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Research Article

An H^1-Galerkin Expanded Mixed Finite Element Approximation of Second-Order Nonlinear Hyperbolic Equations

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We investigate an H^1-Galerkin expanded mixed finite element approximation of nonlinear second-order hyperbolic equations, which model a wide variety of phenomena that involve wave motion or convective transport process. This method possesses some features such as approximating the unknown scalar, its gradient, and the flux function simultaneously, the finite element space being free of LBB condition, and avoiding the difficulties arising from calculating the inverse of coefficient tensor. The existence and uniqueness of the numerical solution are discussed. Optimal-order error estimates for this method are proved without introducing curl operator. A numerical example is also given to illustrate the theoretical findings.

1. Introduction

The objective of this paper is to present and analyze an H^1-Galerkin expanded mixed finite element method for the following second-order nonlinear hyperbolic equation:

$$u_{tt} - \nabla \cdot (A(u) \nabla u) = f, \quad (\mathbf{x}, t) \in \Omega \times J, \tag{1a}$$

$$u(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times J, \tag{1b}$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{1c}$$

$$u_t(\mathbf{x}, 0) = u_1(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{1d}$$

where Ω is a bounded convex polygonal domain in R^2 with boundary $\partial\Omega$ and $J = [0, T]$ with $T < \infty$. $u(\mathbf{x}, t)$ denotes the sound pressure, $f(\mathbf{x}, t)$ is the external force, and $A(u)$ is the coefficient, which is supposed to satisfy the following conditions.

(A_1) There exist positive constants α_i, $i = 0, 1, 2$, such that

$$\begin{gathered} 0 < \alpha_0 \le A(u) \le \alpha_1, \\ \left|A_u(u)\right| + \left|A_{uu}(u)\right| \le \alpha_2. \end{gathered} \tag{2}$$

(A_2) $A(u)$, $A_u(u)$, and $A_{uu}(u)$ are Lipschitz continuous with respect to u.

The primary interests in engineering application for the mathematical model (1a)–(1d) are the sound pressure u, the gradient of sound pressure $\mathbf{p}$, and the acceleration of sound transmission $\boldsymbol{\sigma}$. Extensive research has been carried out on the numerical methods and corresponding numerical analysis for model (1a)–(1d), including finite difference methods, finite element methods, and mixed finite element methods. One can refer to [1–4] and the references cited herein.

The standard finite difference or finite element methods solve the sound pressure u directly, then differentiate it to determine ∇u, and multiply the gradient of u by $A(u)$ to determine the acceleration of sound transmission $\boldsymbol{\sigma}$. Therefore, the resulting acceleration of sound transmission $\boldsymbol{\sigma}$ and the gradient of sound pressure ∇u are often inaccurate, which then reduces the accuracy of the prediction, as well as the accuracy of the adjoint vector $\boldsymbol{\sigma}$. The mixed finite element method can approximate both u and $\boldsymbol{\sigma}$ simultaneously and yields an accurate $\boldsymbol{\sigma}$. However, the mixed formulation has to face numerical difficulties arising in a low permeability zone because the inversion and the finite element spaces need to satisfy the LBB conditions.

In order to overcome the above problems, we propose an H^1-Galerkin expanded mixed finite element method for model (1a)–(1d) which can solve the sound pressure u, the gradient of sound pressure $\mathbf{p}$, and the acceleration of sound transmission $\boldsymbol{\sigma}$ directly and avoid inverting $A(u)$ explicitly. In this formulation the finite element spaces are free of LBB conditions as required by the standard mixed finite element methods. Another feature of the new procedure we have found so far is that it avoids the trouble which resulted from representation of the time derivatives for nonlinear problems and leads to optimal error estimates without introducing curl operator. We prove the equivalence of the problem (1a)–(1d), the H^1-Galerkin expanded mixed variational formulation and the existence and uniqueness of the semidiscrete H^1-Galerkin expanded mixed finite element procedure. By introducing some projection and interpolation operators as well as lemmas, optimal-order error estimates for this formulation are deduced. The theoretical findings are verified by one numerical example. In recent years, there exist lots of work in the literature on the development and analysis of H^1-Galerkin mixed finite element method. One can refer to [5–8] for linear parabolic type equations, [9] for regularized long wave equation, and [10] for linear second-order hyperbolic equation.

The rest of the paper is organized as follows. In Section 2, we describe the H^1-Galerkin expanded mixed finite element variational form and prove the equivalence between primal problem and the variational formulation. In Section 3, the H^1-Galerkin expanded mixed finite element procedure is presented, and the existence and uniqueness of the solution are proved. In Section 4, we prove the main error estimates. A numerical example is given in Section 5 to illustrate the theoretical findings.

Throughout this paper, C denotes a generic constant which does not depend on mesh parameter h. We use $(\cdot,\cdot)$ and $\|\cdot\|$ to denote the inner product and the norm, respectively, in $L^2(\Omega)$ or $(L^2(\Omega))^2$. Also we will denote the norms in usual Sobolev spaces $W^{k,p}(\Omega)$ by $\|\cdot\|_{k,p}$ and the norms in $L^q(0,t;W^{k,p}(\Omega))$ by $\|\cdot\|_{L^q(0,t;W^{k,p}(\Omega))}$ with $p = 2$ being omitted.

2. An H^1-Galerkin Expanded Mixed Variational Formulation

In order to derive an H^1-Galerkin expanded mixed variational formulation we split (1a)–(1d) into a first-order system by introducing $\mathbf{p} = \nabla u$ and $\boldsymbol{\sigma} = A(u)\mathbf{p}$:

$$u_{tt} - \nabla\cdot\boldsymbol{\sigma} = f, \tag{3a}$$

$$\mathbf{p} = \nabla u, \tag{3b}$$

$$\boldsymbol{\sigma} = A(u)\,\mathbf{p}, \tag{3c}$$

$$\boldsymbol{\sigma}(\mathbf{x},0) = A(u_0)\,\nabla u_0(\mathbf{x}), \tag{3d}$$

$$\mathbf{p}(\mathbf{x},0) = \nabla u_0(\mathbf{x}), \qquad \mathbf{p}_t(\mathbf{x},0) = \nabla u_1(\mathbf{x}). \tag{3e}$$

Define the following spaces:

$$\begin{gathered}\mathbf{H} = \mathbf{H}(\operatorname{div},\Omega) = \left\{\mathbf{w}\in\left(L^2(\Omega)\right)^d; \nabla\cdot\mathbf{w}\in L^2(\Omega)\right\},\\ V = H_0^1(\Omega) = \left\{v\in H^1(\Omega); v=0 \ \text{ on } \partial\Omega\right\}.\end{gathered} \tag{4}$$

Multiplying (3a) by $\nabla\cdot\mathbf{q}$ for $\mathbf{q}\in\mathbf{H}(\operatorname{div},\Omega)$ and integrating Ω lead to the weak form (5a). Multiplying (3b) by ∇v for $v\in H_0^1(\Omega)$ leads to the weak form (5b). Multiplying (3c) by $\mathbf{w}$ for $\mathbf{w}\in\mathbf{H}(\operatorname{div},\Omega)$ and integrating on Ω result in the weak form (5c). Then the H^1-Galerkin expanded variational problem is to find $(u,\mathbf{p},\boldsymbol{\sigma})\in H_0^1(\Omega)\times\mathbf{H}(\operatorname{div},\Omega)\times\mathbf{H}(\operatorname{div},\Omega)$ such that

$$\left(\mathbf{p}_{tt}, q\right) + (\nabla\cdot\boldsymbol{\sigma}, \nabla\cdot\mathbf{q}) = -\left(f, \nabla\cdot\mathbf{q}\right), \quad \forall\mathbf{q}\in\mathbf{H}, \tag{5a}$$

$$(\mathbf{p},\nabla v) = (\nabla u, \nabla v), \quad \forall v\in V, \tag{5b}$$

$$(\boldsymbol{\sigma},\mathbf{w}) = (A(u)\,\mathbf{p},\mathbf{w}), \quad \forall\mathbf{w}\in\mathbf{H}, \tag{5c}$$

$$\boldsymbol{\sigma}(\mathbf{x},0) = A(u_0)\,\nabla u_0(\mathbf{x}), \quad \mathbf{x}\in\Omega, \tag{5d}$$

$$\mathbf{p}(\mathbf{x},0) = \nabla u_0(\mathbf{x}), \quad \mathbf{p}_t(\mathbf{x},0) = \nabla u_1(\mathbf{x}), \quad \mathbf{x}\in\Omega. \tag{5e}$$

In order to prove the equivalence of problem (3a)–(3e) and variational problem (5a)–(5e), we need the following lemmas (see [11] or Theorem 3.5, Chapter 1 of [12]).

Lemma 1. *For* $\mathbf{p}\in\mathbf{H}(\operatorname{div},\Omega)$, *there exist a* $\phi\in H^2(\Omega)\bigcap H_0^1(\Omega)$ *and a* $\boldsymbol{\psi}\in\mathbf{H}(\operatorname{div},\Omega)$ *satisfying* $\nabla\cdot\boldsymbol{\psi}=0$, *such that* $\mathbf{p}=\nabla\phi+\boldsymbol{\psi}$.

Lemma 2. *For* $g\in L^2(\Omega)$, *there exists a* $\mathbf{p}\in(H^1(\Omega))^d\subset\mathbf{H}(\operatorname{div},\Omega)$, *such that* $\nabla\cdot\mathbf{p}=g$.

With the help of these lemmas we can prove the following theorem.

Theorem 3. $(u,\mathbf{p},\boldsymbol{\sigma})\in H_0^1(\Omega)\times\mathbf{H}(\operatorname{div};\Omega)\times\mathbf{H}(\operatorname{div};\Omega)$ *is a solution to the system* (3a)–(3e) *if and only if it is a solution to the variational formulation* (5a)–(5e).

