Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 939405, 9 pages
http://dx.doi.org/10.1155/2013/939405

Research Article

Global Solutions for an m-Component System of Activator-Inhibitor Type

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Received 15 May 2013; Accepted 17 July 2013

Academic Editor: Khalil Ezzinbi

This paper deals with a reaction-diffusion system with fractional reactions modeling m-substances into interaction following activator-inhibitor's scheme. The existence of global solutions is obtained via a judicious Lyapunov functional that generalizes the one introduced by Masuda and Takahashi.

1. Introduction

In this paper, we are concerned with the existence of global solutions to a reaction-diffusion system with m components generalizing the activator-inhibitor system:

$$\begin{aligned} \partial_t u_1 - a_1 \Delta u_1 &= f_1(u) = \sigma - b_1 u_1 + \frac{u_1^{p_{11}}}{\prod_{j=2}^m u_j^{p_{1j}}}, \\ \partial_t u_i - a_i \Delta u_i &= f_i(u) = -b_i u_i + \frac{u_1^{p_{i1}}}{\prod_{j=2}^m u_j^{p_{ij}}}, \qquad x \in \Omega,\ t > 0, \\ & i = 2, \dots, m, \end{aligned} \tag{1}$$

supplemented with Neumann boundary conditions

$$\frac{\partial u_i}{\partial \eta} = 0, \quad \text{on } \partial\Omega \times \{t > 0\},\ i = 1, \dots, m, \tag{2}$$

and the positive initial data

$$u_i(x, 0) = \varphi_i(x) \quad \text{on } \Omega,\ i = 1, \dots, m. \tag{3}$$

Here $u = (u_1, u_2, \dots, u_m)$, Ω is an open bounded domain of class C^1 in $\mathbb{R}^N$, with boundary $\partial\Omega$, and $\partial/\partial\eta$ denotes the outward normal derivative on $\partial\Omega$.

Throughout the paper, we make the following hypotheses:

The indexes a_i, p_{ij} are nonnegative for all $i, j = 1, \dots, m$, with $\sigma > 0$:

$$\begin{aligned} 0 &< p_{11} - 1 \\ &< \max_{k=2,3,\dots,m} \left\{ p_{k1} \min \left\{ 1, \frac{p_{1k}}{p_{kk}+1}, \frac{p_{1j}}{p_{kj}}, \right.\right. \\ &\qquad \left.\left. j = 2, \dots, m,\ j \neq k \right\} \right\}; \end{aligned} \tag{4}$$

we set $A_{ij} = (a_i + a_j)/(2\sqrt{a_i\, a_j})$ for all $i, j = 1, \dots, m$. Let α_i, $i = 1, \dots, m$, be positive constants such that

$$\alpha_1 > 2 \max \left\{ 1, \sum_{i=1}^m \frac{b_i}{b_1} \right\}, \tag{5}$$

$$S_l^l > 0, \quad l = 2, \dots, m, \tag{6}$$

where

$$S_l^r = S_{r-1}^{r-1} \cdot S_l^{r-1} - \left[H_l^{r-1} \right]^2, \quad r = 3, \dots, l,$$

$$H_l^r = \det_{1 \le i,j \le l} \left(\left(a_{i,j} \right)_{\substack{i \neq l, \dots r+1 \\ j \neq l-1, \dots r}} \right) \prod_{k=1}^{k=r-2} (\det[k])^{2^{(r-k-2)}},$$

$$r = 3, \dots, l-1,$$

$$S_l^2 = \alpha_1^2 \alpha_l^2 a_1 a_l \left[\frac{1}{2\alpha_l} - A_{1l}^2 \right],$$
$$H_l^2 = \alpha_1^2 \alpha_2 \alpha_l a_1 \sqrt{a_2 a_l} \left[\frac{\alpha_1 - 1}{\alpha_1} A_{2l} - A_{12} A_{1l} \right], \tag{7}$$

where $\det_{1\leq i,j\leq l}((a_{i,j})_{i\neq l,\ldots,r+1, j\neq l-1,\ldots,r})$ stands for the determinant of the r-square symmetric matrix obtained from the matrix $(a_{i,j})_{1\leq i,j\leq m}$ by removing the $(r+1)$th, $(r+2)$th, $\ldots$, lth rows and the rth, $(r+1)$th, $\ldots$, $(l-1)$th columns, where $\det[1], \ldots, \det[m]$ are the minors of the matrix $(a_{ij})_{1\leq i,j\leq m}$. The elements of the matrix are as follows:

$$\begin{gathered} a_{11} = a_1 \alpha_1 (\alpha_1 - 1), \\ a_{1i} = -\alpha_1 \alpha_i \frac{(a_1 + a_i)}{2}, \quad i = 2, \ldots, m, \\ a_{ii} = a_i \alpha_i (\alpha_i + 1), \quad i = 2, \ldots, m, \\ a_{ij} = \alpha_i \alpha_j \frac{(a_i + a_j)}{2}, \quad i, j = 2, \ldots, m, \ i \neq j. \end{gathered} \tag{8}$$

The main result of the paper is the following.

