Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 493204, 9 pages
http://dx.doi.org/10.1155/2013/493204

Research Article

Exponential Collocation Method for Solutions of Singularly Perturbed Delay Differential Equations

Şuayip Yüzbaşı[1] and Mehmet Sezer[2]

[1] *Department of Mathematics, Faculty of Science, Akdeniz University, Antalya 07058, Turkey*
[2] *Department of Mathematics, Faculty of Science, Celal Bayar University, Manisa 45040, Turkey*

Correspondence should be addressed to Şuayip Yüzbaşı; syuzbasi@akdeniz.edu.tr

Received 16 June 2013; Accepted 22 July 2013

Academic Editor: Valery Y. Glizer

This paper deals with the singularly perturbed delay differential equations under boundary conditions. A numerical approximation based on the exponential functions is proposed to solve the singularly perturbed delay differential equations. By aid of the collocation points and the matrix operations, the suggested scheme converts singularly perturbed problem into a matrix equation, and this matrix equation corresponds to a system of linear algebraic equations. Also, an error analysis technique based on the residual function is introduced for the method. Four examples are considered to demonstrate the performance of the proposed scheme, and the results are discussed.

1. Introduction

The mathematical models of many practical phenomena in many areas of sciences often result in boundary-value problems of singularly perturbed delay differential equations, for example, the study of bistable devices [1], description of the human pupil-light reflex [2], a variety of models for physiological processes or diseases [3, 4], evolutionary biology [4], variational problems in control theory [5, 6], and so forth. These problems mainly depend on a small positive parameter and a delay parameter in such a way that the solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain. Also, this class of problems possesses boundary layers, that is, regions of rapid change in the solution near one of the boundary points.

In the recent years, many researchers have a great interest in singularly perturbed delay differential equation problems. For example, for these problems, Patidar and Sharma [7] have studied nonstandard finite difference methods, Kadalbajoo and Ramesh [8] have applied the hybrid method, Kadalbajoo and Sharma [9] have presented a numerical study, Rai and Sharma [10] have worked on a numerical scheme based fitted operator methods, Kadalbajoo and Ramesh [11] have studied the finite difference scheme, the hybrid method, and the fitted mesh methods, Amiraliyev and Erdogan [12] have presented a numerical study based on finite difference scheme and piecewise-uniform mesh, Kadalbajoo and Kumar [13] have applied the fitted mesh B-spline collocation method, and Kadalbajoo and Sharma [14] have presented a numerical study involved which finite difference scheme. In addition, Rai and Sharma [15] have solved the singularly perturbed differential difference equation arising in the modeling of neuronal variability by using fitted operator scheme, Lange and Miura [16] have given the singular perturbation analysis of boundary-value problems for differential difference equations and boundary-value problems, and Kadalbajoo and Sharma [17, 18] have studied the numerical solutions of singularly perturbed delay differential equations by various methods.

On the other hand, exponential polynomials or exponential functions have interesting applications in many optical and quantum electronics [19], some nonlinear phenomena modeled by partial differential equations [20], many statistical discussions (especially in data analysis) [21], the safety analysis of control synthesis [22], the problem of expressing mean-periodic functions [23], and the study of spectral synthesis [24, 25]. These polynomials are based on the exponential base set $\{1, e^{-x}, e^{-2x}, \ldots\}$.

Recently, Yüzbaşı and Sezer have studied the exponential polynomial solutions of the systems of linear differential equations in [26].

In this study, we consider the singularly perturbed delay differential equation

$$\begin{aligned} L[y(x)] &= \varepsilon y''(x) + p(x) y'(x-\delta) + r(x) y(x) \\ &= g(x), \quad 0 \le x \le b \end{aligned} \tag{1}$$

with the boundary conditions

$$y(0) = \alpha, \qquad y(b) = \beta, \tag{2}$$

where ε is a small positive parameter $(0 < \varepsilon \ll 1)$, δ is a small shifting parameter $0 < \delta \ll 1$, α and β are given constants, $y(x)$ is an unknown function, and $p(x)$ and $r(x)$ are the known functions defined on interval $0 \le x \le b < \infty$.

The aim of this paper is to give an approximate solution of the problems (1)-(2) in the form

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n e^{-nx}, \quad 0 \le x \le b, \tag{3}$$

where the exponential basis set is defined by $\{1, e^{-x}, e^{-2x}, \ldots, e^{-Nx}\}$ and a_n, $(n = 0, 1, 2, \ldots, N)$ are unknown coefficients.

To find a solution in the form (3) of (1) under the conditions (2), we will use the equally spaced collocation points

$$x_i = \frac{b}{N} i, \quad i = 0, 1, \ldots, N, \ 0 \le x \le b. \tag{4}$$

2. Matrix Relations for Exponential Functions

In this section, we construct the matrix relations related to the exponential solution (3). Note that these relations will be used in Section 3.

Firstly, the approximate solution $y_N(x)$ defined by (3) of (1) can be written in the matrix form

$$y(x) = \mathbf{E}(x)\mathbf{A}, \tag{5}$$

where

$$\begin{aligned} \mathbf{E}(x) &= \begin{bmatrix} 1 & e^{-x} & e^{-2x} & \cdots & e^{-Nx} \end{bmatrix}, \\ \mathbf{A} &= \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T. \end{aligned} \tag{6}$$

Also, the relation between $\mathbf{E}(x)$ and its first derivative $\mathbf{E}^{(1)}(x)$ is given by

$$\mathbf{E}^{(1)}(x) = \mathbf{E}(x)\mathbf{M} \tag{7}$$

and the relation between $\mathbf{E}(x)$ and its second derivative $\mathbf{E}^{(2)}(x)$ is in the form

$$\mathbf{E}^{(2)}(x) = \mathbf{E}(x)\mathbf{M}^2, \tag{8}$$

where

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & -2 & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & -N \end{bmatrix}. \tag{9}$$

