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Research Article

Multiple Solutions for Boundary Value Problems of *n*th-Order Nonlinear Integrodifferential Equations in Banach Spaces

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The boundary value problems of a class of nth-order nonlinear integrodifferential equations of mixed type in Banach space are considered, and the existence of three solutions is obtained by using the fixed-point index theory.

Guo [1] considered the initial value problems of a class of integrodifferential equations of Volterra type and obtained the existence of maximal and minimal solutions by establishing a comparison result. In [2], the author and Qin investigated a first-order impulsive singular integrodifferential equation on the half line in a Banach space and proved the existence of two positive solutions by means of the fixed-point theorem of cone expansion and compression with norm type. For other results related to integrodifferential equations in Banach spaces please see also [3–6] and the references therein. It is worth pointing out that the nonlinear terms involved in the equations they considered are either sublinear or superlinear globally.

In this paper, by using fixed-point index theory (for details please see [7]), we consider the nth-order integrodifferential equations with nonlinear terms neither sublinear nor superlinear globally and prove the existence of three solutions.

Let E be a real Banach space and P a cone in E which defines a partial ordering in E by $x \le y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$, where θ denotes the zero element of E and the smallest N is called the normal constant of P. If $x \le y$ and $x \ne y$, we write $x < y$. P is said to be solid if its interior is not empty; that is, $\operatorname{int}(P) \ne \phi$. In case of $y - x \in \operatorname{int}(P)$, we write $x \ll y$. For details on cone theory, please see [8].

We consider the following boundary value problem (BVP for short) in E:

$$\begin{aligned} -u^{(n)}(t) &= f\left(t, u(t), u'(t), \ldots, u^{(n-1)}(t),\right. \\ &\qquad \left.(Tu)(t), (Su)(t)\right), \quad \forall t \in J, \\ u^{(i)}(0) &= \theta \quad (i = 0, 1, \ldots, n-2), \\ u^{(n-1)}(a) &= \theta, \end{aligned} \tag{1}$$

where $J := [0, a] (a > 0)$, $f \in C[J \times \underbrace{P \times P \times \cdots \times P}_{n+2}, P]$, θ denotes the zero element of E, and

$$\begin{aligned} (Tu)(t) &= \int_0^t k(t,s)\, u(s)\, ds, \\ (Su)(t) &= \int_0^a h(t,s)\, u(s)\, ds, \quad \forall t \in J, \end{aligned} \tag{2}$$

with $k \in C[D, R_+]$, $h \in C[J \times J, R_+]$, $D := \{(t,s) \in J \times J : t \ge s\}$, and R_+ the set of all nonnegative numbers. Let

$$\begin{gathered} k_0 := \max_{(t,s)\in D} k(t,s), \qquad h_0 := \max_{(t,s)\in J\times J} h(t,s), \\ \eta := 2\max\{1, a^n\}. \end{gathered} \tag{3}$$

Denote that $C^{n-1}[J,E] := \{u : u \text{ is a map from } J \text{ into } E$ and $u^{(n-1)}(t)$ is continuous on $J\}$. It is clear that $C^{n-1}[J,E]$ is a Banach space with norm defined by

$$\|u\|_{n-1} := \max_{i=0,1,\ldots,n-1}\left\|u^{(i)}\right\|_c, \quad \text{where } \left\|u^{(i)}\right\|_c := \max_{t\in J}\left\|u^{(i)}(t)\right\|. \tag{4}$$

Let

$$\begin{aligned} C^n[J,P] &:= \left\{u \in C^n[J,E] : u^{(i)}(t) \geq \theta \ (i = 0,\ldots,n-1),\right.\\ &\qquad \left. u^{(n)}(t) \leq \theta\right\},\\ C^{n-1}[J,P] &:= \left\{u \in C^{n-1}[J,E] : u^{(i)}(t)\right.\\ &\qquad \left.\geq \theta \ (i = 0,\ldots,n-1)\right\}. \end{aligned} \tag{5}$$

It is obvious that $C^n[J,P]$ and $C^{n-1}[J,P]$ are two cones in $C^n[J,E]$ and $C^{n-1}[J,E]$, respectively.