Theorem 1. *Assume that condition* (4) *is satisfied. Let u be a solution of* (1)–(3) *with positive and bounded initial data, and let*

$$L(t) = \int_\Omega \frac{u_1^{\alpha_1}(t,x)}{\prod_{j=2}^m u_j^{\alpha_j}(t,x)} dx. \tag{9}$$

Then the functional L is uniformly bounded on the interval $[0, T^*]$, $T^* < T_{\max}$, where $T_{\max}(\|\varphi_1\|_\infty, \|\varphi_2\|_\infty, \ldots, \|\varphi_m\|_\infty)$ denotes the eventual blow-up time.

Corollary 2. *Under the assumptions of Theorem 1 all solutions of* (1)–(3) *with positive initial data in $C(\overline{\Omega})$ are global. If in addition $b_1, \ldots, b_m$, $\sigma > 0$, then u is uniformly bounded in $\overline{\Omega} \times [0, \infty)$.*

Before we prove our results, let us dwell a while on the existing literature concerning Gierer-Meinhardt's type systems.

In 1972, following an ingenious idea of Turing [1], Gierer and Meinhardt [2] proposed a mathematical model for pattern formations of spatial tissue structures of hydra in morphogenesis. It is a system of reaction-diffusion equations of the form:

$$\begin{aligned} \partial_t u - a_1 \Delta u &= \sigma - \mu u + \frac{u^p}{v^q}, \\ \partial_t v - a_2 \Delta v &= -\nu v + \frac{u^r}{v^s}, \end{aligned} \qquad x \in \Omega, \ t > 0, \tag{10}$$

with Neumann boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, \quad x \in \partial\Omega, \ t > 0, \tag{11}$$

and initial conditions

$$\begin{aligned} u(x,0) &= \varphi_1(x) > 0, \\ v(x,0) &= \varphi_2(x) > 0, \end{aligned} \qquad x \in \Omega, \tag{12}$$

where $\Omega \subset \mathbb{R}^N$ ($N = 1, 2, 3$ in practice) is a bounded domain with smooth boundary $\partial\Omega$, $a_1, a_2, \mu, \nu, \sigma > 0$, and p, q, r and s are non negative windexes with $p > 1$. Here u is the activator, and v is the inhibitor.

Global existence of solutions in $(0, \infty)$ was proved by Rothe [3], more than ten years after Gierer and Meinhardt's original paper with special choice of the parameters: $p = 2$, $q = 1$, $r = 2$, $s = 0$, and $N = 3$. Masuda and Takahashi [4] were able to prove global estimates and bounds of the solution for Gierer and Meinhardt's system in its general form. They proceeded by first proving lower bounds, then L^p bounds (for any $p > 1$), then uniform estimates and bounds in appropriate Sobolev spaces. The key point is represented by the L^p bounds, which are derived using in a subtle way the specific structure of the equations.

Li et al. [5] also studied the activator-inhibitor model.

Very recently, Bernasconi [6] considered the larger system:

$$\begin{aligned} \partial_t a(x,t) &= d_a a_{xx} + \frac{a^2(x,t)}{h(x,t)} - \mu a(x,t) + \rho, \\ \partial_t h(x,t) &= d_h h_{xx}(x,t) + a^2(x,t) - \nu h(x,t) + \varepsilon s(x,t), \\ \partial_t s(x,t) &= d_s s_{xx}(x,t) + a(x,t) - \kappa s(x,t), \end{aligned} \tag{13}$$

and Meinhardt et al. [7] proposed activator-inhibitor models to describe a theory of biological pattern:

$$\begin{aligned} \partial_t a(x,t) &= d_a a_{xx} + \frac{a^2(x,t)}{h(x,t) s(x,t)} - \mu a(x,t) + \rho, \\ \partial_t h(x,t) &= d_h h_{xx}(x,t) + a^2(x,t) - \nu h(x,t), \\ \partial_t s(x,t) &= d_s s_{xx}(x,t) + a(x,t) - \kappa s(x,t), \end{aligned} \tag{14}$$

which is Gierer and Meinhardt's system supplemented with a third equation, where $a(x,t)$ is the activator, $h(x,t)$ is the inhibitor, and $s(x,t)$ is a source that acts as an inhomogeneous inhibitor.

Our paper generalizes the system in [5] to m-components.

2. Preliminary Observations and Notations

The usual norms in the spaces $L^p(\Omega)$, $L^\infty(\Omega)$, and $C(\overline{\Omega})$ are denoted, respectively, by the following:

$$\begin{gathered} \|u\|_p^p = \frac{1}{|\Omega|} \int_\Omega |u(x)|^p dx, \\ \|u\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|, \\ \|u\|_{C(\overline{\Omega})} = \max_{x\in\overline{\Omega}} |u(x)|. \end{gathered} \tag{15}$$

It is well known that to prove global existence of solutions to (1)–(3), it suffices to derive a uniform estimate of $\|f_i(u_1,u_2,\ldots,u_m)\|_p$, $i=1,\ldots,m$ on $[0;T_{\max})$ in the space $L^p(\Omega)$ for some $p>n/2$ (see Henry [8]).

Since the functions f_i are continuously differentiable on $\mathbb{R}_+^m$ for all $i=1,\ldots,m$, then for any initial data in $C(\overline{\Omega})$, the system (1)–(3) admits a unique, classical solution $(u_1,u_2,\ldots,u_m)$ on $(0,T_{\max})\times\Omega$ with the alternative

(i) either $T_{\max}=\infty$;

(ii) or $T_{\max}<\infty$, and $\lim_{t\nearrow T_{\max}}\sum_{i=1}^m\|u_i(t,\cdot)\|_\infty=\infty$.