By placing the relation (7) into first derivative of (5), we have

$$y^{(1)}(x) = \mathbf{E}(x)\mathbf{M}\mathbf{A}. \tag{10}$$

Similarly, from the relations (5) and (8), we obtain the matrix form

$$y^{(2)}(x) = \mathbf{E}(x)\mathbf{M}^2\mathbf{A}. \tag{11}$$

By writing $x \to x - \delta$ in (10), we get the relation

$$y^{(1)}(x-\delta) = \mathbf{E}(x-\delta)\mathbf{M}\mathbf{A}. \tag{12}$$

The relation between $\mathbf{E}(x-\delta)$ and $\mathbf{E}(x)$ is as follows:

$$\mathbf{E}(x-\delta) = \mathbf{E}(x)\c{S}_\delta\mathbf{A}, \tag{13}$$

where

$$\begin{aligned} \mathbf{E}(x) &= \begin{bmatrix} 1 & e^{-x} & e^{-2x} & \cdots & e^{-Nx} \end{bmatrix}, \\ \c{S}_\delta &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & e^{\delta} & 0 & \cdots & 0 \\ 0 & 0 & e^{2\delta} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{N\delta} \end{bmatrix}. \end{aligned} \tag{14}$$

By substituting (13) into (12), we have the matrix form

$$y^{(1)}(x-\delta) = \mathbf{E}(x)\c{S}_\delta\mathbf{M}\mathbf{A}. \tag{15}$$

3. Exponential Collocation Method

In this section, to compute the unknown coefficients in the approximate solution (3), we use the following procedure by using the matrix relations in Section 2.

Firstly, let us substitute the matrix relations (5), (11), (12), and (15) into (1) as follows:

$$\varepsilon \mathbf{E}(x)\mathbf{M}^2\mathbf{A} + p(x)\mathbf{E}(x)\text{Ş}_\delta\mathbf{M}\mathbf{A} + r(x)\mathbf{E}(x)\mathbf{A} = g(x). \quad (16)$$

The collocation points defined by (4) are placed into (16), and we have the system of the matrix equations as

$$\varepsilon \mathbf{E}(x_i)\mathbf{M}^2\mathbf{A} + p(x_i)\mathbf{E}(x_i)\text{Ş}_\delta\mathbf{M}\mathbf{A} + r(x_i)\mathbf{E}(x_i)\mathbf{A} = g(x_i), \quad i = 0,1,\ldots,N. \quad (17)$$

The system can be written in the matrix form

$$\left\{\boldsymbol{\varepsilon}\mathbf{E}\mathbf{M}^2 + \mathbf{P}\mathbf{E}\text{Ş}_\delta\mathbf{M} + \mathbf{R}\mathbf{E}\right\}\mathbf{A} = \mathbf{G}, \quad (18)$$

where

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \varepsilon & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \varepsilon \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} \mathbf{E}(x_0) \\ \mathbf{E}(x_1) \\ \vdots \\ \mathbf{E}(x_N) \end{bmatrix} = \begin{bmatrix} 1 & e^{-x_0} & e^{-2x_0} & \cdots & e^{-Nx_0} \\ 1 & e^{-x_1} & e^{-2x_1} & \cdots & e^{-Nx_1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & e^{-x_N} & e^{-2x_N} & \cdots & e^{-Nx_N} \end{bmatrix},$$

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & -2 & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & -N \end{bmatrix},$$

$$\mathbf{P} = \begin{bmatrix} p(x_0) & 0 & 0 & 0 \\ 0 & p(x_1) & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & p(x_N) \end{bmatrix},$$

$$\text{Ş}_\delta = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & e^{\delta} & 0 & \cdots & 0 \\ 0 & 0 & e^{2\delta} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{N\delta} \end{bmatrix},$$

$$\mathbf{R} = \begin{bmatrix} r(x_0) & 0 & 0 & 0 \\ 0 & r(x_1) & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & r(x_N) \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}. \quad (19)$$

We note that the matrix equation (18) corresponds to a system of $(N+1)$ algebraic equations with the $(N+1)$ unknown coefficients $a_0, a_1, \ldots, a_N$.

Briefly, (18) can be expressed in the form

$$\mathbf{W}\mathbf{A} = \mathbf{G} \quad \text{or} \quad [\mathbf{W};\mathbf{G}], \quad (20)$$

where

$$\mathbf{W} = \boldsymbol{\varepsilon}\mathbf{E}\mathbf{M}^2 + \mathbf{P}\mathbf{E}\text{Ş}_\delta\mathbf{M} + \mathbf{R}\mathbf{E}. \quad (21)$$

From the relation (5), the matrix forms of the conditions (2) are written as

$$y(0) = \mathbf{E}(0)\mathbf{A} = [\alpha], \qquad y(b) = \mathbf{E}(b)\mathbf{A} = [\beta]. \quad (22)$$

Briefly, we write the above matrix forms of the conditions as follows:

$$\begin{aligned} \mathbf{C}_1\mathbf{A} = [\alpha] \quad &\text{or} \quad [\mathbf{C}_1;\alpha], \\ \mathbf{C}_2\mathbf{A} = [\beta] \quad &\text{or} \quad [\mathbf{C}_2;\beta], \end{aligned} \quad (23)$$

where

$$\begin{aligned} \mathbf{C}_1 &= \mathbf{E}(0) = [c_{1\,0} \;\; c_{1\,1} \;\; c_{1\,2} \;\; \cdots \;\; c_{1\,N}], \\ \mathbf{C}_2 &= \mathbf{E}(b) = [c_{2\,0} \;\; c_{2\,1} \;\; c_{2\,2} \;\; \cdots \;\; c_{2\,N}]. \end{aligned} \quad (24)$$

