_0(u-\tau))\big|\, d\tau\Bigg) du\Bigg) ds \\
&\leq L_\gamma \int_{t_0-b}^{T_1} \Bigg(\frac{1}{r(s)} (s-t_0+b) \times \Bigg(\Bigg|\int_a^b p(s,\tau)\Bigg| \times \big|f(x_n(s-\tau)) - f(x_0(s-\tau))\big|\, d\tau\Bigg)\Bigg) ds \\
&\quad + 2\theta_\gamma L_\gamma \int_{T_1}^{\infty} \frac{1}{r(s)} \int_s^{\infty}\Bigg(\int_a^b p(u,\tau)\, d\tau\Bigg) du\Bigg) ds \\
&< \frac{\epsilon}{3} + \frac{2\epsilon}{3} = \epsilon.
\end{aligned} \tag{9}
$$

The continuity of F on S is proved.

Moreover, for all $t_2, t_1 > t_0 - b$,

$$
\begin{aligned}
&(Fx)(t_2) - (Fx)(t_1) \\
&\quad = \int_{t_1}^{t_2} \Phi^{-1}\Bigg(\frac{1}{r(s)} \times \int_s^{\infty}\Bigg(q(u) - \int_a^b p(u,\tau) \times f(x(t-\tau))\, d\tau\Bigg) du\Bigg) ds \\
&\quad \leq \int_{t_1}^{t_2} \Phi^{-1}\Bigg(\frac{1}{r(s)} \int_s^{\infty}\Bigg(q(u) + \theta_\gamma \int_a^b p(u,\tau)\, d\tau\Bigg) du\Bigg) ds \\
&\quad \leq \alpha(t_2 - t_1),
\end{aligned} \tag{10}
$$

where $\alpha = \sup_{t \geq t_0} \Phi^{-1}(1/r(s) \int_t^{\infty}(q(u) + \theta_\gamma \int_a^b p(u,\tau)d\tau)du)$;

$$
\begin{aligned}
&(Fx)(t_2) - (Fx)(t_1) \\
&\quad = \int_{t_1}^{t_2} \Phi^{-1}\Bigg(\frac{1}{r(s)} \times \int_s^{\infty}\Bigg(q(u) + \int_a^b p(u,\tau) \times f(x(t-\tau))\, d\tau\Bigg) du\Bigg) ds \\
&\quad \geq \int_{t_1}^{t_2} \Phi^{-1}\Bigg(\frac{1}{r(s)} \int_s^{\infty}\Bigg(q(u) - \theta_\gamma \int_a^b p(u,\tau)\, d\tau\Bigg) du\Bigg) ds \\
&\quad \geq \beta(t_2 - t_1),
\end{aligned} \tag{11}
$$

where $\beta = \inf_{t \geq t_0} \Phi^{-1}(1/r(s) \int_t^{\infty}(q(u) - \theta_\gamma \int_a^b p(u,\tau)d\tau)du)$. Thus,

$$
\left|(Fx)(t_2) - (Fx)(t_1)\right| \leq M\left|t_2 - t_1\right|, \tag{12}
$$

where $M = \max\{|\alpha|, |\beta|\}$. This implies that Fx is equicontinuous. Hence, by the Ascoli-Arzela Theorem, the operator is completely continuous on S. By Lemma 1, there exists $\tilde{x} \in S$ satisfying

$$\tilde{x}(t) = (F\tilde{x})(t) = \begin{cases} \int_t^\infty \Phi^{-1}\left(\frac{1}{r(s)} \times \int_s^\infty \left(q(u) - \int_a^b p(u,\tau) \times f(\tilde{x}(t-\tau))\, d\tau\right) du\right) ds, & t \geq T, \\ (F\tilde{x})(T), & t_0 - b \leq t < T. \end{cases} \tag{13}$$

On the other hand, from (3), we find

$$\begin{aligned} \tilde{x}(t_n) &\leq \int_{t_n}^\infty \Phi^{-1} \times \left(\frac{1}{r(s)} \int_s^\infty \left(q(u) + \theta_\gamma \int_a^b p(u,\tau)\, d\tau\right) du\right) ds < 0, \\ \tilde{x}(s_n) &\geq \int_{s_n}^\infty \Phi^{-1} \times \left(\frac{1}{r(s)} \int_s^\infty \left(q(u) - \theta_\gamma \int_a^b p(u,\tau)\, d\tau\right) du\right) ds > 0, \end{aligned} \tag{14}$$

which implies that $\tilde{x}(t)$ is a bounded oscillatory solution of (1) and $\lim_{t\to\infty}\tilde{x}(t) = 0$. The proof is complete. □