Proof. The proof of the "only if" part is pretty straightforward. It remains to prove the "if" part. We insert $\mathbf{w}=\boldsymbol{\sigma}-A(u)\mathbf{p}$ into (5c) to get (3c). By (3a)–(3e) and taking $\mathbf{p}=\nabla\phi+\boldsymbol{\psi}$ in (5b), we conclude

$$(\nabla\phi,\nabla v) = (\nabla u,\nabla v), \quad \forall v\in V. \tag{6}$$

Then we have

$$\mathbf{p} = \nabla u + \boldsymbol{\psi}, \tag{7}$$

which, together with (5a) and (3c), yields the following equation:

$$\begin{aligned}&\left(u_{tt},\nabla\cdot q\right) - \left(\nabla\cdot\left(A(u)\nabla u\right),\nabla\cdot\mathbf{q}\right) - \left(\boldsymbol{\psi}_{tt},\mathbf{q}\right)\\ &\quad - \left(\nabla\cdot\left(A(u)\,\boldsymbol{\psi}\right),\nabla\cdot\mathbf{q}\right) = \left(f,\nabla\cdot\mathbf{q}\right), \quad \forall\mathbf{q}\in\mathbf{H}.\end{aligned} \tag{8}$$

Note that $\nabla\cdot\boldsymbol{\psi}=0$. We obtain that $\nabla\cdot\boldsymbol{\psi}_t=0$. Setting $\mathbf{q}=\boldsymbol{\psi}_t$ in (8) we obtain $(\boldsymbol{\psi}_{tt},\boldsymbol{\psi}_t)=0$, which implies

$$\left(\boldsymbol{\psi}_t(t),\boldsymbol{\psi}_t(t)\right) = \left(\boldsymbol{\psi}_t(0),\boldsymbol{\psi}_t(0)\right). \tag{9}$$

Then by

$$\begin{aligned}\mathbf{p}_t(\mathbf{x},0) &= \nabla u_t(\mathbf{x},0) = \nabla u_1(\mathbf{x},0),\\ &= \nabla u_1(\mathbf{x},0) + \boldsymbol{\psi}_t(\mathbf{x},0)\end{aligned} \tag{10}$$

we can conclude that

$$\psi_t(\mathbf{x}, t) \equiv 0. \tag{11}$$

By (5c) we obtain

$$\boldsymbol{\sigma}(\mathbf{x}, 0) = A(u_0)\nabla u(\mathbf{x}, 0) + A(u_0)\boldsymbol{\psi}(\mathbf{x}, 0). \tag{12}$$

Here we select the initial value $\boldsymbol{\sigma}(\mathbf{x}, 0)$ as in (5d) to have $\boldsymbol{\psi}(\mathbf{x}, 0) = 0$. Then, we derive $\boldsymbol{\psi}(\mathbf{x}, t) = 0$ and (7) reduces to

$$\mathbf{p} = \nabla u. \tag{13}$$

Further we get

$$(u_{tt}, \nabla\cdot\mathbf{q}) - (\nabla\cdot(A(u)\nabla u), \nabla\cdot\mathbf{q}) = (f, \nabla\cdot\mathbf{q}), \quad \forall\mathbf{q}\in\mathbf{H}. \tag{14}$$

By lemma 2, there exists a $F \in \mathbf{H}(\mathrm{div}, \Omega)$ such that $\nabla\cdot F = u_{tt} - f$. Therefore, we have

$$(\nabla\cdot F, \nabla\cdot\mathbf{q}) - (\nabla\cdot(A(u)\nabla u), \nabla\cdot\mathbf{q}) = 0, \tag{15}$$

which implies

$$F = A(u)\nabla u = \boldsymbol{\sigma}. \tag{16}$$

That is,

$$u_{tt} - f = \nabla\cdot\boldsymbol{\sigma}. \tag{17}$$

This completes the proof. □

3. H^1-Galerkin Expanded Mixed Finite Element Procedure

In this section we will present the numerical scheme for (5a)–(5e). Let $\mathcal{T}_h$ be a quasiuniform partition of domain Ω; that is, $\Omega = \bigcup_{K\in\mathcal{T}_h} K$ with $h = \max\{\mathrm{diam}(K); K \in \mathcal{T}_h\}$. Let $\mathbf{H}_h$ and V_h be the finite dimensional subspaces of $\mathbf{H}(\mathrm{div};\Omega)$ and $H_0^1(\Omega)$ defined by

$$\begin{aligned} \mathbf{H}_h &= \left\{\mathbf{q}_h \in \mathbf{H}(\mathrm{div};\Omega); \mathbf{q}_h|_K \in (\mathbf{P}_k(K))^d, \forall K\in\mathcal{T}_h\right\}, \\ V_h &= \left\{v_h \in H_0^1(\Omega); v_h|_K \in \mathbf{P}_m(K), \forall K\in\mathcal{T}_h\right\}, \end{aligned} \tag{18}$$

where $\mathbf{P}_j(K)$ denotes the set of polynomials of degree at most j. Assume that $\mathbf{H}_h$, and V_h satisfy the following approximation properties. For integers $k \geq 0$, $m \geq 1$,

$$\begin{gathered} \inf_{\mathbf{q}_h\in\mathbf{H}_h} \|\mathbf{q} - \mathbf{q}_h\| \leq Ch^{k+1}\|\mathbf{q}\|_{k+1,\Omega}, \\ \mathbf{q} \in \left(H^{k+1}(\Omega)\right)^d \cap \mathbf{H}, \\ \inf_{\mathbf{q}_h\in\mathbf{H}_h} \|\nabla\cdot(\mathbf{q} - \mathbf{q}_h)\| \leq Ch^{k_1}\|\mathbf{q}\|_{k_1+1,\Omega}, \\ \mathbf{q} \in \left(H^{k+1}(\Omega)\right)^d \cap \mathbf{H}, \\ \inf_{v_h\in V_h} \left\{\|v - v_h\| + h\|v - v_h\|_{1,\Omega}\right\} \leq Ch^{m+1}\|v\|_{m+1,\Omega}, \\ v \in H^{m+1}(\Omega) \cap V. \end{gathered} \tag{19}$$

Here $k_1 = k + 1$ when H_h is one of the *Raviart-Thomas* elements or the *Nedelec* elements, and $k_1 = k \geq 1$, when H_h is one of the other classical mixed elements, such as *Breezi-Douglas-Fortin-Marini* elements and *Breezi-Douglas-Marini* elements.

Then the H^1-Galerkin expanded mixed finite element procedure for the system (3a)–(3e) is to find $(u_h, \mathbf{p}_h, \boldsymbol{\sigma}_h) \in V_h \times \mathbf{H}_h \times \mathbf{H}_h$ such that

$$(\mathbf{p}_{htt}, \mathbf{q}_h) + (\nabla\cdot\boldsymbol{\sigma}_h, \nabla\cdot\mathbf{q}_h) = -(f, \nabla\cdot\mathbf{q}_h), \quad \forall\mathbf{q}_h\in\mathbf{H}_h, \tag{20a}$$

$$(\mathbf{p}_h, \nabla v_h) = (\nabla u_h, \nabla v_h), \quad \forall v_h\in V_h, \tag{20b}$$

$$(\boldsymbol{\sigma}_h, \mathbf{w}_h) = (A(u_h)\mathbf{p}_h, \mathbf{w}_h), \quad \forall\mathbf{w}_h\in\mathbf{H}_h, \tag{20c}$$

$$\boldsymbol{\sigma}_h(\mathbf{x}, 0) = \boldsymbol{\Pi}_h\boldsymbol{\sigma}(\mathbf{x}, 0), \quad \forall\mathbf{x}\in\Omega, \tag{20d}$$

$$\mathbf{p}_h(\mathbf{x}, 0) = \boldsymbol{\Pi}_h\mathbf{p}(\mathbf{x}, 0), \qquad \mathbf{p}_{ht}(\mathbf{x}, 0) = \boldsymbol{\Pi}_h\mathbf{p}_t(\mathbf{x}, 0), \quad \forall\mathbf{x}\in\Omega, \tag{20e}$$

where $\boldsymbol{\Pi}_h$ denotes the Raviart-Thomas projection. We next prove the existence and uniqueness of solutions of the scheme (20a)–(20e).

Theorem 4. *There exists a unique solution* $(u_h, \mathbf{p}_h, \boldsymbol{\sigma}_h) \in V_h \times \mathbf{H}_h \times \mathbf{H}_h$ *to the* H^1*-Galerkin expanded mixed finite element procedure* (20a)–(20e).

Proof. Let $\mathbf{H}_h = \mathrm{span}\{\boldsymbol{\psi}_i\}_{i=1}^M$ and $V_h = \mathrm{span}\{\phi_i\}_{i=1}^N$; then $\boldsymbol{\sigma}_h \in \mathbf{H}_h$, $\mathbf{p}_h \in \mathbf{H}_h$, and $u_h \in V_h$ have the following expressions:

$$\mathbf{p}_h = \sum_{i=1}^M p_i\boldsymbol{\psi}_i, \qquad \boldsymbol{\sigma}_h = \sum_{i=1}^M \lambda_i\boldsymbol{\psi}_i, \qquad u_h = \sum_{i=1}^N u_i\phi_i. \tag{21}$$

Then the scheme (20a)–(20e) can be written in the following matrix form:

$$A\mathbf{P}_{tt} + B\boldsymbol{\Lambda} = \mathbf{F}, \tag{22a}$$

$$D\mathbf{U} = C\mathbf{P}, \tag{22b}$$

$$A\boldsymbol{\Lambda} = G(\mathbf{U})\mathbf{P}, \tag{22c}$$

$$\mathbf{P}(0), \mathbf{P}_t(0) \text{ are given}, \tag{22d}$$

where

$$
\begin{aligned}
A &= \left(\left(\boldsymbol{\psi}_i, \boldsymbol{\psi}_j\right)\right)_{M\times M},\\
B &= \left(\left(\nabla\cdot\boldsymbol{\psi}_i, \nabla\cdot\boldsymbol{\psi}_j\right)\right)_{M\times M},\\
C &= \left(\left(\boldsymbol{\psi}_i, \nabla\phi_j\right)\right)_{N\times M},\\
D &= \left(\left(\nabla\phi_i, \nabla\phi_j\right)\right)_{N\times N},\\
G(\mathbf{U}) &= \left(\left(A(\mathbf{U})\boldsymbol{\psi}_i, \boldsymbol{\psi}_j\right)\right)_{M\times M},\\
\mathbf{F} &= \left(\left(-f, \boldsymbol{\psi}_j\right)\right)_{M\times 1},\\
\mathbf{P} &= \left(p_1, p_2, \ldots, p_M\right)^T,\\
\boldsymbol{\Lambda} &= \left(\lambda_1, \lambda_2, \ldots, \lambda_M\right)^T,\\
\mathbf{U} &= \left(u_1, u_2, \ldots, u_N\right)^T.
\end{aligned}
\tag{23}
$$

Noting that A and D are positive definite. We can rewrite (22b) and (22c) as

$$
\begin{aligned}
\mathbf{U} &= D^{-1}C\mathbf{P},\\
\boldsymbol{\Lambda} &= A^{-1}G(\mathbf{U})\mathbf{P}.
\end{aligned}
\tag{24}
$$

Then the system (22a)–(22d) can be characterized as follows:

$$
A\mathbf{P}_{tt} + BA^{-1}G\left(D^{-1}C\mathbf{P}\right)\mathbf{P} = \mathbf{F}, \tag{25a}
$$

$$
\mathbf{P}(0), \mathbf{P}_t(0) \text{ are given.} \tag{25b}
$$

Recalling the assumptions on $A(u)$, we can deduce that the coefficients of $\mathbf{P}_{tt}$ and $\mathbf{P}$ are all Lipschitz continuous with respect to $\mathbf{P}(t)$. By the standard theory for the initial-value problems of nonlinear ordinary differential equations, we can deduce that there exists a unique solution $(u_h, \mathbf{p}_h, \boldsymbol{\sigma}_h) \in V_h \times \mathbf{H}_h \times \mathbf{H}_h$ to the H^1-Galerkin expanded mixed finite element scheme (20a)–(20e). □

4. Convergence Analysis

In this section we will prove the error estimates for the H^1-Galerkin expanded mixed finite element discretization scheme. We begin by reviewing some preliminary knowledge that will be used in the following theoretical analysis.