To obtain the solution of (1) under conditions (2), we replace the row matrices (23) with any two rows of the matrix (20), and thus we have the augmented matrix

$$\widetilde{\mathbf{W}}\mathbf{A} = \widetilde{\mathbf{G}}. \quad (25)$$

For simplicity, if the last two rows of the matrix (20) are replaced, the augmented matrix (25) becomes

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}] = \begin{bmatrix} w_{0\,0} & w_{0\,1} & w_{0\,2} & \cdots & w_{0\,N} & ; & g(x_0) \\ w_{1\,0} & w_{1\,1} & w_{1\,2} & \cdots & w_{1\,N} & ; & g(x_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{N-2\,0} & w_{N-2\,1} & w_{N-2\,2} & \cdots & w_{N-2\,N} & \vdots & g(x_{N-2}) \\ c_{1\,0} & c_{1\,1} & c_{1\,2} & \cdots & c_{1\,N} & ; & \alpha \\ c_{2\,0} & c_{2\,1} & c_{2\,2} & \cdots & c_{2\,N} & ; & \beta \end{bmatrix}. \quad (26)$$

However, we do not have to replace the last rows. For example, if the matrix $\mathbf{W}$ is singular, then the rows that have the same factor or all zeros are replaced.

If $\operatorname{rank}\widetilde{\mathbf{W}} = \operatorname{rank}[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}] = N+1$, then the coefficients $a_0, a_1, \ldots, a_N$ are uniquely determined by

$$\mathbf{A} = \widetilde{\mathbf{W}}^{-1}\widetilde{\mathbf{G}}, \quad (27)$$

where

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T. \quad (28)$$

By substituting the determined coefficients $a_0, a_1, \ldots, a_N$ into (3), we obtain the approximate solution

$$y_N(x) = \sum_{n=0}^{N} a_n e^{-nx}. \quad (29)$$

On the other hand, when $|\widetilde{\mathbf{W}}| = 0$, if rank $\widetilde{\mathbf{W}}$ = rank$[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] < N + 1$, then we may find a particular solution. Otherwise if rank $\widetilde{\mathbf{W}} \neq$ rank$[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] < N + 1$, then there is not a solution.

4. Error Estimation Based on Residual Function and Improvement of Solution

In this section, we apply the error estimation technique in [27–29] and the residual correction method in [30, 31] for our method and the problems (1)-(2). For our purpose, let us define the residual function for the present method as

$$R_N(x) = L[y_N(x)] - g(x), \quad (30)$$

where $y_N(x)$ denotes the approximate solution (29) of the problems (1)-(2). Therefore, $y_N(x)$ satisfies

$$\begin{aligned} L[y_N(x)] &= \varepsilon y_N^{(2)}(x) + p(x)\, y_N^{(1)}(x-\delta) + q(x)\, y_N(x) \\ &= g(x) + R_N(x) \end{aligned} \quad (31)$$

and the conditions

$$y_N(0) = \alpha, \qquad y_N(b) = \beta. \quad (32)$$

If $y(x)$ is the exact solution of the problems (1)-(2), then

$$e_N(x) = y(x) - y_N(x) \quad (33)$$

becomes the error function. By substituting (33) into the problem (1)-(2) and by using (30), we obtain the error differential equation

$$L[e_N(x)] = L[y(x)] - L[y_N(x)] = -R_N(x). \quad (34)$$

By using (33), the inhomogeneous conditions (2) and (32) are reduced to the homogeneous conditions

$$e_N(0) = 0, \qquad e_N(b) = 0. \quad (35)$$

From (34) and (35), we can clearly write the error problem

$$\begin{gathered} \varepsilon e_N^{(2)}(x) + p(x)\, e_N^{(1)}(x-\delta) + q(x)\, e_N(x) = -R_N(x), \\ e_N(0) = 0, \qquad e_N(b) = 0. \end{gathered} \quad (36)$$

By solving the problem (36) in the same way as Section 3, the approximation $e_{N,M}(x)$ is obtained for $e_N(x)$.

Consequently, by summing the exponential polynomial solution $y_N(x)$ and the estimated error function $e_{N,M}(x)$, we obtain the corrected exponential solution

$$y_{N,M}(x) = y_N(x) + e_{N,M}(x). \quad (37)$$

We note that the errors $e_N(x_i) = y(x_i) - y_N(x_i)$, $(0 \leq x_i \leq b)$ can be estimated by the error function $e_{N,M}(x)$ when the exact solution of (1) is unknown.

5. Numerical Examples

In this section, we apply the presented method to some examples. In examples, the terms $y(x)$, $y_N^{\varepsilon,\delta}(x)$, $y_{N,M}^{\varepsilon,\delta}(x)$, and $|e_{N,M}^{\varepsilon,\delta}(x)|$, respectively, represent the exact solution, the approximate solution, the corrected approximate solution, and the estimated absolute error function. Also, $e_{N,M}^{\varepsilon,\delta} = \max\{|e_{N,M}^{\varepsilon,\delta}(x)|, 0 \leq x \leq b\}$ denotes the estimated maximum error for the values ε, δ, N, and M.