Corollary 3. *Assume that* (2) *of Theorem 2 holds, and specially* $\Phi(u) = u^\alpha, \alpha \geq 1$ *is the ratio of two positive odd integers, there exist two increasing divergent sequences* $\{t_n\}$ *and* $\{s_n\}, t_n, s_n$ *such that*

$$\begin{aligned} &\int_{t_n}^\infty \left(\frac{1}{r(s)} \int_s^\infty \left(q(u) + \theta_\gamma \int_a^b p(u,\tau)\, d\tau\right) du\right)^{1/\alpha} ds < 0, \\ &\int_{s_n}^\infty \left(\frac{1}{r(s)} \int_s^\infty \left(q(u) - \theta_\gamma \int_a^b p(u,\tau)\, d\tau\right) du\right)^{1/\alpha} ds > 0. \end{aligned} \tag{15}$$

Then, (1) *has an oscillatory solution* $x(t)$ *defined on* $[t_0, \infty)$ *with* $|x| \leq \gamma$, *and* $\lim_{t\to\infty} x(t) = 0$.

3. Remark

When $r(t) \equiv 1, \Phi(u) = u, f(v) = v$, and $q(t) = 0$. Let $s = t - \tau$, (1) becomes

$$x''(t) - \int_{t-b}^{t-a} k(t,s)\, x(s)\, ds = 0. \tag{16}$$

According to the results of this paper, $\int_t^\infty \int_a^b k(u, u-\tau) d\tau\, du$ is integrable on $[t_0, \infty)$, and there exist two sequences $\{t_n\}, \{s_n\}$ with $t_n \to \infty, s_n \to \infty$ such that

$$\begin{aligned} &\int_{t_n}^\infty \left(\int_s^\infty \int_a^b k(u, u-\tau)\, d\tau du\right) ds < 0, \\ &\int_{s_n}^\infty \left(\int_s^\infty \int_a^b k(u, u-\tau)\, d\tau du\right) ds > 0. \end{aligned} \tag{17}$$

Equation (16) has oscillatory solution.

In paper [14], it was demonstrated that the inequality $b(b-a)\max_{0\leq s\leq t<\infty} k(t,s) \leq 2/e$ implied the existence of positive solutions. Neither of the two conditions can be deduced from each other. Moreover, when $k(t,s) = s/1+t^2 (s = t-\tau)$ is bounded, but $\int_t^\infty \int_a^b k(u, u-\tau) d\tau\, du$ cannot exist. Likewise, when $k(t,s) = 5/1+t^2$ and $b(b-a) \geq 2/e$ claim, $\int_t^\infty \int_a^b k(u, u-\tau) d\tau\, du$ can exist, but $b(b-a)\max_{0\leq s\leq t<\infty} k(t,s) \leq 2/e$ cannot be hold.

4. Examples

Example 1. Consider second order delay differential equations

$$\begin{aligned} &\left(e^{-t} x'(t)\right)' + \frac{2}{e^{2\pi} + e^\pi} \int_\pi^{2\pi} e^{-2t} x(t-\tau)\, d\tau \\ &\quad = e^{-2t}(\sin t - 3\cos t) + e^{-3t}(\cos t + \sin t). \end{aligned} \tag{18}$$

Here, $r(t) = e^{-t}$, $\Phi(u) = u$, $f(v) = v$, $p(t,\tau) = 2/((e^{2\pi} + e^\pi)e^{-2t})$, $a = \pi$, $b = 2\pi$, and $q(t) = e^{-2t}(\sin t - 3\cos t) + e^{-3t}(\cos t + \sin t)$.

It is easy to see that $r(t) \geq \eta > 0$.

We have

$$\begin{aligned} Q(t) &:= \int_t^\infty \Phi^{-1} \times \left(\frac{1}{r(s)} \int_s^\infty \left(q(u) + \frac{2\gamma}{e^{2\pi} + e^\pi} \int_\pi^{2\pi} e^{-2u} d\tau\right) du\right) ds \\ &= \int_t^\infty e^s \int_s^\infty \left(e^{-2u}(\sin u - 3\cos u) + e^{-3u}(\cos u + \sin u) + \frac{2\pi\gamma}{e^{2\pi} + e^\pi} e^{-2u}\right) du\, ds \\ &= e^{-t}\sin t + \frac{4}{25} e^{-2t}\left(\cos t + \frac{3}{4}\sin t\right) - \frac{\pi\gamma}{e^{2\pi} + e^\pi} e^{-t}, \end{aligned}$$

$$\begin{aligned}
P(t) &:= \int_t^\infty \Phi^{-1} \\
&\quad \times \left(\frac{1}{r(s)} \int_s^\infty \left(q(u) - \frac{2\gamma}{e^{2\pi}+e^{\pi}} \int_\pi^{2\pi} e^{-2u} d\tau \right) du \right) ds \\
&= \int_t^\infty e^s \int_s^\infty \Big(e^{-2u}(\sin u - 3\cos u) + e^{-3u}(\cos u + \sin u) \\
&\qquad - \frac{2\pi\gamma}{e^{2\pi}+e^{\pi}} e^{-2u} \Big) du\, ds \\
&= e^{-t}\sin t + \frac{4}{25} e^{-2t} \left(\cos t + \frac{3}{4}\sin t \right) + \frac{\pi\gamma}{e^{2\pi}+e^{\pi}} e^{-t}. \quad (19)
\end{aligned}$$