Lemma 1. *$u \in C^n[J,P]$ is the solution of problem* (1) *if and only if $u \in C^{n-1}[J,P]$ is the fixed point of operator A defined by*

$$\begin{aligned} (Au)(t) &= \frac{1}{(n-1)!}\left[\int_0^a t^{n-1} f\left(s,u(s),u'(s),\ldots,u^{(n-1)}(s),\right.\right.\\ &\qquad \left.(Tu)(s),(Su)(s)\right)ds\\ &\quad - \int_0^t (t-s)^{n-1} f\left(s,u(s),u'(s),\ldots,\right.\\ &\qquad u^{(n-1)}(s),(Tu)(s),\\ &\qquad \left.\left.(Su)(s)\right)ds\right]. \end{aligned} \tag{6}$$

Proof. For $u \in C^n[J,E]$, Taylor's formula with the integral remainder term gives

$$\begin{aligned} u(t) &= \sum_{k=0}^{n-1}\frac{(t-a)^k}{k!}u^{(k)}(a)\\ &\quad - \frac{1}{(n-1)!}\int_t^a (t-s)^{n-1}u^{(n)}(s)\,ds, \quad \forall t \in J. \end{aligned} \tag{7}$$

Taking $a = 0$, we have

$$\begin{aligned} u(t) &= \sum_{i=0}^{n-1}\frac{t^i}{i!}u^{(i)}(0)\\ &\quad + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}u^{(n)}(s)\,ds, \quad \forall t \in J. \end{aligned} \tag{8}$$

Substituting

$$u^{(n-1)}(0) = u^{(n-1)}(a) - \int_0^a u^{(n)}(s)\,ds \tag{9}$$

into (8), we get

$$\begin{aligned} u(t) &= \sum_{i=0}^{n-2}\frac{(t)^i}{i!}u^{(i)}(0) + \frac{t^{(n-1)}}{(n-1)!}u^{(n-1)}(a) - \frac{1}{(n-1)!}\\ &\quad \times\left(\int_0^a t^{(n-1)}u^{(n)}(s)\,ds\right.\\ &\quad \left. - \int_0^t (t-s)^{n-1}u^{(n)}(s)\,ds\right), \quad \forall t \in J. \end{aligned} \tag{10}$$

Let $u \in C^n[J,P]$ be the solution of BVP (1). Then (10) implies

$$\begin{aligned} &u(t)\\ &= \frac{1}{(n-1)!}\\ &\quad \times\left[\int_0^a t^{n-1} f\left(s,u(s),u'(s),\ldots,u^{(n-1)}(s),(Tu)(s),\right.\right.\\ &\qquad \left.(Su)(s)\right)ds\\ &\quad - \int_0^t (t-s)^{n-1} f\left(s,u(s),u'(s),\ldots, u^{(n-1)}(s),\right.\\ &\qquad \left.\left.(Tu)(s),(Su)(s)\right)ds\right]. \end{aligned} \tag{11}$$

Comparing this with (6), we have $u(t) = (Au)(t)$, which means that $u(t)$ is the fixed point of the operator A in $C^{n-1}[J,P]$.