Using the maximum principle, one derives the lower bounds of the components of the solution u of (1)–(3):

$$u_i(t,x)\ge e^{-b_it}\min(\varphi_i(x))>0,\quad i=1,\ldots,m. \tag{16}$$

Our aim is to construct a Lyapunov functional that allows us to obtain L^p-bounds on u_i leading to global existence.

3. Preparatory Lemmas

For the proof of Theorem 1, we need some preparatory lemmas whose proofs will be in the appendix.

Lemma 3. *Assume that the constants q_{ij} satisfy*

$$\frac{q_{11}-1}{q_{k1}}<\min\left\{1,\frac{q_{1k}}{q_{kk}+1},\frac{q_{1j}}{q_{kj}},\ j=2,\ldots,m,\ j\neq k\right\}. \tag{17}$$

Then for all $h_{i-1},\alpha_i>0$, $j,i=1,\ldots,m$, there exist $C=C(h_{i-1},\alpha_i)>0$ and $\theta=\theta(\alpha_1)\in(0,1)$, such that

$$\alpha_1\frac{u_1^{q_{11}-1+\alpha_1}}{\prod_{j=2}^m u_j^{q_{1j}+\alpha_j}}\le\alpha_k\frac{u_1^{q_{k1}+\alpha_1}}{u_k^{q_{kk}+1+\alpha_k}\prod_{j=2,j\neq k}^m u_j^{q_{kj}+\alpha_j}}+C\left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^{\theta}, \tag{18}$$

$u_1\ge0$, $u_i\ge h_{i-1}$, $i=1,\ldots,m$, $k\in\{2,\ldots,m\}$.

Lemma 4 (see [9]). *Let $A=(a_{ij})_{1\le i,j\le m}$. Then one has:*

$$K_m^m=\det[m]\cdot\prod_{k=1}^{k=m-2}(\det[k])^{2^{(m-k-2)}},\quad m>2,\qquad K_2^2=\det[2], \tag{19}$$

where

$$K_m^l=K_{l-1}^{l-1}\cdot K_m^{l-1}-\left(H_m^{l-1}\right)^2,\quad l=3,\ldots,m, \tag{20}$$

$$H_m^l=\det_{1\le i,j\le m}\left(\left(a_{i,j}\right)_{\substack{i\neq(l+1),\ldots,m,\\ j\neq l,\ldots,(m-1)}}\right)\cdot\prod_{k=1}^{k=l-2}(\det[k])^{2^{(l-k-2)}},\quad l=3,\ldots,m-1, \tag{21}$$

$$K_m^2=a_{11}a_{mm}-\left(a_{1m}\right)^2, \tag{22}$$

$$H_m^2=a_{11}a_{2m}-a_{12}a_{1m}. \tag{23}$$

Lemma 5. *Let $\alpha_1>2\max\{1,\sum_{i=1}^m b_i/b_1\}$. One has*

$$K_l^l>S_l^l,\quad l=2,\ldots,m, \tag{24}$$

where

$$K_l^r=K_{r-1}^{r-1}\cdot K_l^{r-1}-\left[H_l^{r-1}\right]^2,\quad r=3,\ldots,l,$$

$$H_l^r=\det_{1\le i,j\le l}\left(\left(a_{i,j}\right)_{\substack{i\neq l,\ldots r+1\\ j\neq l-1,\ldots r}}\right)\cdot\prod_{k=1}^{k=r-2}(\det[k])^{2^{(r-k-2)}},\quad r=3,\ldots,l-1, \tag{25}$$

$$K_l^2=\alpha_1^2\alpha_l^2a_1a_l\left[\frac{\alpha_1-1}{\alpha_1}\frac{\alpha_l+1}{\alpha_l}-A_{1l}^2\right],$$

$$H_l^2=\alpha_1^2\alpha_2\alpha_la_1\sqrt{a_2a_l}\left[\frac{\alpha_1-1}{\alpha_1}A_{2l}-A_{12}A_{1l}\right].$$

Lemma 6 (see Masuda and Takahashi [4]). *Let $\mu,T>0$ and let $f_j=f_j(t)$ be a nonnegative integrable function on $[0,T)$ and $0<\theta_j<1$ $(j=1,\ldots,J)$. Let $W=W(t)$ be a positive function on $[0,T)$ satisfying the differential inequality*

$$\frac{dW(t)}{dt}\le-\mu W(t)+\sum_{j=1}^J f_j(t)W^{\theta_j}(t),\quad 0\le t<T. \tag{26}$$

Then, one has

$$W(t)\le\kappa,\quad 0\le t<T, \tag{27}$$

where κ is the maximal root of the algebraic equation:

$$x-\sum_{j=1}^J\left(\sup_{0<t<T}\int_0^t e^{-\mu(t-\xi)}f_j(\xi)\,d\xi\right)x^{\theta_j}=W(0). \tag{28}$$

4. Proofs

Proof of Theorem 1. Since u_1 satisfies $\partial_tu_1-a_1\Delta u_1>0$ on $\{u_1<\sigma/b_1\}$, the maximum principle implies $u_1\ge\delta:=\min(\sigma/b_1,\min u_0(x))>0$.