Let $t_n = (2n-1)\pi, s_n = (2n+1)\pi, n = 1, 2, \ldots,$

$$\begin{aligned}
Q(t_n) &= -e^{-t_n} \left(\frac{4}{25} e^{-t_n} + \frac{\pi\gamma}{e^{2\pi}+e^{\pi}} \right) < 0, \\
P(s_n) &= e^{-s_n} \left(\frac{4}{25} e^{-s_n} + \frac{\pi\gamma}{e^{2\pi}+e^{\pi}} \right) > 0.
\end{aligned} \quad (20)$$

It is easy to see from (19) that there exists a $N = 1$ such that for all $n \geq N$, $Q(t_n) < 0$ and $P(t_n) > 0$. Thus, by Theorem 2, (18) has an oscillatory solution $x(t)$ on $[t_0, \infty)$ and $\lim_{t\to\infty} x(t) = 0$. It is not difficult to check that (18) has the oscillatory solution $x(t) = e^{-t}\sin t$.

Example 2. Consider second order delay differential equations

$$\begin{aligned}
&\left(e^{-t} \left(x'(t) \right)^{1/3} \right)' + \frac{6}{7\gamma^3} \int_1^2 \tau^2 e^{-2t} x^3 (t-\tau)\, d\tau \\
&\qquad = e^{-2t} (\cos t - 2\sin t).
\end{aligned} \quad (21)$$

Here, $r(t) = e^{-t}, \Phi(u) = u^{1/3}, p(t,\tau) = (6/7\gamma^3)\tau^2 e^{-2t} f(v) = v^3, a = 1, b = 2$, and $q(t) = e^{-2t}(\cos t - 2\sin t)$. It is easy to see that $r(t) \geq \eta > 0$,

$$\begin{aligned}
Q(t) &:= \int_t^\infty \Phi^{-1} \left(\frac{1}{r(s)} \int_s^\infty \left(q(u) + \frac{6}{7} \int_1^2 \tau^2 e^{-2u} d\tau \right) du \right) ds \\
&= \int_t^\infty \left(e^s \int_s^\infty e^{-2u} (\cos u - 2\sin u) + 2e^{-2u} du \right)^3 ds \\
&= \frac{1}{6} e^{-3t} \sin^2 t (-\sin t - \cos t) + e^{-3t} (-3\sin t - \cos t) \\
&\quad - \frac{3}{13} e^{-3t} \sin t (-3\sin t - 2\cos t) + \frac{19}{39} e^{-3t},
\end{aligned}$$

$$\begin{aligned}
P(t) &:= \int_t^\infty \Phi^{-1} \left(\frac{1}{r(s)} \int_s^\infty \left(q(u) + \frac{6}{7} \int_1^2 \tau^2 e^{-2u} d\tau \right) du \right) ds \\
&= \int_t^\infty \left(e^s \int_s^\infty e^{-2u} (\cos u - 2\sin u) - 2e^{-2u} du \right)^3 ds \\
&= \frac{1}{6} e^{-3t} \sin^2 t (-\sin t - \cos t) + e^{-3t} (-3\sin t - \cos t) \\
&\quad + \frac{3}{13} e^{-3t} \sin t (-3\sin t - 2\cos t) + \frac{7}{39} e^{-3t}. \quad (22)
\end{aligned}$$

Let $t_n = 2n\pi, s_n = (2n+1)\pi, n = 1, 2, \ldots,$

$$Q(t_n) = -\frac{20}{39} e^{-3t_n}, \qquad P(s_n) = \frac{46}{39} e^{-3s_n}. \quad (23)$$

It is easy to see from (22) that there exists a $N = 1$ such that for all $n \geq N$, $Q(t_n) < 0$ and $P(s_n) > 0$. Thus, by Theorem 2, (21) has an oscillatory solution $x(t)$ on $[t_0, \infty)$ and $\lim_{t\to\infty} x(t) = 0$.

Acknowledgments

The author thanks both referees for a careful reading of the paper and useful suggestions that helped to improve the presentation. This research is supported by Natural Sciences Foundation of China (no. 11172194), Natural Sciences Foundation of Shanxi Province (no. 2010011008), and Scientific Research Project Shanxi Datong University (no. 2011 K3).

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