On the other hand, let $u(t) \in C^{n-1}[J,P]$ be the fixed point of the operator A. By (6),

$$\begin{aligned} &u^{(j)}(t) = (Au)^{(j)}(t)\\ &= \frac{1}{(n-1-j)!}\\ &\quad \times\left[\int_0^a t^{n-1-j} f\left(s,u(s),u'(s),\ldots,\right.\right.\\ &\qquad \left.u^{(n-1)}(s),(Tu)(s),(Su)(s)\right)ds\\ &\quad - \int_0^t (t-s)^{n-1-j} f\left(s,u(s),u'(s),\ldots,u^{(n-1)}(s),\right.\\ &\qquad \left.\left.(Tu)(s),(Su)(s)\right)ds\right], \end{aligned} \tag{12}$$

where $j = 1,2,\ldots,n-1$. It follows by taking $t = 0$ and $t = a$ in (12) that

$$\begin{aligned} &u^{(j)}(0) = \theta \quad (j = 0,1,\ldots,n-2), \qquad u^{(n-1)}(a) = \theta,\\ &\qquad u^{(j)}(t) \geq \theta \quad (j = 0,1,\ldots,n-2),\ t \in J. \end{aligned} \tag{13}$$

It is also clear from (12) that

$$u^{(n-1)}(t) = \int_t^a f\left(s, u(s), u'(s), \ldots, u^{(n-1)}(s), (Tu)(s), (Su)(s)\right) ds, \quad t \in J,$$
$$u^{(n)}(t) = -f\left(t, u(t), u'(t), \ldots, u^{(n-1)}(t), (Tu)(t), (Su)(t)\right), \quad t \in J. \tag{14}$$

Hence, $u^{(n)}(t) \le \theta$. Then (13)–(14) imply that u is the solution for BVP (1) in $C^n[J, P]$. □

To continue, let us formulate some conditions.

(H_1) Let $f(t, v_0, v_1, \ldots, v_{n+1})$ be bounded and uniformly continuous in t on $J \times \underbrace{B_r \times B_r \times \cdots \times B_r}_{n+2}$, $\forall r > 0$. There exist nonnegative constants c_i $(i = 0, 1, \ldots, n+1)$ such that

$$\eta k^* \left(\sum_{i=0, i \ne n-1}^{n+1} c_i + 2c_{n-1} \right) < 1, \tag{15}$$

$$\alpha\left(f\left(J, V_0, V_1, \ldots, V_{n-1}, V_n, V_{n+1}\right)\right) \le \sum_{i=0}^{n+1} c_i \alpha\left(V_i\right), \quad \forall V_i \subset B_r, \tag{16}$$

where $k^* := \max\{1, k_0 a, h_0 a\}$, α denotes the Kuratowski measure of noncompactness, and $B_r = \{u \in E : \|u\| \le r\}$.

(H_2) Assume that

$$\overline{\lim_{r \to \infty}} \frac{M(r)}{r} < \frac{\eta^*}{k^*}, \tag{17}$$

$$\overline{\lim_{r \to 0^+}} \frac{M(r)}{r} < \frac{\eta^*}{k^*}, \tag{18}$$

where

$$M(r) := \sup\left\{\left\|f\left(t, v_0, v_1, \ldots, v_{n-1}, v_n, v_{n+1}\right)\right\| : \left(t, v_0, v_1, \ldots, v_{n-1}, v_n, v_{n+1}\right) \in J \times P_r \times P_r \times \cdots \times P_r \times P_r \times P_r\right\}, \tag{19}$$

$P_r := \{u \in P : \|u\| \le r\}$, $\eta^* := \eta^{-1}$, and k^* is defined by (H_1).

(H_3) There exist $u^* \in \operatorname{int}(P)$, $0 < t_0 < t_1 < a$, and $F(t) \in C[J, R_+]$ such that

$$f\left(t, v_0, v_1, \ldots, v_{n-1}, v_n, v_{n+1}\right) \ge F(t) u^*,$$
$$\int_{t_1}^a F(s)\, ds > \max\left\{1, \frac{1}{t_0}, \frac{2!}{t_0^2}, \ldots, \frac{(n-1)!}{t_0^{n-1}}\right\}, \tag{20}$$

for $v_i \ge u^*$ $(i = 0, 1, \ldots, n-1)$, $v_n \ge \theta$, $v_{n+1} \ge \theta$, and $t \in [t_0, t_1]$.

Remark 2. By (H_2) and (H_3), one can see that f is neither sublinear nor superlinear globally.