Example 1 (see [13]). Firstly, let us consider the singularly-perturbed delay differential equation

$$\begin{gathered} \varepsilon y''(x) + (1+x)\, y'(x-\delta) - e^{-x} y(x) = 1, \\ 0 \leq x \leq 1 \end{gathered} \quad (38)$$

with the boundary conditions

$$y(0) = 0, \qquad y(1) = 1. \quad (39)$$

Firstly, we obtain the approximate solutions $y_N^{\varepsilon}(x)$ for various values of N by the presented method in Section 3. Secondly, the approximate solutions are corrected by the residual correction technique for various values of M. Hence, the corrected approximate solutions $y_{N,M}^{\varepsilon,\delta}(x)$ are obtained. In Table 1, we give the estimated maximum absolute errors for various values of $\varepsilon = 2\delta$, N, and M. Figures 1(a), 1(b), and 1(c) display the corrected approximate solutions $y_{N,M}^{\varepsilon}(x)$ for some values of $\varepsilon = 2\delta$, N, and M.

Example 2 (see [7]). Now, we consider the singularly perturbed delay differential equation

$$\varepsilon y''(x) - e^{x} y'(x-\delta) - y(x) = 0, \quad 0 \leq x \leq 1 \quad (40)$$

with the boundary conditions

$$y(0) = 1, \qquad y(1) = 1. \quad (41)$$

In Table 2, the estimated maximum absolute errors for various values of $\varepsilon = 2\delta$, N, and M are presented. For various values of $\varepsilon = 2\delta$, N, and M, Figures 2(a), 2(b), 2(c), and 2(d) show the graphs of the corrected approximate solutions $y_{N,M}^{\varepsilon,\delta}(x)$.

Table 1: Estimated maximum absolute errors for various values of $\varepsilon = 2\delta$, N, and M of (38).

ε	Estimated absolute errors					
	$e^{\varepsilon}_{3,8}$	$e^{\varepsilon}_{5,10}e_{5,10}$	$e^{\varepsilon}_{8,12}$	$e^{\varepsilon}_{11,16}$	$e^{\varepsilon}_{14,19}$	$e^{\varepsilon}_{20,22}$
2^{-6}	$5.7489e-002$	$5.4037e-002$	$3.1683e-002$	$2.2311e-001$	$5.1323e-002$	$8.3183e-002$
2^{-7}	$5.6434e-002$	$5.6931e-002$	$3.2954e-002$	$2.4047e-001$	$4.8550e-002$	$9.8938e-002$
2^{-8}	$5.5986e-002$	$5.8328e-002$	$3.3947e-002$	$2.5350e-001$	$6.1626e-002$	$1.4590e-001$
2^{-9}	$5.5792e-002$	$5.8965e-002$	$3.4384e-002$	$2.5973e-001$	$4.5761e-002$	$2.1930e-001$
2^{-10}	$5.5703e-002$	$5.9264e-002$	$3.4579e-002$	$2.6261e-001$	$7.3528e-002$	$2.4933e-001$
2^{-11}	$5.5661e-002$	$5.9409e-002$	$3.4670e-002$	$2.6405e-001$	$4.4186e-002$	$1.9391e-001$
2^{-12}	$5.5641e-002$	$5.9480e-002$	$3.4714e-002$	$2.6472e-001$	$5.6664e-002$	$2.1338e-001$