Let $\Pi_h : \mathbf{H} \rightarrow \mathbf{H}_h$ be the *Raviart-Thomas* projection defined by

$$
\left(\nabla\cdot\left(\mathbf{q} - \Pi_h\mathbf{q}\right), \nabla\cdot\mathbf{q}_h\right) = 0, \quad \forall \mathbf{q}_h \in \mathbf{H}_h. \tag{26}
$$

The following error estimates [13–15] hold for Π_h and $2 \le p \le \infty$:

$$
\begin{aligned}
\left\|\mathbf{q} - \Pi_h\mathbf{q}\right\|_{p,\Omega} &\le Ch^{k+1}\|\mathbf{q}\|_{k+1,p,\Omega},\\
\left\|\nabla\cdot\left(\mathbf{q} - \Pi_h\mathbf{q}\right)\right\|_{p,\Omega} &\le Ch^{k_1}\|\mathbf{q}\|_{k_1+1,p,\Omega}.
\end{aligned}
\tag{27}
$$

Let $R_h : V \rightarrow V_h$ denote the elliptic projection defined by

$$
\left(\nabla\left(w - R_h w\right), \nabla v_h\right) = 0, \quad \forall v_h \in V_h \tag{28}
$$

which satisfies the following error estimates (see Theorems 3.2.2 and 3.2.5, Chapter 3 of [16]):

$$
\begin{aligned}
&\left\|w - R_h w\right\| + \left\|\left(w - R_h w\right)_t\right\| + \left\|\left(w - R_h w\right)_{tt}\right\|\\
&\quad + \left\|\left(w - R_h w\right)_{ttt}\right\| + h\left\|\nabla\left(w - R_h w\right)\right\| \le Ch^{m+1},
\end{aligned}
\tag{29}
$$

$$
\max\left\{\left\|R_h w\right\|_{0,\infty}, \left\|\left(R_h w\right)_t\right\|_{0,\infty}, \left\|\left(R_h w\right)_{tt}\right\|_{0,\infty}\right\} \le C. \tag{30}
$$

To derive the main error estimates we also need the following lemma.

Lemma 5. *Suppose that $\boldsymbol{\xi}, \boldsymbol{\zeta} \in \mathbf{H}_h$, $\boldsymbol{\theta} \in \mathbf{H}$, and $\beta_{tt} \in V_h$ satisfy*

$$
\left(\boldsymbol{\xi}_{tt} + \boldsymbol{\theta}_{tt}, \mathbf{q}_h\right) + \left(\nabla\cdot\boldsymbol{\zeta}, \nabla\cdot\mathbf{q}_h\right) = 0, \quad \forall \mathbf{q}_h \in \mathbf{H}_h, \tag{31}
$$

$$
\left(\boldsymbol{\xi}_{tt} + \boldsymbol{\theta}_{tt}, \nabla v_h\right) = \left(\nabla\beta_{tt}, \nabla v_h\right), \quad \forall v_h \in V_h. \tag{32}
$$

Then there exists a constant C such that

$$
\left\|\beta_{tt}\right\| \le C\left(\|\nabla\cdot\boldsymbol{\zeta}\| + h\left\|\boldsymbol{\theta}_{tt}\right\|\right). \tag{33}
$$

Proof. Assume that $\phi \in H^2(\Omega)$ is the solution of the following equation with $\psi \in L^2(\Omega)$:

$$
\begin{aligned}
-\Delta\phi &= \psi, \quad x \in \Omega,\\
\phi &= 0, \quad x \in \partial\Omega.
\end{aligned}
\tag{34}
$$

Recalling that Ω is convex, we have

$$
\|\phi\|_{2,\Omega} \le C\|\psi\|. \tag{35}
$$

Then by (31) and (32) we deduce

$$
\begin{aligned}
\left(\beta_{tt}, \psi\right) &= -\left(\beta_{tt}, \Delta\phi\right)\\
&= \left(\nabla\beta_{tt}, \nabla\phi\right)\\
&= \left(\nabla\beta_{tt}, \nabla\phi - \nabla R_h\phi\right) + \left(\nabla\beta_{tt}, \nabla R_h\phi\right)\\
&= \left(\nabla\beta_{tt}, \nabla\phi - \nabla R_h\phi\right) - \left(\boldsymbol{\xi}_{tt} + \boldsymbol{\theta}_{tt}, \nabla\phi - \nabla R_h\phi\right)\\
&\quad + \left(\boldsymbol{\xi}_{tt} + \boldsymbol{\theta}_{tt}, \nabla\phi\right)\\
&= T_1 + T_2 + T_3.
\end{aligned}
\tag{36}
$$

Using the estimate of R_h we have

$$
\begin{aligned}
T_1 &\le Ch\left\|\nabla\beta_{tt}\right\|\|\phi\|_{2,\Omega} \le Ch\left\|\nabla\beta_{tt}\right\|\|\psi\|,\\
T_2 &\le Ch\left\|\boldsymbol{\xi}_{tt} + \boldsymbol{\theta}_{tt}\right\|\|\phi\|_{2,\Omega} \le Ch\left\|\boldsymbol{\xi}_{tt} + \boldsymbol{\theta}_{tt}\right\|\|\psi\|.
\end{aligned}
\tag{37}
$$

By (31), we obtain

$$
\begin{aligned}
T_3 &= \left(\boldsymbol{\xi}_{tt} + \boldsymbol{\theta}_{tt}, \nabla\phi\right)\\
&= \left(\boldsymbol{\xi}_{tt} + \boldsymbol{\theta}_{tt}, \nabla\phi - \boldsymbol{\Pi}_h(\nabla\phi)\right) + \left(\boldsymbol{\xi}_{tt} + \boldsymbol{\theta}_{tt}, \boldsymbol{\Pi}_h(\nabla\phi)\right)\\
&= \left(\boldsymbol{\xi}_{tt} + \boldsymbol{\theta}_{tt}, \nabla\phi - \boldsymbol{\Pi}_h(\nabla\phi)\right) - \left(\nabla\cdot\boldsymbol{\zeta}, \nabla\cdot\boldsymbol{\Pi}_h(\nabla\phi)\right)\\
&= \left(\boldsymbol{\xi}_{tt} + \boldsymbol{\theta}_{tt}, \nabla\phi - \boldsymbol{\Pi}_h(\nabla\phi)\right) - \left(\nabla\cdot\boldsymbol{\zeta}, \nabla\cdot(\nabla\phi)\right)\\
&\le C\left(h\left\|\boldsymbol{\xi}_{tt} + \boldsymbol{\theta}_{tt}\right\| + \|\nabla\cdot\boldsymbol{\zeta}\|\right)\|\psi\|.
\end{aligned}
\tag{38}
$$

Inserting the estimates of T_1, T_2, and T_3 into (36) leads to

$$\left|(\beta_{tt}, \psi)\right| \le C\left\{h\left(\|\boldsymbol{\xi}_{tt} + \boldsymbol{\theta}_{tt}\| + \|\nabla\beta_{tt}\|\right) + \|\nabla\cdot\boldsymbol{\zeta}\|\right\}\|\psi\|. \quad (39)$$

By (32) we derive

$$\|\nabla\beta_{tt}\| \le \|\boldsymbol{\xi}_{tt} + \boldsymbol{\theta}_{tt}\|. \quad (40)$$

Further we have

$$\left|(\beta_{tt}, \psi)\right| \le C\left\{h\|\boldsymbol{\xi}_{tt} + \boldsymbol{\theta}_{tt}\| + \|\nabla\cdot\boldsymbol{\zeta}\|\right\}\|\psi\|, \quad (41)$$

which implies

$$\|\beta_{tt}\| \le C\left(h\|\boldsymbol{\xi}_{tt}\| + h\|\boldsymbol{\theta}_{tt}\| + \|\nabla\cdot\boldsymbol{\zeta}\|\right). \quad (42)$$

Further, taking $q_h = \boldsymbol{\xi}_{tt}$ in (31) and by *Hölder* inequalities as well as inverse property of the finite element spaces V_h and $\mathbf{H}_h$ yield

$$\begin{aligned}\|\boldsymbol{\xi}_{tt}\|^2 &\le \|\nabla\cdot\boldsymbol{\zeta}\|\,\|\nabla\cdot\boldsymbol{\xi}_{tt}\| + \|\boldsymbol{\theta}_{tt}\|\,\|\boldsymbol{\xi}_{tt}\| \\ &\le C\left\{h^{-1}\|\nabla\cdot\boldsymbol{\zeta}\|\,\|\boldsymbol{\xi}_{tt}\| + \|\boldsymbol{\theta}_{tt}\|\,\|\boldsymbol{\xi}_{tt}\|\right\}.\end{aligned} \quad (43)$$