Lemma 3 (see [8]). *Let H be a bounded set of $C^m[J, E]$. Then*

$$\alpha_m(H) \ge \alpha(H(J)), \alpha_m(H) \ge \alpha\left(H'(J)\right), \ldots, \alpha_m(H) \ge \alpha\left(H^{(m-1)}(J)\right), \alpha_m(H) \ge \frac{1}{2}\alpha\left(H^{(m)}(J)\right), \tag{21}$$

where $H^{(i)}(J) := \{u^{(i)}(t) : t \in J, u \in H\}$ $(i = 0, 1, 2, \ldots, m)$.

Lemma 4 (see [8]). *Let H be a bounded set of $C^m[J, E]$. Suppose that $H^{(m)} := \{u^{(m)} : u \in H\}$ is equicontinuous. Then*

$$\alpha_m(H) = \max_{i=0,1,\ldots,m}\left\{\alpha\left(H^{(i)}(J)\right)\right\} = \max_{i=0,1,\ldots,m}\left\{\max_{t \in J}\left\{\alpha\left(H^{(i)}(t)\right)\right\}\right\}, \tag{22}$$

where $H^{(i)}(J)$ $(i = 0, 1, 2, \ldots, m)$ is defined by Lemma 3 and $H^{(i)}(t) := \{u^{(i)}(t) : u \in H\}$ $(i = 0, 1, 2, \ldots, m)$.

Lemma 5. *Let (H_1) hold. Then operator A defined by (6) is a strict set contraction from $C^{n-1}[J, P]$ into $C^{n-1}[J, P]$.*

Proof. It is easy to see that $A : C^{n-1}[J, P] \to C^{n-1}[J, P]$ and A is a bounded operator by (6), (12), and (H_1).

Now we check that operator A is continuous from $C^{n-1}[J, P]$ into $C^{n-1}[J, P]$. Let $\{u_m\}_{m=1}^{\infty} \subset C^{n-1}[J, P]$, $u \in C^{n-1}[J, P]$, and

$$\left\|u_m - u\right\|_{n-1} \longrightarrow 0 \quad (m \longrightarrow \infty). \tag{23}$$

For any $t \in J$, by (6),

$$\left\|\left(Au_m\right)(t) - (Au)(t)\right\| \le \frac{1}{(n-1)!} \times \left[\int_0^a t^{n-1} \left\|f\left(s, u_m(s), u_m'(s), \ldots,\right.\right.\right.$$

$$u_m^{(n-2)}(s),(Tu_m)(s),(Su_m)(s)\Big) - f\Big(s,u(s),u'(s),\dots,u^{(n-2)}(s), (Tu)(s),(Su)(s)\Big)\Big\|\,ds + \int_0^t (t-s)^{n-1}\Big\|f\Big(s,u_m(s),u_m'(s),\dots,u_m^{(n-2)}(s), (Tu_m)(s),(Su_m)(s)\Big) - f\Big(s,u(s),u'(s),\dots, u^{(n-1)}(s), (Tu)(s),(Su)(s)\Big)\Big\|\,ds\Bigg]. \tag{24}$$

Then the Lebesgue dominated convergence theorem gives

$$\begin{aligned}\max_{t\in J}&\left\|(Au_m)(t)-(Au)(t)\right\| \\ &\le \frac{1}{(n-1)!} \times\Bigg[\int_0^a a^{n-1}\Big\|f\Big(s,u_m(s),u_m'(s),\dots, u_m^{(n-2)}(s),(Tu_m)(s),(Su_m)(s)\Big) - f\Big(s,u(s),u'(s),\dots,u^{(n-2)}(s), (Tu)(s),(Su)(s)\,u_m'(s)\Big)\Big\|\,ds \\ &\quad+\int_0^a (a-s)^{n-1} \Big\|f\Big(s,u_m(s),u_m'(s),\dots,u_m^{(n-2)}(s), (Tu_m)(s),(Su_m)(s)\Big) - f\Big(s,u(s),u'(s),\dots,u^{(n-1)}(s),(Tu)(s), (Su)(s)\Big)\Big\|\,ds\Bigg]\longrightarrow 0,\quad (m\longrightarrow\infty).\end{aligned} \tag{25}$$