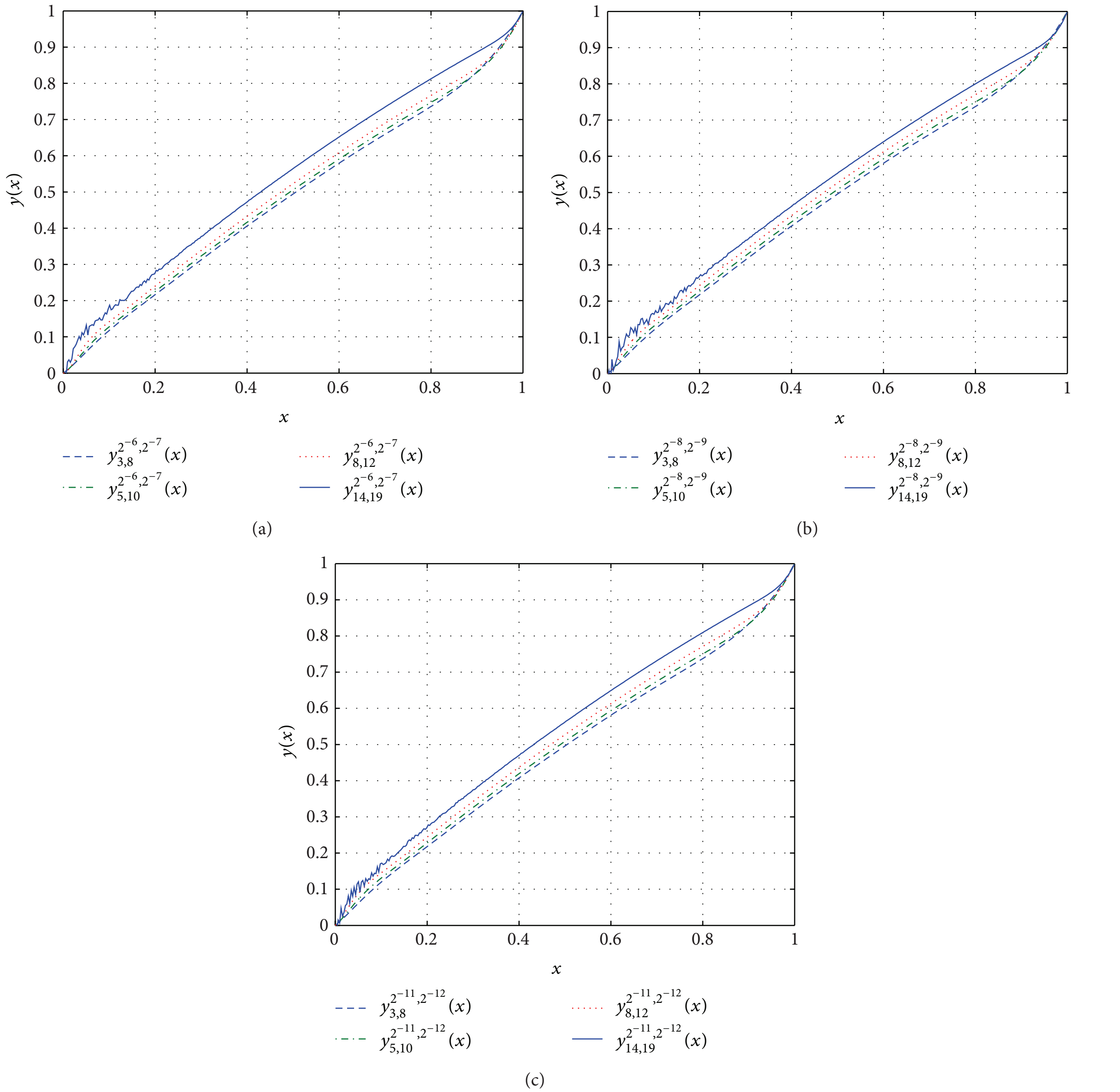


Figure 1: For $\delta = 0.5\varepsilon$ and various values of N and M of (38), (a) graph of the approximate solutions for $\varepsilon = 2^{-6}$, (b) graph of the approximate solutions for $\varepsilon = 2^{-8}$, and (c) graph of the approximate solutions for $\varepsilon = 2^{-11}$.

Table 2: Estimated maximum absolute errors for various values of $\varepsilon = 2\delta$, N, and M of (40).

ε	Estimated absolute errors				
	$e^{\varepsilon}_{3,6}$	$e^{\varepsilon}_{6,8}$	$e^{\varepsilon}_{9,12}$	$e^{\varepsilon}_{12,15}$	$e^{\varepsilon}_{16,19}$
2^{-5}	$3.3762e-001$	$3.5893e-001$	$4.5710e-001$	$3.0570e-001$	$3.1328e-001$
2^{-6}	$3.6939e-001$	$3.7298e-001$	$5.2769e-001$	$4.4360e-001$	$5.4311e-001$
2^{-7}	$3.8483e-001$	$3.8095e-001$	$5.7945e-001$	$9.0343e-001$	$6.8159e-001$
2^{-8}	$3.9191e-001$	$3.8524e-001$	$6.0724e-001$	$1.4068e+000$	$9.6851e-001$
2^{-9}	$3.9515e-001$	$3.8739e-001$	$6.1946e-001$	$1.7759e+000$	$1.0094e+000$
2^{-10}	$3.9668e-001$	$3.8844e-001$	$6.2446e-001$	$1.9697e+000$	$8.9508e-001$
2^{-11}	$3.9741e-001$	$3.8895e-001$	$6.2654e-001$	$2.0530e+000$	$8.4631e-001$