Therefore we obtain

$$\|\boldsymbol{\xi}_{tt}\| \le C\left\{h^{-1}\|\nabla\cdot\boldsymbol{\zeta}\| + \|\boldsymbol{\theta}_{tt}\|\right\}, \quad (44)$$

which, together with (42), yields the desired result. □

Theorem 6. *Let* $(u, \mathbf{p}, \boldsymbol{\sigma})$ *and* $(u_h, \mathbf{p}_h, \boldsymbol{\sigma}_h)$ *be the solutions of* (5a)–(5e) *and* (20a)–(20e)*, respectively. Assume that* $(u, \mathbf{p}, \boldsymbol{\sigma})$ *satisfies the following regularities:*

$$\begin{aligned}u &\in H^2\left(J; H^{m+1}(\Omega)\right) \cap L^\infty\left(J; H^{m+1}(\Omega)\right), \\ \mathbf{p} &\in H^3\left(J; H^{k+1}(\Omega)\right) \cap L^\infty\left(J; H^{k+1}(\Omega)\right), \\ \boldsymbol{\sigma} &\in H^2\left(J; H^{k+1}(\Omega)\right) \cap L^\infty\left(J; H^{k+1}(\Omega)\right)\end{aligned} \quad (45)$$

and $u_h(\mathbf{x}, 0) = R_h u_0(\mathbf{x})$, $u_{ht}(\mathbf{x}, 0) = R_h u_1(\mathbf{x})$, *and* $\mathbf{p}_h(\mathbf{x}, 0) = \Pi_h \mathbf{p}(\mathbf{x}, 0)$. *Then there exists a positive constant C independent of h such that*

$$\begin{aligned}&\left\|(u - u_h)(t)\right\|_1 \le Ch^{\min(k+1, m)}, \\ &\left\|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t)\right\| \le Ch^{\min(k_1, m+1)}, \\ &\left\|(u - u_h)(t)\right\| + \left\|(\mathbf{p} - \mathbf{p}_h)(t)\right\| + \left\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(t)\right\| \\ &\quad \le Ch^{\min(k+1, m+1)}\end{aligned} \quad (46)$$

with $k \ge 0$, $m \ge 1$, *for* $d = 1$, *and* $k \ge 1$, $m \ge 1$, *for* $d = 2, 3$.

Proof. In order to derive the error estimates, we decompose the errors as follows:

$$\begin{aligned}\Pi_h\mathbf{p} - \mathbf{p}_h &= \boldsymbol{\xi}, \qquad \mathbf{p} - \Pi_h\mathbf{p} = \boldsymbol{\theta}, \\ \Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h &= \boldsymbol{\zeta}, \qquad \boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma} = \boldsymbol{\eta}, \\ R_h u - u_h &= \beta, \qquad u - R_h u = \gamma.\end{aligned} \quad (47)$$

Subtracting the numerical scheme (20a)–(20e) from the weak formulation (5a)–(5e), we can derive the following error equations:

$$\begin{aligned}&\left(\boldsymbol{\xi}_{tt}, \mathbf{q}_h\right) + \left(\nabla\cdot\boldsymbol{\zeta}, \nabla\cdot\mathbf{q}_h\right) = -\left(\boldsymbol{\theta}_{tt}, \mathbf{q}_h\right), \quad \forall \mathbf{q}_h \in \mathbf{H}_h, \\ &\left(\boldsymbol{\xi}, \nabla v_h\right) - \left(\nabla\beta, \nabla v_h\right) = -\left(\boldsymbol{\theta}, \nabla v_h\right), \quad \forall v_h \in V_h, \\ &\left(\boldsymbol{\zeta}, \mathbf{w}_h\right) - \left(A(u_h)\boldsymbol{\xi}, \mathbf{w}_h\right) = \left(\left(A(u) - A(u_h)\right)\mathbf{p} - \boldsymbol{\eta}, \mathbf{w}_h\right) \\ &\qquad + \left(A(u_h)\boldsymbol{\theta}, \mathbf{w}_h\right), \quad \forall \mathbf{w}_h \in \mathbf{H}_h.\end{aligned} \quad (48)$$

Choosing $v_h = \beta$ in the second equation of (48) leads to

$$\|\nabla\beta\| \le \|\boldsymbol{\theta}\| + \|\boldsymbol{\xi}\|. \quad (49)$$

By setting $w_h = \boldsymbol{\zeta}$ in the third equation of (48) and using the assumption on $A(u)$ we deduce

$$\|\boldsymbol{\zeta}\| \le C\left(\|\boldsymbol{\theta}\| + \|\boldsymbol{\xi}\| + \|\boldsymbol{\eta}\| + \|\gamma\|\right). \quad (50)$$

In the following we will estimate $\|\boldsymbol{\xi}_t\|$. Differentiating the third equation in (48) gives

$$\begin{aligned}&\left(\boldsymbol{\zeta}_t, \mathbf{w}_h\right) - \left(A(u_h)\boldsymbol{\xi}_t, \mathbf{w}_h\right) \\ &\quad = \left(A_u(u_h)u_{ht}\boldsymbol{\xi}, w_h\right) - \left(\boldsymbol{\eta}_t, \mathbf{w}_h\right) + \left(A(u_h)\boldsymbol{\theta}_t, \mathbf{w}_h\right) \\ &\qquad + \left(A_u(u_h)u_{ht}\boldsymbol{\theta}, \mathbf{w}_h\right) + \left(\left(A(u) - A(u_h)\right)\mathbf{p}_t, \mathbf{w}_h\right) \\ &\qquad - \left(\left(A_u(u)u_t - A_u(u_h)u_{ht}\right)\mathbf{p}, \mathbf{w}_h\right).\end{aligned} \quad (51)$$

Taking $\mathbf{q}_h = \boldsymbol{\zeta}_t$ in the first equation of (48) and $\mathbf{w}_h = \boldsymbol{\xi}_{tt}$ in (51) and then subtracting the resulting equations lead to

$$\begin{aligned}&\left(\nabla\cdot\boldsymbol{\zeta}, \nabla\cdot\boldsymbol{\zeta}_t\right) + \left(A(u_h)\boldsymbol{\xi}_t, \boldsymbol{\xi}_{tt}\right) \\ &\quad = -\left(\boldsymbol{\theta}_{tt}, \boldsymbol{\zeta}_t\right) + \left(\boldsymbol{\eta}_t, \boldsymbol{\xi}_{tt}\right) - \left(A(u_h)\boldsymbol{\theta}_t, \boldsymbol{\xi}_{tt}\right) \\ &\qquad - \left(A_u(u_h)u_{ht}\boldsymbol{\theta}, \boldsymbol{\xi}_{tt}\right) - \left(A_u(u_h)u_{ht}\boldsymbol{\xi}, \boldsymbol{\xi}_{tt}\right) \\ &\qquad - \left(\left(A(u) - A(u_h)\right)\mathbf{p}_t, \boldsymbol{\xi}_{tt}\right) \\ &\qquad - \left(\left(A_u(u)u_t - A_u(u_h)u_{ht}\right)\mathbf{p}, \boldsymbol{\xi}_{tt}\right) \\ &\quad = \sum_{i=1}^{7} B_i.\end{aligned} \quad (52)$$

The left terms can be dealt with as follows:

$$\begin{aligned}\left(\nabla\cdot\boldsymbol{\zeta}, \nabla\cdot\boldsymbol{\zeta}_t\right) &= \frac{1}{2}\frac{d}{dt}\left(\nabla\cdot\boldsymbol{\zeta}, \nabla\cdot\boldsymbol{\zeta}\right), \\ \left(A(u_h)\boldsymbol{\xi}_t, \boldsymbol{\xi}_{tt}\right) &= \frac{1}{2}\frac{d}{dt}\left(A(u_h)\boldsymbol{\xi}_t, \boldsymbol{\xi}_t\right) - \frac{1}{2}\left(A_u(u_h)u_{ht}\boldsymbol{\xi}_t, \boldsymbol{\xi}_t\right).\end{aligned} \quad (53)$$

The terms on the right side can be rewritten as follows by integral formula by parts:

$$\begin{aligned}
B_1 &= -\frac{d}{dt}(\boldsymbol{\theta}_{tt}, \boldsymbol{\zeta}) + (\boldsymbol{\theta}_{ttt}, \boldsymbol{\zeta}), \\
B_2 &= \frac{d}{dt}(\boldsymbol{\eta}_t, \boldsymbol{\xi}_t) - (\boldsymbol{\eta}_{tt}, \boldsymbol{\xi}_t), \\
B_3 &= -\frac{d}{dt}(A(u_h)\boldsymbol{\theta}_t, \boldsymbol{\xi}_t) + (A_u(u_h)u_{ht}\boldsymbol{\theta}_t, \boldsymbol{\xi}_t) \\
&\quad + (A(u_h)\boldsymbol{\theta}_{tt}, \boldsymbol{\xi}_t), \\
B_4 &= -\frac{d}{dt}(A_u(u_h)u_{ht}\boldsymbol{\theta}, \boldsymbol{\xi}_t) + (A_u(u_h)u_{ht}\boldsymbol{\theta}_t, \boldsymbol{\xi}_t) \\
&\quad + \left(A_{uu}(u_h)u_{ht}^2\boldsymbol{\theta}, \boldsymbol{\xi}_t\right) + (A_u(u_h)u_{htt}\boldsymbol{\theta}, \boldsymbol{\xi}_t), \\
B_5 &= -\frac{d}{dt}(A_u(u_h)u_{ht}\boldsymbol{\xi}, \boldsymbol{\xi}_t) + (A_u(u_h)u_{htt}\boldsymbol{\xi}, \boldsymbol{\xi}_t) \\
&\quad + \left(A_{uu}(u_h)u_{ht}^2\boldsymbol{\xi}, \boldsymbol{\xi}_t\right) + (A_u(u_h)u_{ht}\boldsymbol{\xi}_t, \boldsymbol{\xi}_t), \\
B_6 &= -\frac{d}{dt}((A(u) - A(u_h))\mathbf{p}_t, \boldsymbol{\xi}_t) \\
&\quad + ((A_u(u)u_t - A_u(u_h)u_{ht})\mathbf{p}_t, \boldsymbol{\xi}_t) \\
&\quad + ((A(u) - A(u_h))\mathbf{p}_{tt}, \boldsymbol{\xi}_t), \\
B_7 &= -\frac{d}{dt}((A_u(u)u_t - A_u(u_h)u_{ht})\mathbf{p}, \boldsymbol{\xi}_t) \\
&\quad + ((A_u(u)u_t - A_u(u_h)u_{ht})\mathbf{p}_t, \boldsymbol{\xi}_t) \\
&\quad + \Big(\big(A_{uu}(u)u_t^2 + A_u(u)u_{tt} - A_{uu}(u_h)u_{ht}^2 \\
&\qquad - A_u(u_h)u_{htt}\big)\mathbf{p}, \boldsymbol{\xi}_t\Big).
\end{aligned} \tag{54}$$