Differentiating $L(t)$ with respect to t yields

$$L'(t) = \int_\Omega \frac{d}{dt}\left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)dx$$
$$= \int_\Omega \left(\alpha_1 \frac{u_1^{\alpha_1-1}}{\prod_{j=2}^m u_j^{\alpha_j}}\partial_t u_1 - \sum_{i=2}^m \alpha_i \frac{u_1^{\alpha_1}}{u_i^{\alpha_i+1}\prod_{j=2,j\neq i}^m u_j^{\alpha_j}}\partial_t u_i\right)dx. \quad (29)$$

Replacing $\partial_t u_i$, $i = 1,\ldots,m$, by its expression from (1), we get

$$L'(t) = \int_\Omega \left(a_1\alpha_1 \frac{u_1^{\alpha_1-1}}{\prod_{j=2}^m u_j^{\alpha_j}}\Delta u_1 - \sum_{i=2}^m \alpha_i a_i \frac{u_1^{\alpha_1}}{u_i^{\alpha_i+1}\prod_{j=2,j\neq i}^m u_j^{\alpha_j}}\Delta u_i - b_1\alpha \frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}} + \sum_{i=2}^m b_i\alpha_i \frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}} + \alpha_1 \frac{u_1^{p_{11}-1+\alpha_1}}{\prod_{j=2}^m u_j^{p_{1j}+\alpha_j}} - \sum_{i=2}^m \alpha_i \frac{u_1^{p_{i1}+\alpha_1}}{u_k^{p_{ii}+1+\alpha_i}\prod_{j=2,j\neq i}^m u_j^{p_{i,j}+\alpha_j}} + \sigma\alpha_1 \frac{u_1^{\alpha_1-1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)dx \quad (30)$$
$$:= I + J,$$

where we have set

$$I = a_1\alpha_1 \int_\Omega \frac{u_1^{\alpha_1-1}}{\prod_{j=2}^m u_j^{\alpha_j}}\Delta u_1\,dx - \sum_{i=2}^m \alpha_i a_i \int_\Omega \frac{u_1^{\alpha_1}}{u_i^{\alpha_i+1}\prod_{j=2,j\neq i}^m u_j^{\alpha_j}}\Delta u_i\,dx, \quad (31)$$

$$J = \left(-b_1\alpha_1 + \sum_{i=2}^m b_i\alpha_i\right)L(t) + \alpha_1 \int_\Omega \frac{u_1^{p_{11}-1+\alpha_1}}{\prod_{j=2}^m u_j^{p_{1j}+\alpha_j}}dx - \sum_{i=2}^m \alpha_i \int_\Omega \frac{u_1^{p_{i1}+\alpha_1}}{u_k^{p_{ii}+1+\alpha_i}\prod_{j=2,j\neq i}^m u_j^{p_{ij}+\alpha_j}}dx + \sigma\alpha_1 \int_\Omega \frac{u_1^{\alpha_1-1}}{\prod_{j=2}^m u_j^{\alpha_j}}dx. \quad (32)$$

Estimation of I. We are going to show that $I \le 0$.

Using Green's formula, we obtain

$$I = \int_\Omega \left(a_1\alpha_1\left[-(\alpha_1-1)\frac{u_1^{\alpha_1-2}}{\prod_{j=2}^m u_j^{\alpha_j}}|\nabla u_1|^2 + \sum_{i=2}^m \alpha_i \frac{u_1^{\alpha_1-1}}{u_i^{\alpha_i+1}\prod_{j=2,j\neq i}^m u_j^{\alpha_j}}\nabla u_1\nabla u_i\right] + \sum_{i=2}^m a_i\,\alpha_i \cdot \left[\alpha_1 \frac{u_1^{\alpha_1-1}}{u_i^{\alpha_i+1}\prod_{j=2,j\neq i}^m u_j^{\alpha_j}}\nabla u_1\nabla u_i - (\alpha_i+1)\frac{u_1^{\alpha_1}}{u_i^{\alpha_i+2}\prod_{j=2,j\neq i}^m u_j^{\alpha_j}}|\nabla u_i|^2 - \sum_{\substack{k=2\\k\neq i}}^m \alpha_k \times \frac{u_1^{\alpha_1}}{u_k^{\alpha_k+1}u_i^{\alpha_i+1}\prod_{j=2,j\neq i,j\neq k}^m u_j^{\alpha_j}} \times \nabla u_k\nabla u_i\right]\right)dx,$$
$$= -\int_\Omega \left(\frac{u_1^{\alpha_1-2}}{\prod_{j=2}^m u_j^{\alpha_j+2}}(QT)\cdot T\right)dx, \quad (33)$$

where $Q = (a_{i,j})_{1\le i,j\le m}$ is defined in (8) and

$$T = \left(\prod_{j=2}^m u_j\nabla u_1, \ldots, \prod_{\substack{j=1\\j\neq i}}^m u_j\nabla u_i, \ldots, \prod_{j=1}^{m-1} u_j\nabla u_m\right)^t. \quad (34)$$

The matrix Q is positive definite if and only if all its associated minor matrices $\Delta_1, \Delta_2, \ldots, \Delta_m$ are positive. To see this, we have the following.

(1) $\Delta_1 = a_1\alpha_1(\alpha_1 - 1) > 0$. Using (5), we get $\det[1] > 0$.

(2) According to Lemma 4, we have

$$\det[2] = K_2^2 = \alpha_1^2\alpha_2^2 a_1 a_2\left[\frac{\alpha_1-1}{\alpha_1}\frac{\alpha_2+1}{\alpha_2} - A_{12}^2\right]. \quad (35)$$

Using (6) and (24) for $l = 2$, we get $\det[2] > 0$.