Hence,

$$\|Au_m-Au\|_c = \max_{t\in J}\left\|(Au_m)(t)-(Au)(t)\right\|\longrightarrow 0\quad (m\longrightarrow\infty). \tag{26}$$

Similarly, in view of (12), we get

$$\left\|(Au_m)^{(i)}-(Au)^{(i)}\right\|_c\longrightarrow 0, \quad (m\longrightarrow\infty);\ (i=1,2,\dots,n-1). \tag{27}$$

Then

$$\|Au_m-Au\|_{n-1} = \max_{i=0,1,\dots,n-1}\left\|(Au_m)^{(i)}-(Au)^{(i)}\right\|_c\longrightarrow 0,\quad (m\longrightarrow\infty). \tag{28}$$

Consequently, the continuity of operator A is proved.

Let $Q\subset C^{n-1}[J,P]$ be bounded. Then $A(Q)\subset C^n[J,P]$ is bounded. We prove that $(A(Q))^{(n-1)}$ is equicontinuous on J. In fact, $\forall (A(u))^{(n-1)}\in (A(Q))^{(n-1)}$, by (12),

$$\begin{aligned}&\left\|(A(u))^{(n-1)}(t_1)-(A(u))^{(n-1)}(t_2)\right\| \\ &\le \int_{t_1}^{t_2}\Big\|f\Big(s,u(s),u'(s),\dots,u^{(n-1)}(s), (Tu)(s),(Su)(s)\Big)\Big\|\,ds.\end{aligned} \tag{29}$$

According to the absolute continuity of Lebesgue integral, $(A(Q))^{(n-1)}$ is equicontinuous on J. Therefore, Lemma 4 implies that

$$\alpha_{n-1}(A(Q)) = \max_{i=0,1,\dots,n-1}\left\{\max_{t\in J}\left\{\alpha\left((A(Q))^{(i)}(t)\right)\right\}\right\}, \tag{30}$$

where $\alpha((A(Q))^{(i)}(t)) = \alpha(\{(Au)^{(i)}(t) : u\in Q\})$ (t is fixed, $i=0,1,\dots,n-1$). By (6), we see that

$$\alpha((Au)(t))\le \eta\alpha\Big(f\Big(s,Q(J),Q'(J),\dots, Q^{(n-1)}(J)\Big),(TQ)(J),(SQ)(J)\Big), \tag{31}$$

where $Q^{(i)}(J)=\{u^{(i)}(s): s\in J, u\in Q\}$ $(i=0,1,\dots,n-1)$,

$$\begin{aligned}(TQ)(J) &= \{(Tu)(s): s\in J, u\in Q\},\\ (SQ)(J) &= \{(Su)(s): s\in J, u\in Q\}.\end{aligned} \tag{32}$$

It follows from (31) and (H_1) that

$$\begin{aligned}\alpha&((Au)(t)) \\ &\le \eta\left(\sum_{i=0}^{n-1}c_i\alpha\left(Q^{(i)}(J)\right)+c_nk_0a\alpha(Q(J))+c_{n+1}h_0a\alpha(Q(J))\right) \\ &\le \eta k^*\left(\sum_{i=0}^{n-2}c_i\alpha\left(Q^{(i)}(J)\right)+c_{n-1}\alpha\left(Q^{(n-1)}(J)\right) +c_n\alpha(Q(J))+c_{n+1}\alpha(Q(J))\right),\end{aligned} \tag{33}$$

which implies, according to Lemma 3, that

$$\begin{aligned}\alpha((Au)(t)) &\le \eta k^*\left(\sum_{i=0,i\neq n-1}^{n+1}c_i+2c_{n-1}\right)\alpha_{n-1}(Q) \\ &= \gamma\alpha_{n-1}(Q),\end{aligned} \tag{34}$$

where $\gamma = \eta k^*(\sum_{i=0,i\neq n-1}^{n+1}c_i+2c_{n-1})<1$ in view of (15).