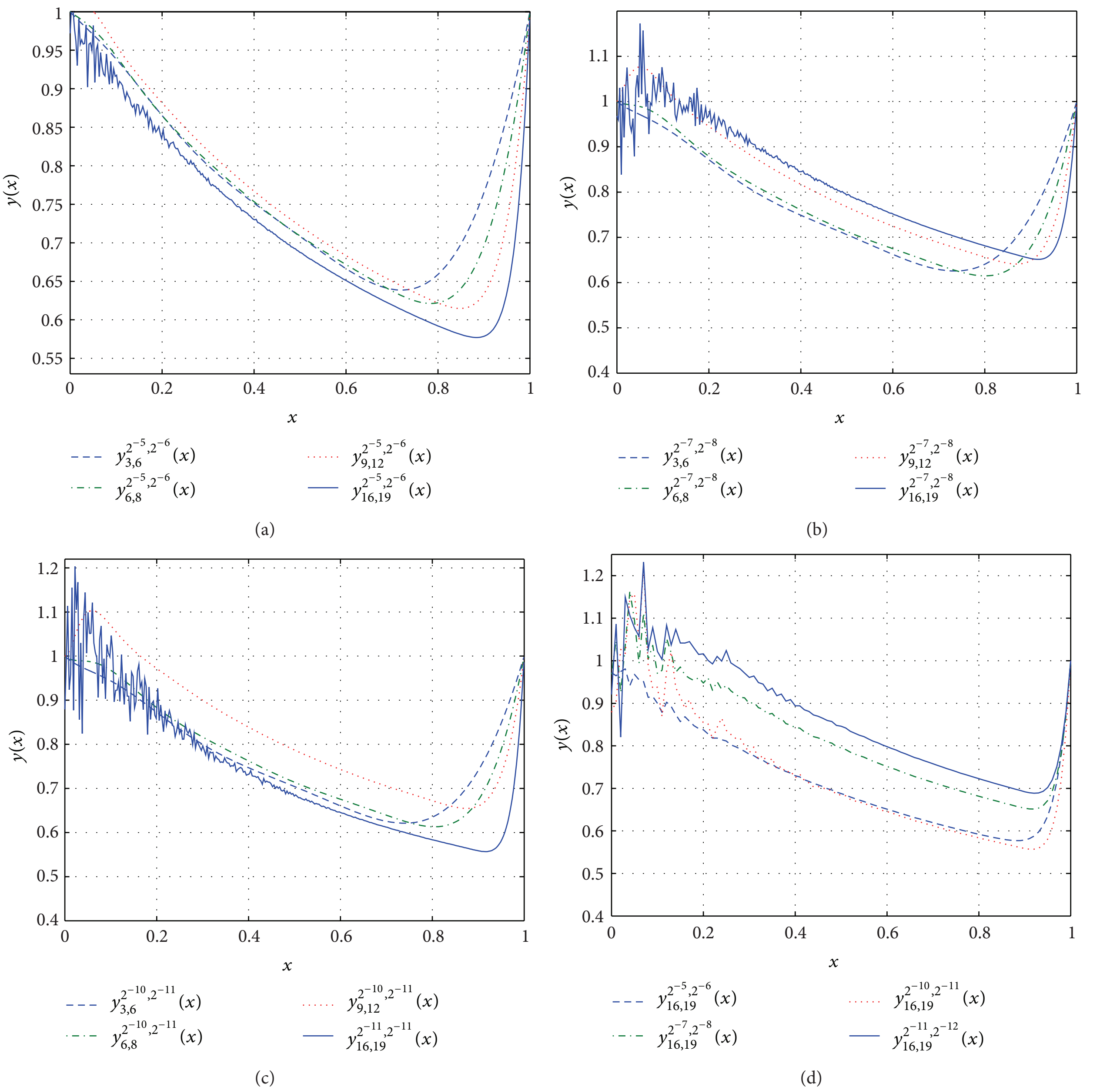
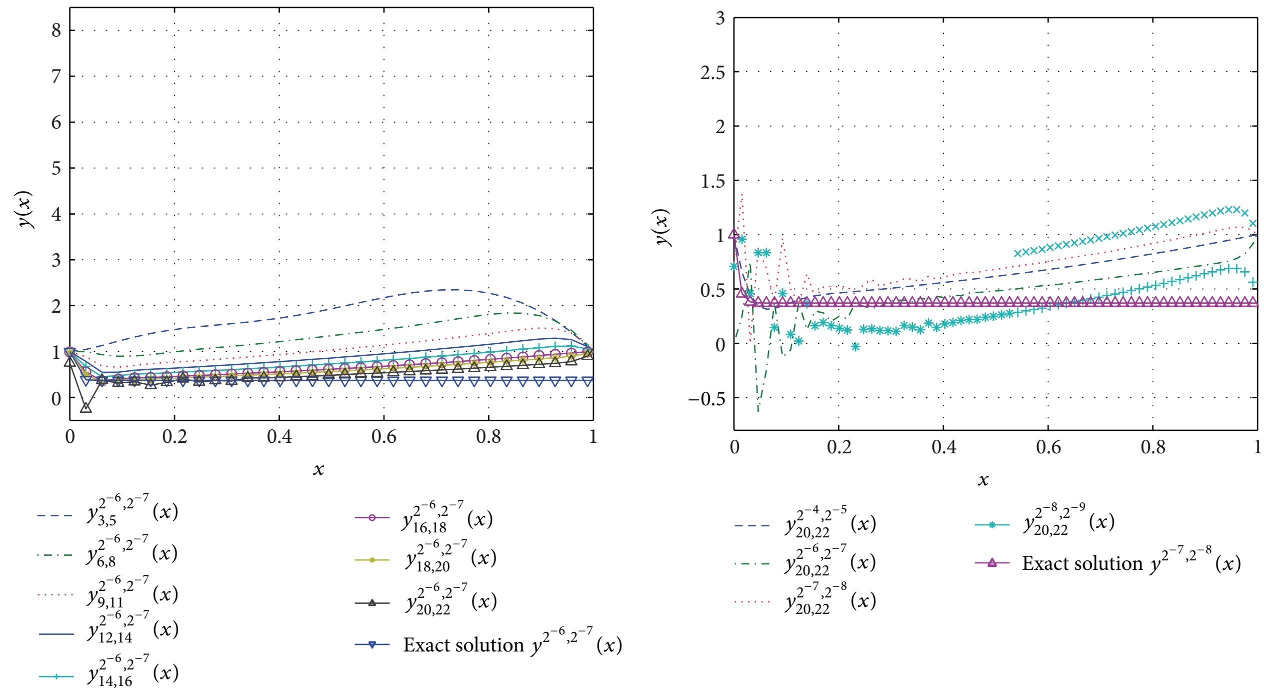


Figure 2: Graphs of the approximate solutions of (40), (a) for $\varepsilon = 2^{-5}$, $\delta = 0.5\varepsilon$, and various values of N and M, (b) for $\varepsilon = 2^{-7}$, $\delta = 0.5\varepsilon$, and various values of N and M, (c) for $\varepsilon = 2^{-10}$, $\delta = 0.5\varepsilon$, and various values of N and M, and (d) for $\varepsilon = 2^{-5}, 2^{-7}, 2^{-10}$, and 2^{-11}, $\delta = 0.5\varepsilon$, $N = 16$, and $M = 19$.



(a) (b)

Figure 3: For (42), (a) comparison of the exact solution and the approximate solutions for $\varepsilon = 2^{-6}$, $\delta = 0.5\varepsilon$, and various values of N and M and (b) comparison of the exact solution for $\varepsilon = 2^{-7}$, and $\delta = 0.5\varepsilon$, and the approximate solutions for $\varepsilon = 2^{-4}, 2^{-6}, 2^{-7}$, and 2^{-8}, $\delta = 0.5\varepsilon$, $N = 20$, and $M = 22$.

Example 3 (see [9]). Let us consider the boundary-value problem

$$\varepsilon y''(x) + y'(x-\delta) - y(x) = 0, \quad 0 \le x \le 1, \qquad y(0) = 1, \quad y(1) = 1. \tag{42}$$

The exact solution of this problem is given by

$$y(x) = \frac{(1-e^m)e^{kx} + (e^k - 1)e^{mx}}{e^k - e^m}, \tag{43}$$

where

$$k = \frac{-1 + \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)}, \qquad m = \frac{-1 - \sqrt{1 + 4(\varepsilon - \delta)}}{2(\varepsilon - \delta)}. \tag{44}$$

For some values of $\varepsilon = 2\delta$, the corrected approximate solutions $y^{\varepsilon,\delta}_{N,M}(x)$ are compared with the exact solution in Figures 3(a) and 3(b).