(3) Again according to Lemma 4, we have

$$K_3^3 = \det[3]\det[1]. \quad (36)$$

But $\det[1] > 0$, thus $\operatorname{sign}(K_3^3) = \operatorname{sign}(\det[3])$. Using (6) and (24) for $l = 3$, we get $\det[3] > 0$.

(4) We suppose that $\det[k] > 0$, $k = 1, 2, \ldots, l-1$ and prove that $\det[l] > 0$; thus

$$\det[k] > 0, \quad k = 1, \ldots, (l-1)$$
$$\Longrightarrow \prod_{k=1}^{k=l-2} (\det[k])^{2^{(l-k-2)}} > 0. \quad (37)$$

From Lemma 4,

$$K_l^l = \det[l] \cdot \prod_{k=1}^{k=l-2} (\det[k])^{2^{(l-k-2)}}. \quad (38)$$

This along with (37) yields

$$\operatorname{sign}(K_l^l) = \operatorname{sign}(\det[l]). \quad (39)$$

But from (6) and (24) $K_l^l > 0$; thus $\det[l] > 0$.

Consequently, we have $I \le 0$.

Estimation of J. We are going to estimate J by a function of $L(t)$.

According to the maximum principle, there exists C_0 depending on $\varphi_i(x)$, $i = 1, \ldots, m$, such that $u_i \ge C_0 > 0$, $i = 2, \ldots, m$. We then have

$$\frac{u_1^{\alpha_1-1}}{\prod_{j=2}^m u_j^{\alpha_j}} = \left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^{(\alpha_1-1)/\alpha_1} \prod_{j=2}^m \left(\frac{1}{u_j}\right)^{\alpha_j/\alpha_1}$$
$$\le \left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^{(\alpha_1-1)/\alpha_1} \left(\frac{1}{C_0}\right)^{\sum_{j=2}^m \alpha_j/\alpha_1}, \quad (40)$$

whereupon

$$\frac{u_1^{\alpha_1-1}}{\prod_{j=2}^m u_j^{\alpha_j}} \le C_2 \left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^{(\alpha_1-1)/\alpha_1},$$
$$\text{where } C_2 = \left(\frac{1}{C_0}\right)^{\sum_{j=2}^m \alpha_j/\alpha_1}. \quad (41)$$

We have

$$J \le \left(-b_1\alpha_1 + \sum_{i=2}^m b_i\alpha_i\right) L(t)$$
$$+ \alpha_1 \int_\Omega \frac{u_1^{q_{11}-1+\alpha_1}}{\prod_{j=2}^m u_j^{q_{1j}+\alpha_j}}\, dx$$
$$- \sum_{i=2}^m \alpha_i \int_\Omega \frac{u_1^{q_{i1}+\alpha_1}}{u_k^{q_{ii}+1+\alpha_i} \prod_{j=2, j\ne i}^m u_j^{q_{ij}+\alpha_j}}\, dx$$
$$+ \sigma\alpha_1 \int_\Omega C_2 \left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^{(\alpha_1-1)/\alpha_1} dx. \quad (42)$$

Using Lemma 3, we obtain

$$J \le \left(-b_1\alpha_1 + \sum_{i=2}^m b_i\alpha_i\right) L(t)$$
$$+ \int_\Omega C \left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^\theta dx$$
$$+ \sigma\alpha_1 \int_\Omega C_2 \left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^{(\alpha_1-1)/\alpha_1} dx. \quad (43)$$

Applying Hölder's inequality, we obtain

$$\int_\Omega C \left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^\theta dx$$
$$\le \left(\int_\Omega \frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}} dx\right)^\theta C(\operatorname{meas}(\Omega))^{1-\theta}. \quad (44)$$

So

$$\int_\Omega C \left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^\theta dx \le C_3 L^\theta(t),$$
$$C_3 = C(\operatorname{meas}(\Omega))^{1-\theta}. \quad (45)$$

Also, we have

$$\int_\Omega C_2 \left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^{(\alpha_1-1)/\alpha_1} dx$$
$$\le \left(\int_\Omega \left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right) dx\right)^{(\alpha_1-1)/\alpha_1}$$
$$\cdot \left(\int_\Omega (C_2)^{\alpha_1} dx\right)^{1/\alpha_1}. \quad (46)$$

So

$$\int_\Omega C_2 \left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^m u_j^{\alpha_j}}\right)^{(\alpha_1-1)/\alpha_1} dx \le C_4 L^{\frac{(\alpha_1-1)}{\alpha_1}}(t),$$
$$\text{where } C_4 = C_2(\operatorname{meas}(\Omega))^{1/\alpha_1}. \quad (47)$$

We then get

$$J \le \left(-b_1\alpha_1 + \sum_{i=2}^m b_i\alpha_i\right) L(t) + C_3 L^\theta(t)$$
$$+ \alpha_1\sigma C_4 L^{(\alpha_1-1)/\alpha_1}(t), \quad (48)$$

which implies

$$J \leq \left(-b_1\alpha_1 + \sum_{i=2}^{m} b_i\alpha_i\right) L(t) + C_5\left(L^{\theta}(t) + \alpha_1\sigma L^{(\alpha_1-1)/\alpha_1}(t)\right). \tag{49}$$