Similarly, we have

$$\alpha\left((Au)^{(i)}(t)\right) \leq \gamma\alpha_{n-1}(Q) \quad (i = 1, 2, \ldots, n-1). \tag{35}$$

Thus, we get $\alpha_{n-1}(A(Q)) \leq \gamma\alpha_{n-1}(Q)$ by (34) and (35). Noticing that A is bounded and continuous, the conclusion follows. □

Theorem 6. *Let P be a normal solid cone and let (H_1), (H_2), and (H_3) hold. Then BVP (1) has at least three solutions in $C^n[J, P]$.*

Proof. Condition (H_2) implies that there exist $\varepsilon' > 0$ and $r' > 0$, such that, for $r > k^* r'$,

$$\frac{M(r)}{r} < \frac{\eta^*}{k^* + \varepsilon'}. \tag{36}$$

Choose $r^* > \max\{r', 2\|u^*\|\}$. Let

$$U := \left\{u \in C^{n-1}[J, P] : \|u\|_{n-1} < r^*\right\}. \tag{37}$$

For $u \in \overline{U}$, we have $\|u^{(i)}\| \leq r^*$ $(i = 0, 1, \ldots, n-1)$, $\|Tu\| \leq k^* r^*$, and $\|Su\| \leq k^* r^*$. So, it follows from (6), (12), and (36) that

$$\left\|(Au)^{(i)}\right\| \leq \eta M(k^* r^*) < \eta\eta^* \frac{k^*}{k^* + \varepsilon'} r^* < r^* \quad (i = 0, 1, \ldots, n-1). \tag{38}$$

Hence, $\|Au\|_{n-1} < r^*$. Thus, we have shown that

$$A\left(\overline{U}\right) \subset U. \tag{39}$$

Similarly, by (18), it is easy to get that there is a number r_0 such that $0 < r_0 < \|u^*\|/N$ and

$$A\left(\overline{U}_0\right) \subset U_0, \tag{40}$$

where $U_0 = \{u \in C^{n-1}[J, P] : \|u\|_{n-1} < r_0\}$ and N is the normal constant of P.

Let

$$U_1 := \left\{u \in C^{n-1}[J, P] : \|u\|_{n-1} < r^*,\ u^{(i)}(t) \geq \lambda u^* \quad (i = 0, 1, \ldots, n-1), t \in [t_0, t_1], \lambda > 1 \text{ depending on } u\right\}. \tag{41}$$

It is easy to see that U, U_0, and U_1 are all nonempty bounded open convex sets of $C^{n-1}[J, P]$, and

$$U_i \subset U \quad (i = 0, 1), \qquad U_0 \cap U_1 = \emptyset. \tag{42}$$

As the proof of (38), for $u \in \overline{U}_1$, by (H_2),

$$\left\|(Au)^{(i)}\right\| < r^*, \quad (i = 0, 1, \ldots, n-1). \tag{43}$$

On the other hand, according to (H_3), for $t \in [t_0, t_1]$, $u^{(i)}(t) \geq u^*$ $(i = 0, 1, \ldots, n-1)$, $(Tu)(t) \geq \theta$, and $(Su)(t) \geq \theta$, we get by (12) that

$$\begin{aligned}(Au)^{(j-1)}(t) &\geq \frac{1}{(n-j)!}\int_t^a t^{n-j} F(s) u^* ds \\ &\geq \frac{1}{(n-j)!}\int_{t_1}^a t_0^{n-j} F(s) ds u^*, \\ &\qquad (j = 1, 2, \ldots, n).\end{aligned} \tag{44}$$