Combining all the terms mentioned above we arrive at

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\nabla\cdot\boldsymbol{\zeta}, \nabla\cdot\boldsymbol{\zeta}) + \frac{1}{2}\frac{d}{dt}(A(u_h)\boldsymbol{\xi}_t, \boldsymbol{\xi}_t) &= \frac{1}{2}(A_u(u_h)u_{ht}\boldsymbol{\xi}_t, \boldsymbol{\xi}_t) + \sum_{i=1}^{7} B_i \\
&= B_0 + \sum_{i=1}^{7} B_i.
\end{aligned} \tag{55}$$

Now we are in the position to estimate the terms B_i, $i = 0, 1, 2, \ldots, 7$. By Lemma 5 we can deduce

$$\begin{aligned}
\|\beta_t\|^2 &\le C\int_0^t \|\beta_{tt}\|^2 d\tau \\
&\le C\int_0^t \left(\|\nabla\cdot\boldsymbol{\zeta}\|^2 + h^2\|\boldsymbol{\theta}_{tt}\|^2\right) d\tau,
\end{aligned} \tag{56}$$

where $\beta_t(0) = 0$ was used. Notice that

$$B_0 = \frac{1}{2}(A_u(u_h)(R_h u)_t\boldsymbol{\xi}_t, \boldsymbol{\xi}_t) - \frac{1}{2}(A_u(u_h)\beta_t\boldsymbol{\xi}_t, \boldsymbol{\xi}_t). \tag{57}$$

Then using the assumption A_1, (30), and Cauchy-Schwartz inequality gives

$$\begin{aligned}
&\left|\int_0^t B_0 d\tau\right| \\
&\quad \le C\int_0^t \|\boldsymbol{\xi}_t\|^2 d\tau + C\|\boldsymbol{\xi}_t\|_{L^\infty(0,t;L^\infty(\Omega))}\int_0^t \|\boldsymbol{\xi}_t\|\,\|\beta_t\| \\
&\quad \le C\|\boldsymbol{\xi}_t\|_{L^\infty(0,t;L^\infty(\Omega))}\int_0^t \left(\|\nabla\cdot\boldsymbol{\zeta}\|^2 + h^2\|\boldsymbol{\theta}_{tt}\|^2\right) d\tau \\
&\qquad + C\left(1 + \|\boldsymbol{\xi}_t\|_{L^\infty(0,t;L^\infty(\Omega))}\right)\int_0^t \|\boldsymbol{\xi}_t\|^2 d\tau.
\end{aligned} \tag{58}$$

Integrating from 0 to t and using (50) as well as Cauchy-Schwartz inequality yield

$$\begin{aligned}
\left|\int_0^t B_1 d\tau\right| &\le \|\boldsymbol{\theta}_{tt}\|\,\|\boldsymbol{\zeta}\| + \int_0^t \|\boldsymbol{\theta}_{ttt}\|\,\|\boldsymbol{\zeta}\|\, d\tau \\
&\le \frac{1}{2}\left(\|\boldsymbol{\theta}_{tt}\|^2 + \|\boldsymbol{\zeta}\|^2\right) + \frac{1}{2}\int_0^t \|\boldsymbol{\theta}_{ttt}\|^2 d\tau \\
&\quad + \frac{1}{2}\int_0^t \|\boldsymbol{\zeta}\|^2 d\tau \\
&\le \frac{1}{2}\left(\|\boldsymbol{\theta}_{tt}\|^2 + \|\boldsymbol{\theta}\|^2 + \|\boldsymbol{\xi}\|^2 + \|\boldsymbol{\eta}\|^2 + \|\gamma\|^2\right) \\
&\quad + \frac{1}{2}\int_0^t \|\boldsymbol{\theta}_{ttt}\|^2 d\tau \\
&\quad + \frac{1}{2}\int_0^t \left(\|\boldsymbol{\theta}\|^2 + \|\boldsymbol{\xi}\|^2 + \|\boldsymbol{\eta}\|^2 + \|\gamma\|^2\right) d\tau.
\end{aligned} \tag{59}$$

Note that

$$\|\boldsymbol{\xi}\|^2 \le C\int_0^t \|\boldsymbol{\xi}_t\|^2 d\tau \tag{60}$$

for $\boldsymbol{\xi}(0) = 0$. Then we have

$$\begin{aligned}
&\left|\int_0^t B_1 d\tau\right| \\
&\quad \le \frac{1}{2}\left(\|\boldsymbol{\theta}_{tt}\|^2 + \|\boldsymbol{\theta}\|^2 + \|\boldsymbol{\eta}\|^2 + \|\gamma\|^2\right) + \frac{1}{2}\int_0^t \|\boldsymbol{\theta}_{ttt}\|^2 d\tau \\
&\qquad + \frac{1}{2}\int_0^t \left(\|\boldsymbol{\theta}\|^2 + \|\boldsymbol{\eta}\|^2 + \|\gamma\|^2\right) d\tau + C\int_0^t \|\boldsymbol{\xi}_t\|^2 d\tau.
\end{aligned} \tag{61}$$

Similarly, we can estimate the other terms. By ϵ inequality we deduce

$$\begin{aligned}
\left|\int_0^t B_2 d\tau\right| &\le \|\boldsymbol{\eta}_t\|\,\|\boldsymbol{\xi}_t\| + \int_0^t \|\boldsymbol{\eta}_{tt}\|\,\|\boldsymbol{\xi}_t\|\, d\tau \\
&\le C\|\boldsymbol{\eta}_t\|^2 + \epsilon\|\boldsymbol{\xi}_t\|^2 + \frac{1}{2}\int_0^t \|\boldsymbol{\eta}_{tt}\|^2 d\tau \\
&\quad + \frac{1}{2}\int_0^t \|\boldsymbol{\xi}_t\|^2 d\tau.
\end{aligned} \tag{62}$$

For B_3 we can rewrite it as

$$\begin{aligned}B_3 &= -\frac{d}{dt}\left(A\left(u_h\right)\boldsymbol{\theta}_t, \boldsymbol{\xi}_t\right) + \left(A_u\left(u_h\right)\left(u_{ht} - \left(R_h u\right)_t\right)\boldsymbol{\theta}_t, \boldsymbol{\xi}_t\right) \\ &\quad + \left(A_u\left(u_h\right)\left(R_h u\right)_t \boldsymbol{\theta}_t, \boldsymbol{\xi}_t\right) + \left(A\left(u_h\right)\boldsymbol{\theta}_{tt}, \boldsymbol{\xi}_t\right) \\ &= -\frac{d}{dt}\left(A\left(u_h\right)\boldsymbol{\theta}_t, \boldsymbol{\xi}_t\right) - \left(A_u\left(u_h\right)\beta_t \boldsymbol{\theta}_t, \boldsymbol{\xi}_t\right) \\ &\quad + \left(A_u\left(u_h\right)\left(R_h u\right)_t \boldsymbol{\theta}_t, \boldsymbol{\xi}_t\right) + \left(A\left(u_h\right)\boldsymbol{\theta}_{tt}, \boldsymbol{\xi}_t\right).\end{aligned} \tag{63}$$

Then by Cauchy-Schwartz inequality and ϵ inequality we derive

$$\begin{aligned}\left|\int_0^t B_3 d\tau\right| &\le C\left\|\boldsymbol{\theta}_t\right\|\left\|\boldsymbol{\xi}_t\right\| + C\left\|\theta_t\right\|_{0,\infty}\int_0^t \left\|\beta_t\right\|\left\|\boldsymbol{\xi}_t\right\| d\tau \\ &\quad + C\int_0^t \left\|\boldsymbol{\theta}_t\right\|\left\|\boldsymbol{\xi}_t\right\| d\tau + C\int_0^t \left\|\boldsymbol{\theta}_{tt}\right\|\left\|\boldsymbol{\xi}_t\right\| d\tau \\ &\le C\left\|\boldsymbol{\theta}_t\right\|^2 + \epsilon\left\|\boldsymbol{\xi}_t\right\|^2 + C\int_0^t \left\|\boldsymbol{\theta}_t\right\|^2 d\tau + C\int_0^t \left\|\boldsymbol{\theta}_{tt}\right\|^2 d\tau \\ &\quad + C\int_0^t \left\|\beta_t\right\|^2 d\tau + C\int_0^t \left\|\boldsymbol{\xi}_t\right\|^2 d\tau \\ &\le C\left\|\boldsymbol{\theta}_t\right\|^2 + \epsilon\left\|\boldsymbol{\xi}_t\right\|^2 + C\int_0^t \left\|\boldsymbol{\theta}_t\right\|^2 d\tau + C\int_0^t \left\|\boldsymbol{\theta}_{tt}\right\|^2 d\tau \\ &\quad + C\int_0^t \left(\left\|\nabla\cdot\boldsymbol{\zeta}\right\|^2 + h^2\left\|\boldsymbol{\theta}_{tt}\right\|^2\right) d\tau + C\int_0^t \left\|\boldsymbol{\xi}_t\right\|^2 d\tau.\end{aligned} \tag{64}$$

Here the boundedness of $\|\boldsymbol{\theta}_t\|_{0,\infty}$ and the Ritz projection $\|(R_h u)_t\|_{0,\infty}$ were used. For B_4 we have

$$\begin{aligned}B_4 &= -\frac{d}{dt}\left(A_u\left(u_h\right)\left(\left(R_h u\right)_t - \beta_t\right)\boldsymbol{\theta}, \boldsymbol{\xi}_t\right) \\ &\quad + \left(A_u\left(u_h\right)\left(\left(R_h u\right)_t - \beta_t\right)\boldsymbol{\theta}_t, \boldsymbol{\xi}_t\right) \\ &\quad + \left(A_{uu}\left(u_h\right)\left(\left(R_h u\right)_t - \beta_t\right)^2 \boldsymbol{\theta}, \boldsymbol{\xi}_t\right) \\ &\quad + \left(A_u\left(u_h\right)\left(\left(R_h u\right)_{tt} - \beta_{tt}\right)\boldsymbol{\theta}, \boldsymbol{\xi}_t\right).\end{aligned} \tag{65}$$