Example 4 (see [14]). Finally, we consider the problem

$$\varepsilon y''(x) + 0.25y'(x-\delta) - y(x) = 0, \quad 0 \le x \le 1, \qquad y(0) = 1, \quad y(1) = 0. \tag{45}$$

Figure 4(a) displays the corrected approximate solutions $y^{\varepsilon,\delta}_{N,M}(x)$ for $\varepsilon = 0.5$, $\delta = 2^{-2}$, and various values of N and M. The estimated error functions for these approximate solutions are shown in Figure 4(b). It is seen from Figure 4(b) that the absolute errors decrease while value of N increases. In Figure 4(c), we show the corrected approximate solutions $y^{\varepsilon}_{N,M}(x)$ for $N = 18$, $M = 19$, and different values of $\varepsilon = 0.5\delta$. Figure 4(d) shows the estimated error functions for the approximate solutions in Figure 4(c).

6. Conclusions

In this paper, a numerical scheme based on the exponential functions and the collocation points is presented for the singularly perturbed delay differential equations. Numerical examples are given to demonstrate the applicability and the efficiency of the method. Since the exact solutions of the problems in Examples 1, 2, and 4 are not available, we have computed the estimated maximum absolute errors. For Examples 1 and 2, the maximum absolute errors for some values of ε, δ, N, and M are tabulated in Tables 1 and 2. Also, the approximate solutions for different values of ε, δ, N, and M are compared with the exact solution. It is seen from Figure 3(a) that the accuracies of the approximate solutions increase while N and M increase. However, when values of ε and δ are increased, it is observed from Tables 1 and 2 and Figure 4(c) that the errors usually increase. In addition, the estimated absolute error functions are shown in Figure 4(b) for Example 3. It is seen from Figure 4(b) that the errors decrease as N and M increase. Moreover, the approximate