Condition (H_3) also implies that

$$\frac{1}{(n-j)!}\int_{t_1}^a t_0^{n-j} F(s) ds > 1 \quad (j = 1, 2, \ldots, n). \tag{45}$$

Consequently, in view of (43) and (45), we have shown that

$$A\left(\overline{U}_1\right) \subset U_1. \tag{46}$$

It follows from (39), (40), (42), (46), and Lemma 5 that

$$\begin{aligned}&i\left(A, U, C^{n-1}[J, P]\right) = 1, \\ &i\left(A, U_0, C^{n-1}[J, P]\right) = 1, \\ &i\left(A, U_1, C^{n-1}[J, P]\right) = 1, \\ &i\left(A, U \setminus \left(\overline{U}_0 \bigcup \overline{U}_1\right), C^{n-1}[J, P]\right) \\ &\quad = i\left(A, U, C^{n-1}[J, P]\right) - i\left(A, U_0, C^{n-1}[J, P]\right) \\ &\quad\quad - i\left(A, U_1, C^{n-1}[J, P]\right) = -1,\end{aligned} \tag{47}$$

where $i(\cdot,\cdot,\cdot)$ denotes the fixed-point index [7]. Therefore, A has three fixed points $\overline{u}_0 \in U_0$, $\overline{u}_1 \in U_1$, and $\overline{u}_3 \in U \setminus (\overline{U}_0 \bigcup \overline{U}_1)$. By Lemma 1, BVP (1) has at least three solutions in $C^n[J, P]$. □

An application of Theorem 6 is as follows.

Example 7. Consider

$$\begin{aligned}-u_n^{(4)}(t) = 4t\sqrt{u_n(t)} \ln \Big[1 &+ 5u_n(t) + 6u'_{n+1}(t) \\ &+ 7u''_{n-1}(t) + 8u'''_n(t) \\ &+ \int_0^t \left(2e^t + 3t^2 s\right)^{-1} u_{n+1}(s) ds \Big]\end{aligned}$$

$$
\begin{aligned}
&+ \sin^2\left(u_n(t) + 2u'_{n-1}(t) + 3u''_n(t) + 4u'''_{n+1}(t)\right) \\
&+ \frac{1}{33}\left(u'_n(t)\right)^{3/4} \\
&\times \left(\int_0^2 \lg\left(\frac{t+s}{2}+1\right)\cos^2(t-s)\,u_{n+1}(s)\,ds\right)^{1/4}, \\
&\qquad \forall t \in [0,2], \\
&u_n^{(i)}(0) = 0 \quad (i = 0,1,2), \\
&u_n'''(2) = 0, \quad (n = 1,2,\ldots,m),
\end{aligned} \tag{48}
$$

where $u_0 = u_m$ and $u_{m+1} = u_1$.

Obviously, $u_n(t) \equiv 0$ $(n = 1,2,\ldots,m)$ is the trivial solution of BVP (48).

Conclusion. BVP (48) has at least two nontrivial nonnegative C^4 solutions.

Proof. Let $E := \{u = (u_1,\ldots,u_m)\}$, m-dimensional space, with norm $\|u\| := \sup_{n=1,2,\ldots,m}|u_n|$ and

$$P = \{u = (u_1,\ldots,u_m) : u_n \geq 0, n = 1,2,\ldots,m\}. \tag{49}$$

Then P is a normal and solid cone in E and (48) can be regarded as a BVP of the form (1), where

$$
\begin{aligned}
&a = 2, \qquad k(t,s) = \left(2e^t + 3t^2 s\right)^{-1}, \\
&h(t,s) = \lg\left(\frac{t+s}{2}+1\right)\cos^2(t-s), \\
&u = (u_1,\ldots,u_m), \qquad v = (v_1,\ldots,v_m), \\
&w = (w_1,\ldots,w_m), \qquad x = (x_1,\ldots,x_m), \\
&y = (y_1,\ldots,y_m), \qquad z = (z_1,\ldots,z_m),
\end{aligned} \tag{50}
$$