Therefore by Cauchy-Schwartz inequality and ϵ inequality we obtain

$$\begin{aligned}\left|\int_0^t B_4 d\tau\right| &\le C\left\|\boldsymbol{\theta}\right\|\left\|\boldsymbol{\xi}_t\right\| + C\left\|\boldsymbol{\theta}\right\|_{0,\infty}\left\|\beta_t\right\|\left\|\boldsymbol{\xi}_t\right\| \\ &\quad + C\int_0^t \left(\left\|\boldsymbol{\theta}_t\right\| + \left\|\boldsymbol{\theta}\right\|\right)\left\|\boldsymbol{\xi}_t\right\| + C\left\|\boldsymbol{\theta}_t\right\|_{L^\infty(0,t;L^\infty(\Omega))} \\ &\quad \times \int_0^t \left\|\beta_t\right\|\left\|\boldsymbol{\xi}_t\right\| d\tau + C\left\|\boldsymbol{\theta}\right\|_{L^\infty(0,t;L^\infty(\Omega))} \\ &\quad \times \int_0^t \left(\left\|\beta_t\right\| + \left\|\beta_{tt}\right\|\right)\left\|\boldsymbol{\xi}_t\right\| d\tau \\ &\quad + C\left\|\boldsymbol{\theta}\right\|_{L^\infty(0,t;L^\infty(\Omega))}\left\|\boldsymbol{\xi}_t\right\|_{L^\infty(0,t;L^\infty(\Omega))}\int_0^t \left\|\beta_t\right\|^2 d\tau \\ &\le C\left(\left\|\boldsymbol{\theta}\right\| + \left\|\beta_t\right\|\right)\left\|\boldsymbol{\xi}_t\right\| + C\int_0^t \left(\left\|\beta_t\right\| + \left\|\boldsymbol{\theta}_t\right\|\right)\left\|\boldsymbol{\xi}_t\right\| d\tau \\ &\quad + C\int_0^t \left(\left\|\boldsymbol{\theta}\right\| + \left\|\beta_{tt}\right\|\right)\left\|\boldsymbol{\xi}_t\right\| d\tau + C\left\|\xi_t\right\|_{L^\infty(0,t;L^\infty(\Omega))} \\ &\quad \times \int_0^t \left\|\beta_t\right\|^2 d\tau \\ &\le C\left\|\boldsymbol{\theta}\right\|^2 + C\left(1 + \left\|\xi_t\right\|_{L^\infty(0,t;L^\infty(\Omega))}\right) \\ &\quad \times \int_0^t \left(\left\|\nabla\cdot\boldsymbol{\zeta}\right\|^2 + h^2\left\|\boldsymbol{\theta}_{tt}\right\|^2\right) d\tau \\ &\quad + C\int_0^t \left(\left\|\boldsymbol{\theta}\right\|^2 + \left\|\boldsymbol{\theta}_t\right\|^2\right) d\tau \\ &\quad + C\int_0^t \left\|\boldsymbol{\xi}_t\right\|^2 d\tau + \epsilon\left\|\boldsymbol{\xi}_t\right\|^2.\end{aligned} \tag{66}$$

Here we used the boundedness of $\|\boldsymbol{\theta}\|_{L^\infty(0,t;L^\infty(\Omega))}$ to obtain the above estimate. Similarly, we can deduce

$$\begin{aligned}\left|\int_0^t B_5 d\tau\right| &\le C\left\|\boldsymbol{\xi}\right\|\left\|\boldsymbol{\xi}_t\right\| + C\left\|\boldsymbol{\xi}_t\right\|_{0,\infty}\left\|\beta_t\right\|\left\|\boldsymbol{\xi}\right\| \\ &\quad + C\int_0^t \left(\left\|\boldsymbol{\xi}_t\right\| + \left\|\boldsymbol{\xi}\right\|\right)\left\|\boldsymbol{\xi}_t\right\| + C\left\|\boldsymbol{\xi}_t\right\|_{L^\infty(0,t;L^\infty(\Omega))} \\ &\quad \times \int_0^t \left\|\beta_t\right\|\left\|\boldsymbol{\xi}_t\right\| d\tau + C\left\|\boldsymbol{\xi}\right\|_{L^\infty(0,t;L^\infty(\Omega))} \\ &\quad \times \int_0^t \left(\left\|\beta_t\right\| + \left\|\beta_{tt}\right\|\right)\left\|\boldsymbol{\xi}\right\| d\tau \\ &\quad + C\left\|\boldsymbol{\xi}\right\|_{L^\infty(0,t;L^\infty(\Omega))}\left\|\boldsymbol{\xi}_t\right\|_{L^\infty(0,t;L^\infty(\Omega))}\int_0^t \left\|\beta_t\right\|^2 d\tau \\ &\le \epsilon\left\|\boldsymbol{\xi}_t\right\|^2 + C\left\|\boldsymbol{\xi}_t\right\|_{L^\infty(0,t;L^\infty(\Omega))} \\ &\quad \times \int_0^t \left(\left\|\nabla\cdot\boldsymbol{\zeta}\right\|^2 + h^2\left\|\boldsymbol{\theta}_{tt}\right\|^2 + \left\|\boldsymbol{\xi}_t\right\|^2\right) d\tau \\ &\quad + C\left\|\boldsymbol{\xi}_t\right\|_{L^\infty(0,t;L^\infty(\Omega))}\left\|\boldsymbol{\xi}\right\|_{L^\infty(0,t;L^\infty(\Omega))} \\ &\quad \times \int_0^t \left(\left\|\nabla\cdot\boldsymbol{\zeta}\right\|^2 + h^2\left\|\boldsymbol{\theta}_{tt}\right\|^2\right) d\tau + C\int_0^t \left\|\boldsymbol{\xi}_t\right\|^2 d\tau.\end{aligned} \tag{67}$$

Further for B_6 and B_7 by Cauchy-Schwartz inequality and ϵ inequality we have

$$\begin{aligned}\left|\int_0^t B_6 d\tau\right| &\le C\left(\left\|\beta\right\| + \left\|\gamma\right\|\right)\left\|\boldsymbol{\xi}_t\right\| \\ &\quad + C\int_0^t \left(\left\|\beta_t\right\| + \left\|\gamma_t\right\| + \left\|\beta\right\| + \left\|\gamma\right\|\right)\left\|\boldsymbol{\xi}_t\right\| d\tau \\ &\quad + \int_0^t \left(\left\|\beta\right\| + \left\|\gamma\right\|\right)\left\|\boldsymbol{\xi}_t\right\| d\tau\end{aligned}$$

$$
\begin{aligned}
&\leq C\left(\|\beta\|^2+\|\gamma\|^2\right)+\epsilon\|\boldsymbol{\xi}_t\|^2\\
&\quad+C\int_0^t\left(\|\beta_t\|^2+\|\gamma_t\|^2\right)d\tau\\
&\quad+C\int_0^t\left(\|\beta\|^2+\|\gamma\|^2\right)d\tau+C\int_0^t\|\boldsymbol{\xi}_t\|^2d\tau\\
&\leq C\left(\|\boldsymbol{\theta}\|^2+\|\gamma\|^2\right)+\epsilon\|\boldsymbol{\xi}_t\|^2\\
&\quad+C\int_0^t\left(\|\nabla\cdot\boldsymbol{\zeta}\|^2+h^2\|\boldsymbol{\theta}_{tt}\|^2+\|\gamma_t\|^2\right)d\tau\\
&\quad+C\int_0^t\left(\|\boldsymbol{\theta}\|^2+\|\gamma\|^2\right)d\tau+C\int_0^t\|\boldsymbol{\xi}_t\|^2d\tau,\\
\left|\int_0^t B_7 d\tau\right|&\leq C\left(\|\beta_t\|^2+\|\gamma_t\|^2+\|\beta\|^2+\|\gamma\|^2\right)+\epsilon\|\boldsymbol{\xi}_t\|^2\\
&\quad+C\int_0^t\left(\|\beta_t\|^2+\|\gamma_t\|^2+\|\beta\|^2+\|\gamma\|^2\right)d\tau\\
&\quad+C\int_0^t\|\boldsymbol{\xi}_t\|^2d\tau+C\int_0^t\left(\|\beta_{tt}\|^2+\|\gamma_{tt}\|^2\right)d\tau\\
&\leq C\left(\|\gamma_t\|^2+\|\boldsymbol{\theta}\|^2+\|\gamma\|^2\right)+C\int_0^t\|\boldsymbol{\xi}_t\|^2d\tau+\epsilon\|\boldsymbol{\xi}_t\|^2\\
&\quad+C\int_0^t\left(\|\nabla\cdot\boldsymbol{\zeta}\|^2+h^2\|\boldsymbol{\theta}_{tt}\|^2\right)d\tau+C\int_0^t\|\gamma_{tt}\|^2d\tau.
\end{aligned}
\tag{68}
$$

Combining the above estimates leads to

$$
\begin{aligned}
&\|\nabla\cdot\boldsymbol{\zeta}\|^2+\alpha\|\boldsymbol{\xi}_t\|^2\\
&\quad\leq C\left(\|\boldsymbol{\theta}_{tt}\|^2+\|\boldsymbol{\theta}_t\|^2+\|\boldsymbol{\eta}_t\|^2+\|\boldsymbol{\theta}\|^2\right.\\
&\qquad\left.+\|\boldsymbol{\eta}\|^2+\|\gamma\|^2+\|\gamma_t\|^2\right)\\
&\quad+C\int_0^t\left(\|\boldsymbol{\theta}_{ttt}\|^2+\|\boldsymbol{\theta}_t\|^2+\|\boldsymbol{\theta}\|^2+\|\boldsymbol{\eta}\|^2+\|\boldsymbol{\eta}_{tt}\|^2\right.\\
&\qquad\left.+\|\boldsymbol{\theta}_{tt}\|^2+\|\gamma\|^2+\|\gamma_t\|^2+\|\gamma_{tt}\|^2\right)d\tau\\
&\quad+C\left(1+\|\boldsymbol{\xi}_t\|_{L^\infty(0,t;L^\infty(\Omega))}\right)\\
&\quad\times\int_0^t\left(\|\nabla\cdot\boldsymbol{\zeta}\|^2+h^2\|\boldsymbol{\theta}_{tt}\|^2+\|\boldsymbol{\xi}_t\|^2\right)d\tau\\
&\quad+C\|\boldsymbol{\xi}_t\|_{L^\infty(0,t;L^\infty(\Omega))}\|\boldsymbol{\xi}\|_{L^\infty(0,t;L^\infty(\Omega))}\\
&\quad\times\int_0^t\left(\|\nabla\cdot\boldsymbol{\zeta}\|^2+h^2\|\boldsymbol{\theta}_{tt}\|^2\right)d\tau.
\end{aligned}
\tag{69}
$$