This yields the differential inequality:

$$L'(t) \leq \left(-b_1\alpha_1 + \sum_{i=2}^{m} b_i\alpha_i\right) L(t) + C_5\left(L^{\theta}(t) + \alpha_1\sigma L^{(\alpha_1-1)/\alpha_1}(t)\right). \tag{50}$$

Thus under conditions (5), (6), and (8), we obtain $-b_1\alpha_1 + \sum_{i=2}^{m} b_i\alpha_i < 0$; using Lemma 6 we deduce that $L(t)$ is bounded on $(0, T_{\max})$; that is, $L(t) \leq \gamma_1$, where γ_1 depends on $\varphi_i(x)$, $i = 1, \ldots, m$. □

Proof of Corollary 2 (L^{∞}-bounds). By Theorem 1, we have $u_1^{p_{i1}}/\prod_{j=2}^{m} u_j^{p_{ij}} \in L^{\infty}((0, T_{\max}), L^r(\Omega))$, $i = 2, \ldots, m$ for all $r > N/2$. By a simple argument relying on the variation-of-constants formula and the L^p-L^q-estimate (Proposition 48.4 see [10]), we deduce that u is uniformly bounded. Consequently, $T_{\max} = \infty$. □

Appendix

The purpose of this appendix is to prove the lemmas of Section 3 which have been used in the proof of Theorem 1.

Proof of Lemma 3. Inequality (18) is equivalent to

$$\alpha_1 \frac{u_1^{q_{11}-1}}{\prod_{j=2}^{m} u_j^{q_{1j}}} \leq \alpha_k \frac{u_1^{q_{k1}}}{u_k^{q_{kk}+1}\prod_{j=2, j\neq k}^{m} u_j^{q_{kj}}} + C\left(\frac{u_1^{\alpha_1}}{\prod_{j=2}^{m} u_j^{\alpha_j}}\right)^{\theta-1}. \tag{A.1}$$

Let us set $\zeta = (\alpha_k u_1^{q_{k1}})/(u_k^{q_{kk}+1}\prod_{j=2, j\neq k}^{m} u_j^{q_{kj}})$.

Now, we write

$$\alpha_1 \frac{u_1^{q_{11}-1}}{\prod_{j=2}^{m} u_j^{q_{1j}}} = \alpha_1 (\alpha_k)^{-(q_{11}-1)/q_{k1}} (\zeta)^{(q_{11}-1)/q_{k1}} \cdot \prod_{\substack{j=2\\ j\neq k}}^{m} (u_j)^{q_{kj}(q_{11}-1)/q_{k1} - q_{1j}} \cdot (u_k)^{(q_{kk}+1)(q_{11}-1)/q_{k1} - q_{1k}}. \tag{A.2}$$

For each ϵ such that $0 < \epsilon < \min\{1, q_{1k}/(q_{kk}+1), q_{1j}/q_{kj}, j = 2, \ldots, m, \text{and } j \neq k\} - (q_{11}-1)/q_{k1}$,

$$\begin{aligned}
\alpha_1 \frac{u_1^{q_{11}-1}}{\prod_{j=2}^{m} u_j^{q_{1j}}}
&= \alpha_1 (\alpha_k)^{-(q_{11}-1)/q_{k1}} (\zeta)^{(q_{11}-1)/q_{k1}+\epsilon} \times (\zeta)^{-\epsilon} \prod_{\substack{j=2\\ j\neq k}}^{m} (u_j)^{q_{kj}(q_{11}-1)/q_{k1} - q_{1j}} \times (u_k)^{(q_{kk}+1)(q_{11}-1)/q_{k1} - q_{1k}} \\
&= \alpha_1 (\alpha_k)^{-(q_{11}-1)/q_{k1}-\epsilon} (\zeta)^{(q_{11}-1)/q_{k1}+\epsilon} \times \left(\frac{1}{u_1^{\alpha_1}}\right)^{q_{k1}\epsilon/\alpha_1} \times \prod_{\substack{j=2\\ j\neq k}}^{m} (u_j)^{q_{k,j}(q_{11}-1)/q_{k,1} - q_{1j} + \epsilon q_{kj}} \times (u_k)^{(q_{kk}+1)(q_{11}-1)/q_{k,1} - q_{1k} + \epsilon(q_{kk}+1)} \\
&\leq \alpha_1 (\alpha_k)^{-(q_{11}-1)/q_{k1}-\epsilon} (\zeta)^{(q_{11}-1)/q_{k1}+\epsilon} \times \left(\frac{1}{u_1^{\alpha_1}}\right)^{q_{k1}\epsilon/\alpha_1} \times \prod_{\substack{j=2\\ j\neq k}}^{m} (h_j)^{q_{kj}(q_{11}-1)/q_{k1} - q_{1j} + \epsilon q_{kj}} \\
&\quad \times (h_k)^{(q_{kk}+1)(q_{11}-1)/q_{k1} - q_{1k} + \epsilon(q_{kk}+1)} \prod_{j=2}^{m} \left(\frac{u_j}{h_j}\right)^{\alpha j q_{k1}\epsilon/\alpha_1} \\
&\leq C_1 (\zeta)^{(q_{11}-1)/q_{k1}+\epsilon} \left(\frac{\prod_{j=2}^{m} u_j^{\alpha j}}{u_1^{\alpha_1}}\right)^{q_{k1}\epsilon/\alpha_1},
\end{aligned} \tag{A.3}$$

where

$$C_1 = \alpha_1 (\alpha_k)^{-(q_{11}-1)/q_{k1}-\epsilon} \times \prod_{\substack{j=2\\ j\neq k}}^{m} (h_j)^{q_{kj}(q_{11}-1)/q_{k1} - q_{1,j} + \epsilon q_{k,j} - \alpha j q_{k1}\epsilon/\alpha_1} \times (h_k)^{(q_{kk}+1)(q_{11}-1)/q_{k1} - q_{1k} + \epsilon(q_{kk}+1) - \alpha_k q_{k1}\epsilon/\alpha_1}. \tag{A.4}$$