and $f = (f_1,\ldots,f_m)$ with

$$
\begin{aligned}
&f_n(t,u,v,w,x,y,z) \\
&\quad = 4t\sqrt{u_n}\ln\left(1 + 5u_n + 6v_{n+1} + 7w_{n-1} + 8x_n + y_{n+1}\right) \\
&\qquad + \sin^2\left(u_n + 2v_{n-1} + 3w_n + 4x_{n+1}\right) \\
&\qquad + \frac{1}{33}(v_n)^{3/4}(z_{n+1})^{1/4} \quad (n = 1,2,\ldots,m).
\end{aligned} \tag{51}
$$

Obviously, $f \in C[J \times \underbrace{P \times P \times \cdots \times P}_{6}, P]$ $(J = [0,2])$ and (H_1) is satisfied for $c_i = 0$ $(i = 0,1,\ldots,5)$ since E is finite-dimensional.

One can see that

$$
\begin{aligned}
&\left|\sin\left(u_n + 2v_{n-1} + 3w_n + 4x_{n+1}\right)\right| \\
&\quad \leq \min\left\{1, |u_n| + 2|v_{n-1}| + 3|w_n| + 4|x_{n+1}|\right\}.
\end{aligned} \tag{52}
$$

Then (51) implies that

$$
\begin{aligned}
&\|f(t,u,v,w,x,y,z)\| \\
&\quad \leq 4t\sqrt{\|u\|}\ln\left(1 + 5\|u\| + 6\|v\| + 7\|w\| + 8\|x\| + \|y\|\right) \\
&\qquad + \min\left\{1, \left(\|u\| + 2\|v\| + 3\|w\| + 4\|x\|\right)^2\right\} \\
&\qquad + \frac{1}{33}\|v\|^{3/4}\|z\|^{1/4}, \quad \forall t \in J, u,v,w,x,y,z \in P.
\end{aligned} \tag{53}
$$

Therefore,

$$M(r) \leq 4\sqrt{r}\ln(1+26r) + \min\left\{1, 100r^2\right\} + \frac{1}{33}r. \tag{54}$$

Hence,

$$\overline{\lim_{r\to\infty}}\frac{M(r)}{r} < \frac{1}{32}, \qquad \overline{\lim_{r\to 0^+}}\frac{M(r)}{r} < \frac{1}{32}. \tag{55}$$

On the other hand, it is easy to see that

$$\eta = 32, \qquad \eta^* = \frac{1}{32}, \qquad k^* = 1. \tag{56}$$

Thus, (55) and (56) imply that (H_2) is satisfied.

Now, we check (H_3). Let $u^* = (1,\ldots,1)$, $F(t) = 4\ln 27$ and $t_0 = 1$, $t_1 = 3/2$. Obviously, $u^* \in \mathrm{int}(P)$ and, for $t \in [t_0,t_1]$, $u \geq u^*$, $v \geq u^*$, $w \geq u^*$, $x \geq u^*$, $y \geq \theta$, and $z \geq \theta$ (i.e., $1 \leq t \leq 3/2$, $u_n \geq 1$, $v_n \geq 1$, $w_n \geq 1$, $x_n \geq 1$, $y_n \geq 0$, $z_n \geq 0$, $n = 1,2,\ldots,m$). Then (51) implies that

$$
\begin{aligned}
&f_n(t,u,v,w,x,y,z) \\
&\quad \geq 4t\sqrt{u_n}\ln\left(1 + 5u_n + 6v_{n+1} + 7w_{n-1} + 8x_n\right) \\
&\quad \geq 4\ln 27,
\end{aligned} \tag{57}
$$

where $n = 1,2,\ldots,m$. So, we have $\int_{3/2}^2 4\ln 27\,ds > 6$. Hence, (H_3) is satisfied. And, finally, the conclusion follows from Theorem 6. □

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