To prove the main result we need to make the following induction hypothesis: there exists a constant $0<h_0<1$ such that the following estimate holds for $0<h\leq h_0$:

$$
\max\left\{\|\boldsymbol{\xi}\|_{L^\infty(0,t;L^\infty(\Omega))},\|\boldsymbol{\xi}_t\|_{L^\infty(0,t;L^\infty(\Omega))}\right\}<1,\quad 0\leq t\leq T.
\tag{70}
$$

Then by setting ϵ small enough and using Gronwall's inequality we obtain the following estimate which holds for constant $K>0$:

$$
\begin{aligned}
&\|\nabla\cdot\boldsymbol{\zeta}\|^2+\alpha\|\boldsymbol{\xi}_t\|^2\\
&\quad\leq K\left(\|\boldsymbol{\theta}_{tt}\|^2+\|\boldsymbol{\theta}_t\|^2+\|\boldsymbol{\eta}_t\|^2+\|\boldsymbol{\theta}\|^2\right.\\
&\qquad\left.+\|\boldsymbol{\eta}\|^2+\|\gamma\|^2+\|\gamma_t\|^2\right)\\
&\quad+K\int_0^t\left(\|\boldsymbol{\theta}_{ttt}\|^2+\|\boldsymbol{\theta}_t\|^2+\|\boldsymbol{\theta}\|^2+\|\boldsymbol{\eta}\|^2\right.\\
&\qquad\left.+\|\boldsymbol{\eta}_{tt}\|^2+\|\boldsymbol{\theta}_{tt}\|^2+\|\gamma\|^2\right)d\tau\\
&\quad+K\int_0^t\left(h^2\|\boldsymbol{\theta}_{tt}\|^2+\|\gamma_t\|^2+\|\gamma_{tt}\|^2\right)d\tau.
\end{aligned}
\tag{71}
$$

Further, using (27) and (29) gives

$$
\|\nabla\cdot\boldsymbol{\zeta}\|^2+\|\boldsymbol{\xi}_t\|^2\leq\widetilde{K}h^{2\min\{k+1,m+1\}},
\tag{72}
$$

where constant $\widetilde{K}>0$ is independent of h. We are now in position to prove the inductive hypothesis (70) which holds on $t\in J$. Suppose that there exists a constant $0<h_*\leq h_0$ such that

$$
\max\left\{\|\boldsymbol{\xi}^*\|_{L^\infty(0,t;L^\infty(\Omega))},\|\boldsymbol{\xi}_t^*\|_{L^\infty(0,t;L^\infty(\Omega))}\right\}\geq 1,\quad 0\leq t\leq T.
\tag{73}
$$

Let

$$
\begin{aligned}
t_*=\inf\Big\{t\in J\mid\max\Big\{&\|\boldsymbol{\xi}^*\|_{L^\infty(0,t;L^\infty(\Omega))},\\
&\|\boldsymbol{\xi}_t^*\|_{L^\infty(0,t;L^\infty(\Omega))}\Big\}\geq 1\Big\}.
\end{aligned}
\tag{74}
$$

Then we know that

$$
\max\left\{\|\boldsymbol{\xi}^*\|_{L^\infty(0,t_*;L^\infty(\Omega))},\|\boldsymbol{\xi}_t^*\|_{L^\infty(0,t_*;L^\infty(\Omega))}\right\}=1,
\tag{75}
$$

$$
\max\left\{\|\boldsymbol{\xi}^*\|_{L^\infty(0,t;L^\infty(\Omega))},\|\boldsymbol{\xi}_t^*\|_{L^\infty(0,t;L^\infty(\Omega))}\right\}\leq 1,\quad 0<t\leq t_*.
\tag{76}
$$

By the same arguments for (72) we can prove

$$
\|\nabla\cdot\boldsymbol{\zeta}^*\|^2+\|\boldsymbol{\xi}_t^*\|^2\leq\widetilde{K}_1h^{2\min\{k+1,m+1\}},\quad 0<t\leq t_*.
\tag{77}
$$

Moreover, we can also deduce

$$
\begin{aligned}
\|\boldsymbol{\xi}^*\|^2&\leq C\int_0^t\|\boldsymbol{\xi}_t^*\|^2d\tau\\
&\leq\widetilde{K}_2h^{2\min\{k+1,m+1\}},\quad 0<t\leq t_*.
\end{aligned}
\tag{78}
$$

By inverse inequality of finite element spaces we can conclude

$$
\begin{aligned}
&\|\boldsymbol{\xi}_t^*\|_{L^\infty(0,t_*;L^\infty(\Omega))}\\
&\quad\leq Ch^{-d/2}\|\boldsymbol{\xi}_t^*\|\leq Ch^{-d/2}\widetilde{K}_1h^{\min\{k+1,m+1\}}\\
&\quad\leq C\widetilde{K}_1h^{\min\{k+1,m+1\}-d/2},\\
&\|\boldsymbol{\xi}^*\|_{L^\infty(0,t_*;L^\infty(\Omega))}\\
&\quad\leq Ch^{-d/2}\|\boldsymbol{\xi}^*\|\leq Ch^{-d/2}\widetilde{K}_2h^{\min\{k+1,m+1\}}\\
&\quad\leq C\widetilde{K}_2h^{\min\{k+1,m+1\}-d/2}.
\end{aligned}
\tag{79}
$$

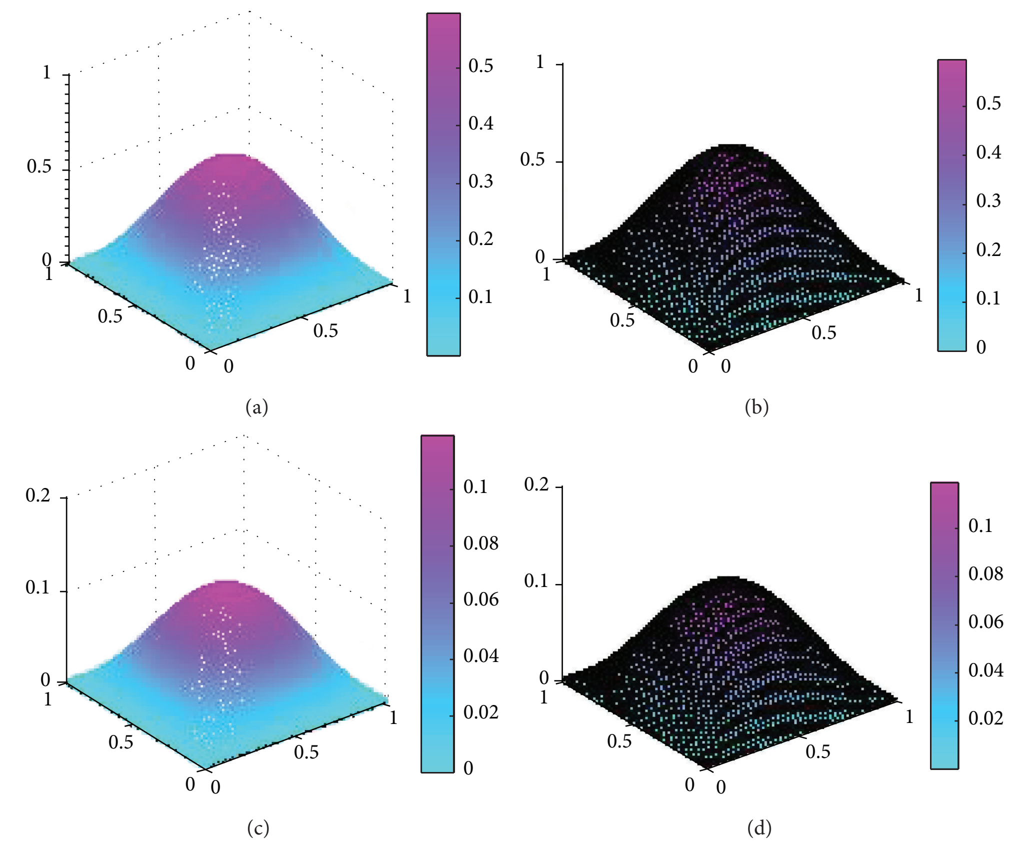

FIGURE 1: The figures of exact solution u and numerical solution u_h at $t = 0.5, 1.0$ ((a), (b) for u and u_h at $t = 1.0$ and (c), (d) for u and u_h at $t = 0.5$).

TABLE 1: The errors of $\|u - u_h\|$ at different times.

Time	$t = 0.2$		$t = 0.4$		$t = 0.8$		$t = 1.0$	
$h = \Delta t$	Error	Order	Error	Order	Error	Order	Error	Order
1/10	0.001	\	0.0025	\	0.0075	\	0.0104	\
1/20	$2.6553e-4$	1.9131	$6.3152e-4$	1.9850	0.0019	1.9809	0.0026	2.0000
1/30	$1.1843e-4$	1.9913	$2.8183e-4$	1.9899	$8.4165e-4$	2.0082	0.0012	1.9069
1/40	$6.6744e-5$	1.9934	$1.5903e-4$	1.9890	$4.7373e-4$	1.9978	$6.6038e-4$	2.0761
1/50	$4.2788e-5$	1.9925	$1.0212e-4$	1.9850	$3.0325e-4$	1.9991	$4.2104e-4$	2.0170

Choose h_0 satisfying

$$\max\left\{C\widetilde{K}_1 h_0^{\min\{k+1,m+1\}-d/2}, C\widetilde{K}_2 h_0^{\min\{k+1,m+1\}-d/2}\right\} \leq \frac{1}{2}, \tag{80}$$

which implies

$$\max\left\{\|\boldsymbol{\xi}^*\|_{L^\infty(0,t_*;L^\infty(\Omega))}, \|\boldsymbol{\xi}_t^*\|_{L^\infty(0,t_*;L^\infty(\Omega))}\right\} \leq \frac{1}{2}. \tag{81}$$

This contradicts with (75). Therefore the induction hypothesis (70) holds.