Using Young's inequality for (A.3) with

$$C = C_1^{1+(q_{11}-1+q_{k1}\epsilon)/(q_{k1}-(q_{11}-1)-q_{k1}\epsilon)}, \qquad \theta = 1 - \frac{q_{k1}\epsilon}{\alpha_1\left(1 - (q_{11}-1)/q_{k1} - \epsilon\right)}, \tag{A.5}$$

where ϵ is sufficiently small, we get inequality (18). □

Proof of Lemma 4. We prove this lemma by induction.

For $m = 2$, we have $K_2^2 = \det[2]$.

We consider the case $m = 3$.

By using the well-known Dodgson condensation [11] for the symmetric 3-square matrix:

$$\det[1]\det[3] = \det[2] \det_{1\le i,j\le 3}\left[\left(a_{i,j}\right)_{i\neq 2, j\neq 2}\right] - \left[\det_{1\le i,j\le 3}\left[\left(a_{i,j}\right)_{i\neq 3, j\neq 2}\right]\right]^2. \tag{A.6}$$

But

$$\det[2] = K_2^2,$$
$$\det_{1\le i,j\le 3}\left[\left(a_{i,j}\right)_{\substack{i\neq 2\\ j\neq 2}}\right] = a_{11}a_{33} - \left(a_{13}\right)^2 = K_3^2, \tag{A.7}$$
$$\det_{1\le i,j\le 3}\left[\left(a_{i,j}\right)_{\substack{i\neq 2\\ j\neq 3}}\right] = a_{11}a_{23} - a_{12}a_{13} = H_3^2.$$

So

$$\det[1]\det[3] = K_2^2 \cdot K_3^2 - \left[H_3^2\right]^2. \tag{A.8}$$

Hence by using formula (20), formula (19) is correct for $m = 3$.

When $m \ge 4$, we suppose that formula (19) is correct for $(m-1), m-2, m-3, \ldots, 4$, and we prove it for m.

It is sufficient to prove that

$$K_m^{m-1} = \det_{1\le i,j\le m}\left(\left(a_{i,j}\right)_{\substack{i\neq m-1\\ j\neq m-1}}\right) \cdot \prod_{k=1}^{k=m-3} \left(\det[k]\right)^{2^{(m-k-3)}}. \tag{A.9}$$

By putting $l = m-1$ in formula (21), we get

$$H_m^{m-1} = \det_{1\le i,j\le m}\left(\left(a_{i,j}\right)_{\substack{i\neq m\\ j\neq m-1}}\right) \cdot \prod_{k=1}^{k=m-3} \left(\det[k]\right)^{2^{(m-k-3)}}. \tag{A.10}$$

From the mathematical induction proof, we have

$$K_{(m-1)}^{(m-1)} = \det[m-1] \cdot \prod_{k=1}^{k=m-3} \left(\det[k]\right)^{2^{(m-k-3)}}. \tag{A.11}$$

By putting $l = m$ in formula (20), we get

$$K_m^m = K_{m-1}^{m-1} \cdot K_m^{m-1} - \left(H_m^{m-1}\right)^2. \tag{A.12}$$

By replacing (A.9), (A.10), and (A.11) in (A.12), we obtain

$$\begin{aligned} K_m^m &= \prod_{k=1}^{k=m-3} \left(\det[k]\right)^{2^{(m-k-2)}} \cdot \det[m-2] \cdot \det[m] \\ &= \det[m] \cdot \prod_{k=1}^{k=m-2} \left(\det[k]\right)^{2^{(m-k-2)}}, \end{aligned} \tag{A.13}$$

and thus formula (19) is correct for m.

Now, we prove formula (A.9); we may generalize formula (A.9) as follows:

$$K_m^l = \det_{1\le i,j\le m}\left(\left(a_{ij}\right)_{\substack{i\neq m-1,\ldots l\\ j\neq m-1,\ldots l}}\right) \cdot \prod_{k=1}^{k=l-2} \left(\det[k]\right)^{2^{((l-2)-k)}}, \quad l = 3, \ldots, m-1. \tag{A.14}$$

Also, we prove formula (A.14) by induction. It is a second inductive proof included in the first one.

It is evident for $l = 2$.

For $l = 3$, formula (20) will be:

$$K_m^3 = K_2^2 \cdot K_m^2 - \left[H_m^2\right]^2. \tag{A.15}$$

Since we already know that

$$K_2^2 = \det[2],$$
$$K_m^2 = \det_{1\le i,j\le m}\left(\left(a_{i,j}\right)_{\substack{i\neq m-1,\ldots 2\\ j\neq m-1,\ldots 2}}\right), \tag{A.16}$$
$$H_m^2 = \det_{1\le i,j\le m}\left(\left(a_{i,j}\right)_{\substack{i\neq m-1,\ldots,2\\ j\neq m,\ldots,3}}\right),$$

simple substitution of these three formulas in the formula (A.15) followed by the application of the modified well-known Dodgson condensation which has been modified in [11] will lead to formula (A.14) for $l = 3$. directly.

When $l \ge 4$, we suppose that formula (A.14) is correct for $l-1$, and we prove it for l.

Formula (20) for $l-1$ reads

$$K_m^l = K_{l-1}^{l-1} \cdot K_m^{l-1} - \left[H_m^{l-1}\right]^2. \tag{A.17}$$

According to the first induction, we have

$$K_{(l-1)}^{(l-1)} = \det[l-1] \prod_{k=1}^{k=l-3} \left(\det[k]\right)^{2^{(l-k-3)}}. \tag{A.18}$$

According to the second induction, we have

$$K_m^{l-1} = \det_{1\le i,j\le m}\left(\left(a_{i,j}\right)_{\substack{i\neq m-1,\ldots,l-1\\ j\neq m-1,\ldots,l-1}}\right) \cdot \prod_{k=1}^{k=(l-3)} \left(\det[k]\right)^{2^{((l-3)-k)}}. \tag{A.19}$$

According to formula (21), we have:

$$H_m^{l-1} = \det_{1\le i,j\le m}\left(\left(a_{i,j}\right)_{\substack{i\neq m,\ldots,l\\ j\neq m-1,\ldots,l-1}}\right) \cdot \prod_{k=1}^{k=l-3} \left(\det[k]\right)^{2^{((l-3)-k)}}. \tag{A.20}$$

By replacing (A.18), (A.19), and (A.20) in (A.17) and by using the well-known Dodgson condensation, we obtain formula (A.14) for l. Therefore, the second inductive proof is finished and consequently the first one. □

Proof of Lemma 5. We prove this lemma by induction:

$$K_l^l > S_l^l, \quad l = 2, \ldots, m. \tag{A.21}$$

For $l = 2$, we have

$$\begin{aligned} K_2^2 &= \alpha_1^2\alpha_2^2 a_1 a_2 \left[\frac{\alpha_1 - 1}{\alpha_1}\frac{\alpha_2 + 1}{\alpha_2} - A_{12}^2\right] \\ &> \alpha_1^2\alpha_2^2 a_1 a_2 \left[\frac{1}{2\alpha_2} - A_{12}^2\right] \\ &= S_2^2. \end{aligned} \tag{A.22}$$

Because

$$\alpha_1 > 2, \text{ then } \frac{\alpha_1 - 1}{\alpha_1}\frac{\alpha_2 + 1}{\alpha_2} > \frac{1}{2\alpha_2}. \tag{A.23}$$

Assuming $l \geq 3$, we suppose (24) is true for $(l-1), l-2, l-3, \ldots, 3$, and we prove it for l. Hence, we aim to prove

$$\begin{aligned} &K_2^2 > S_2^2, \qquad K_3^3 > S_3^3, \qquad K_4^4 > S_4^4, \ldots, \\ &\qquad K_{l-1}^{l-1} > S_{l-1}^{l-1} \Longrightarrow K_l^l > S_l^l. \end{aligned} \tag{A.24}$$

Recall that

$$K_l^l = K_{l-1}^{l-1} K_l^{l-1} - \left[H_l^{l-1}\right]^2. \tag{A.25}$$

It is then sufficient to prove

$$K_l^{l-1} > S_l^{l-1}, \tag{A.26}$$

which will satisfy the inequality

$$\begin{aligned} K_l^l &= K_{l-1}^{l-1} K_l^{l-1} - \left[H_l^{l-1}\right]^2 \\ &> S_{l-1}^{l-1} S_l^{l-1} - \left[H_l^{l-1}\right]^2 = S_l^l. \end{aligned} \tag{A.27}$$

In order to prove (A.26), we first generalize it in the form

$$K_l^r > S_l^r, \quad r = 2, \ldots, l-1. \tag{A.28}$$

This can be proven by mathematical induction. It is a secondary inductive proof inside the primary one. For $r = 2$, it is evident that

$$K_l^2 > S_l^2. \tag{A.29}$$

For $r = 3$, the formula

$$K_l^3 = K_2^2 K_l^2 - \left[H_l^2\right]^2 > S_2^2 S_l^2 - \left[H_l^2\right]^2 = S_l^3 \tag{A.30}$$

is evident too.

When $r \geq 4$, we suppose formula (A.28) true for $l-2$:

$$K_l^{l-2} > S_l^{l-2} \tag{A.31}$$

and we prove it for $l-1$:

$$K_l^{l-1} > S_l^{l-1}. \tag{A.32}$$

We have

$$\begin{aligned} K_l^{l-1} &= K_{l-2}^{l-2} K_l^{l-2} - \left[H_l^{l-2}\right]^2 \\ &> S_{l-2}^{l-2} S_l^{l-2} - \left[H_l^{l-2}\right]^2 \\ &= S_l^{l-1}. \end{aligned} \tag{A.33}$$

Then

$$K_l^{l-1} > S_l^{l-1}. \tag{A.34}$$

Accordingly, we have

$$K_l^l > S_l^l. \tag{A.35}$$

This finishes the proof. □

Acknowledgment

The authors would like to express their deepest gratitude for Professor KIRANE (Université de la Rochelle, France) for all his valuable input and guidance throughout the research and authoring of this paper.

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