By *Poincaré*'s inequality and (49) we have

$$\|\beta\|^2 \leq C\|\nabla\beta\|^2 \leq C\left(\|\boldsymbol{\xi}\|^2 + \|\boldsymbol{\theta}\|^2\right). \tag{82}$$

Combining (50), (60), (72), (82), the estimates of projections (27), (29), and triangle inequality leads to the desired theorem result. □

5. Numerical Examples

The goal of this section is to carry out two numerical experiments to illustrate our theoretical findings. We consider the following second-order nonlinear hyperbolic problem:

$$\begin{gathered} u_{tt} - \nabla\cdot(A(u)\nabla u) = f, \quad (\mathbf{x}, t) \in \Omega \times [0,1], \\ u(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times [0,1], \\ u(\mathbf{x}, 0) = 0, \quad u_t(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \Omega, \end{gathered} \tag{83}$$

where $\Omega = [0,1] \times [0,1]$.

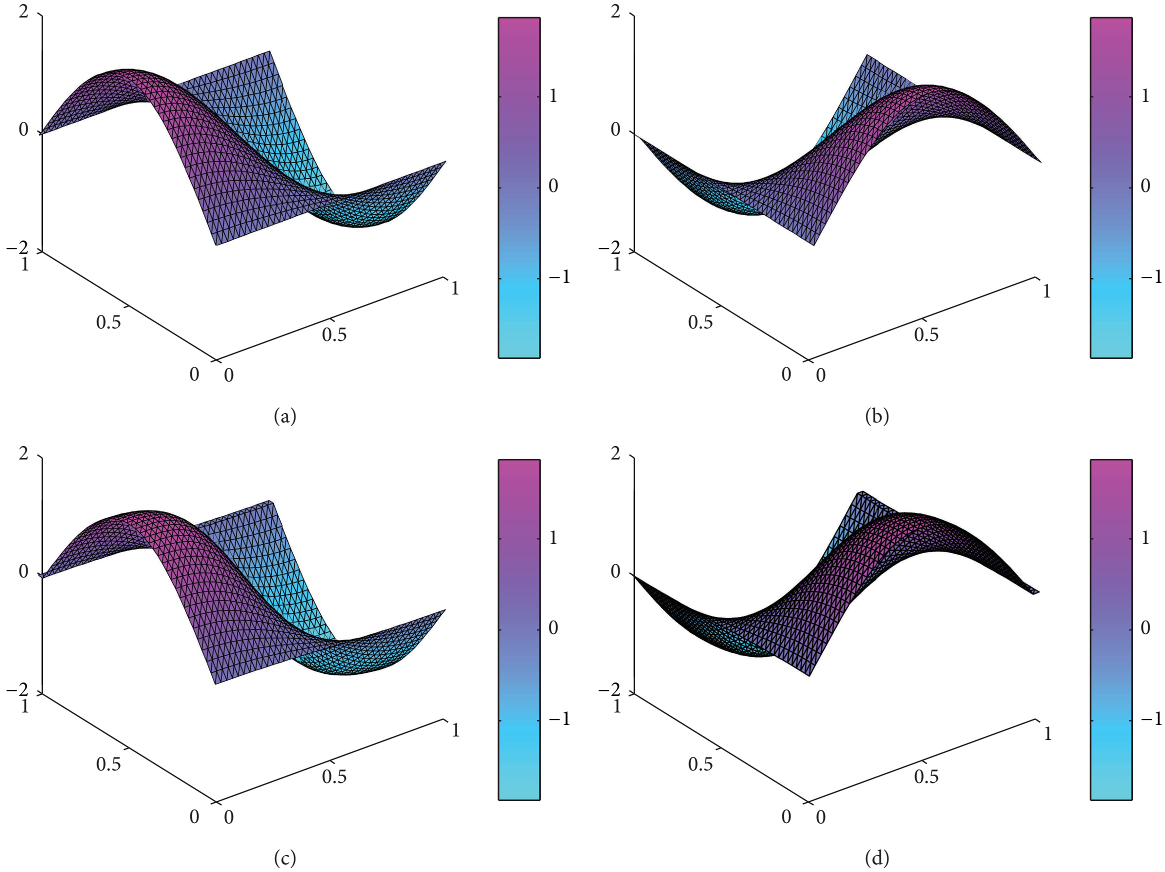

FIGURE 2: The figures of exact solution $\mathbf{p} = (P_1, P_2)$ and numerical solution $\mathbf{p}_h = (P_{h1}, P_{h2})$ at $t = 1.0$ ((a), (b) for $\mathbf{p}$ and (c), (d) for $\mathbf{p}_h$).

TABLE 2: The errors of $\|\mathbf{p} - \mathbf{p}_h\|$ at different times.

Time	$t = 0.2$		$t = 0.4$		$t = 0.8$		$t = 1.0$	
$h = \Delta t$	Error	Order	Error	Order	Error	Order	Error	Order
1/10	0.0045	\	0.0137	\	0.0745	\	0.1203	\
1/20	0.0013	1.7914	0.0062	1.1438	0.0372	1.0019	0.0601	1.0012
1/30	$7.0149e-4$	1.5215	0.0040	1.0809	0.0248	1.0000	0.0400	1.0041
1/40	$4.7332e-4$	1.3676	0.0030	1.0000	0.0186	1.0000	0.0300	1.0000
1/50	$3.5745e-4$	1.2583	0.0024	1.0000	0.0149	0.9940	0.0240	1.0000

TABLE 3: The errors of $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|$ at different times.

Time	$t = 0.2$		$t = 0.4$		$t = 0.8$		$t = 1.0$	
$h = \Delta t$	Error	Order	Error	Order	Error	Order	Error	Order
1/10	0.0045	\	0.0138	\	0.0769	\	0.1306	\
1/20	0.0013	1.7914	0.0062	1.1543	0.0383	1.0056	0.0653	1.0000
1/30	$7.0242e-4$	1.5182	0.0040	1.0809	0.0255	1.0032	0.0435	1.0019
1/40	$4.7410e-4$	1.3665	0.0030	1.0000	0.0191	1.0045	0.0326	1.0027
1/50	$3.5811e-4$	1.2574	0.0024	1.0000	0.0153	0.9941	0.0261	0.9966

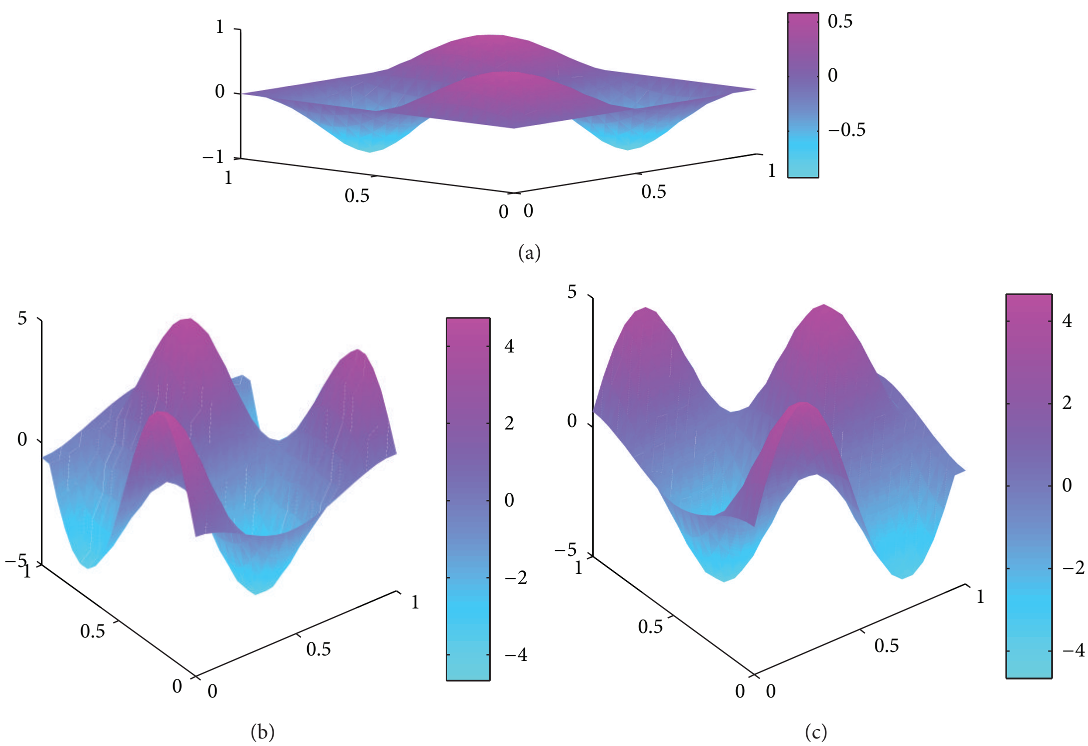

FIGURE 3: The figures of u_h and $\mathbf{p}_h$ ((a) for u_h, and (b), (c) for $\mathbf{p}_h$).

Example 7. In this example the exact solution is chosen as

$$u(x,t) = \sin^3(t)\sin(\pi x)\sin(\pi y). \tag{84}$$

We set $A(u) = u^2 + 1$. Inserting the above functions into the governing equation we can derive the corresponding right term f.

In the first example, we investigate the order of convergence for the H^1-Galerkin expanded mixed finite element method proposed in this paper. Piecewise linear polynomial is used to approximate the unknown function u, while the gradient function $\mathbf{p}$ and the flux function $\boldsymbol{\sigma}$ are approximated by the vector function space of the lowest Raviart-Thomas spaces, respectively. For time discretization we adopt backward Euler method. Here we couple the time step with spatial mesh as $h = \Delta t$.

The errors of $u - u_h$, $\mathbf{p} - \mathbf{p}_h$, and $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$ in L^2 norm at different times and the order of convergence for u, $\mathbf{p}$, and $\boldsymbol{\sigma}$ are presented in Tables 1, 2, and 3, respectively. We can observe that the order of convergence for u approaches 2, and those for $\mathbf{p}$ and $\boldsymbol{\sigma}$ approach 1, which are in agreement with our theoretical results proposed in the previous section.

The figures of the exact solutions u, $\mathbf{p}$ and the numerical solutions u_h, $\mathbf{p}_h$ at $t = 1.0$ are shown in Figures 1 and 2, respectively. We can see that the numerical solutions are accurate and without oscillation compared with the exact solutions.

Example 8. In this example we consider problem (83) with prescribed data $f = 3\sin(2x)\sin(2\pi y)e^{3t}$ and $A(u) = u$.

The profiles of the numerical solutions for u and $\mathbf{p}$ are shown in Figure 3, respectively. From these figures we can see that our method works well for this kind of problems.

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