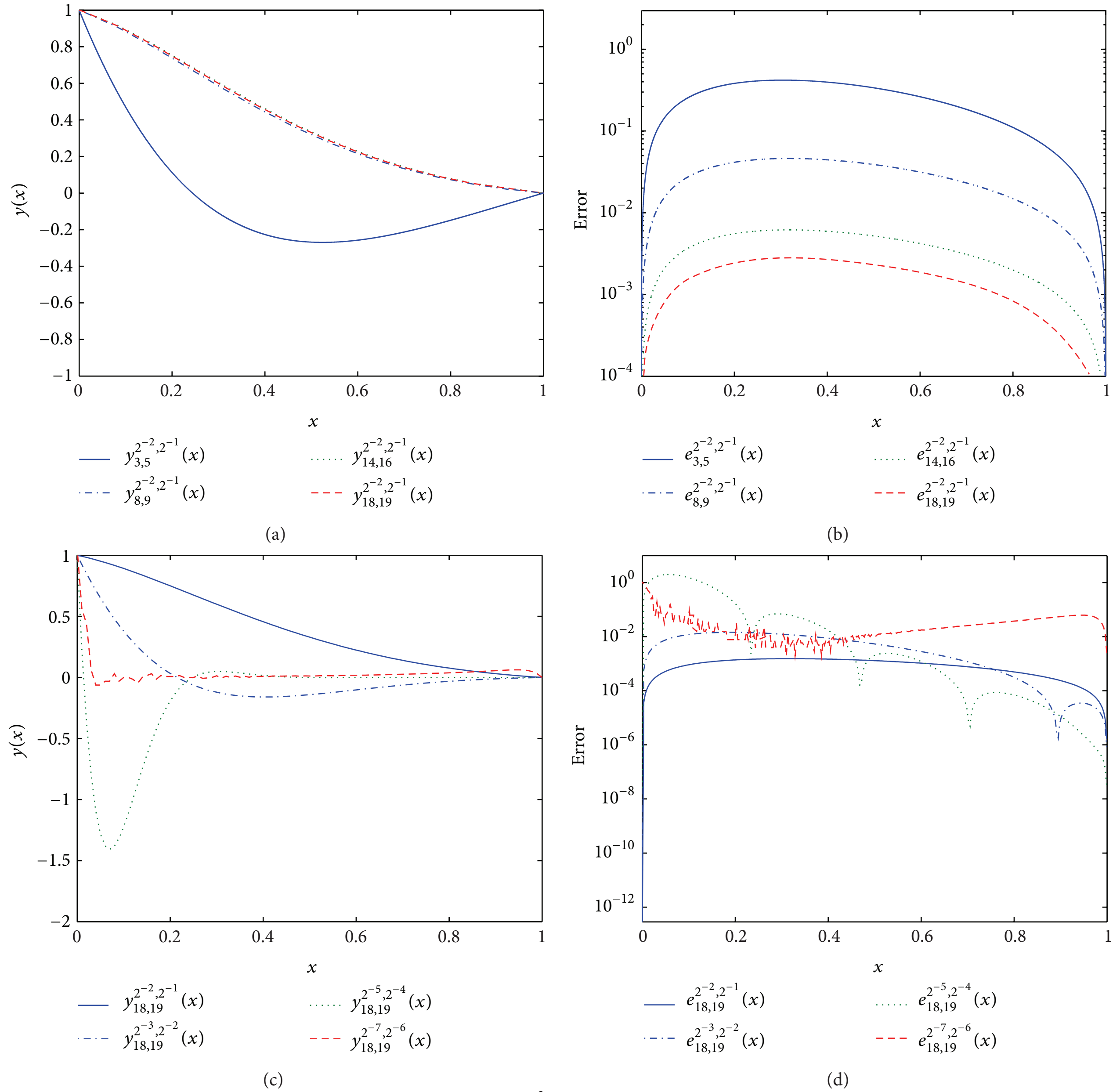


Figure 4: For (45), (a) graph of the approximate solutions for $\varepsilon = 2^{-2}$, $\delta = 2\varepsilon$, and various values of N and M, (b) comparison of the estimated error functions for $\varepsilon = 2^{-2}$, $\delta = 2\varepsilon$, and various values of N and M, (c) graph of the approximate solutions for $\varepsilon = 2^{-2}, 2^{-3}, 2^{-5}, 2^{-7}$, $\delta = 2\varepsilon$, $N = 18$, and $M = 19$, and (d) graph of the estimated error functions for $\varepsilon = 2^{-2}, 2^{-3}, 2^{-5}$, and 2^{-7}, $\delta = 2\varepsilon$, $N = 18$, and $M = 19$.

solutions can be very easily obtained using the software programs.

Acknowledgments

The authors would like to thank the referees for their constructive comments and suggestions to improve the paper. The first author is supported by *the Scientific Research Project Administration* of Akdeniz University.

References

[1] M. W. Derstine, H. M. Gibbs, F. A. Hopf, and D. L. Kaplan, "Bifurcation gap in a hybrid optically bistable system," *Physical Review A*, vol. 26, no. 6, pp. 3720–3722, 1982.

[2] A. Longtin and J. G. Milton, "Complex oscillations in the human pupil light reflex with "mixed" and delayed feedback," *Mathematical Biosciences*, vol. 90, no. 1-2, pp. 183–199, 1988.

[3] M. C. Mackey and L. Glass, "Oscillation and chaos in physiological control systems," *Science*, vol. 197, no. 4300, pp. 287–289, 1977.

[4] M. Ważewska-Czyżewska and A. Lasota, "Mathematical problems of the dynamics of a system of red blood cells," *Matematyka Stosowana*, vol. 6, pp. 23–40, 1976.

[5] V. Y. Glizer, "Asymptotic solution of a singularly perturbed set of functional-differential equations of Riccati type encountered in the optimal control theory," *Nonlinear Differential Equations and Applications*, vol. 5, no. 4, pp. 491–515, 1998.

[6] V. Y. Glizer, "Asymptotic solution of a boundary-value problem for linear singularly-perturbed functional differential equations arising in optimal control theory," *Journal of Optimization Theory and Applications*, vol. 106, no. 2, pp. 309–335, 2000.

[7] K. C. Patidar and K. K. Sharma, "ε-uniformly convergent non-standard finite difference methods for singularly perturbed differential difference equations with small delay," *Applied Mathematics and Computation*, vol. 175, no. 1, pp. 864–890, 2006.

[8] M. K. Kadalbajoo and V. P. Ramesh, "Hybrid method for numerical solution of singularly perturbed delay differential equations," *Applied Mathematics and Computation*, vol. 187, no. 2, pp. 797–814, 2007.

[9] M. K. Kadalbajoo and K. K. Sharma, "Numerical analysis of singularly perturbed delay differential equations with layer behavior," *Applied Mathematics and Computation*, vol. 157, no. 1, pp. 11–28, 2004.

[10] P. Rai and K. K. Sharma, "Numerical analysis of singularly perturbed delay differential turning point problem," *Applied Mathematics and Computation*, vol. 218, no. 7, pp. 3483–3498, 2011.

[11] M. K. Kadalbajoo and V. P. Ramesh, "Numerical methods on Shishkin mesh for singularly perturbed delay differential equations with a grid adaptation strategy," *Applied Mathematics and Computation*, vol. 188, no. 2, pp. 1816–1831, 2007.

[12] G. M. Amiraliyev and F. Erdogan, "Uniform numerical method for singularly perturbed delay differential equations," *Computers & Mathematics with Applications*, vol. 53, no. 8, pp. 1251–1259, 2007.

[13] M. K. Kadalbajoo and D. Kumar, "Fitted mesh B-spline collocation method for singularly perturbed differential-difference equations with small delay," *Applied Mathematics and Computation*, vol. 204, no. 1, pp. 90–98, 2008.

[14] M. K. Kadalbajoo and K. K. Sharma, "A numerical method based on finite difference for boundary value problems for singularly perturbed delay differential equations," *Applied Mathematics and Computation*, vol. 197, no. 2, pp. 692–707, 2008.

[15] P. Rai and K. K. Sharma, "Numerical study of singularly perturbed differential-difference equation arising in the modeling of neuronal variability," *Computers & Mathematics with Applications*, vol. 63, no. 1, pp. 118–132, 2012.

[16] C. G. Lange and R. M. Miura, "Singular perturbation analysis of boundary value problems for differential-difference equations. V. Small shifts with layer behavior," *SIAM Journal on Applied Mathematics*, vol. 54, no. 1, pp. 249–272, 1994.

[17] M. K. Kadalbajoo and K. K. Sharma, "Numerical treatment of boundary value problems for second order singularly perturbed delay differential equations," *Computational & Applied Mathematics*, vol. 24, no. 2, pp. 151–172, 2005.

[18] M. K. Kadalbajoo and K. K. Sharma, "An ε-uniform fitted operator method for solving boundary-value problems for singularly perturbed delay differential equations: layer behavior," *International Journal of Computer Mathematics*, vol. 80, no. 10, pp. 1261–1276, 2003.

[19] F. Alharbi, "Predefined exponential basis set for half-bounded multi domain spectral method," *Applied Mathematics*, vol. 1, pp. 146–152, 2010.

[20] M. M. Alipour, G. Domairry, and A. G. Davodi, "An application of exp-function method to approximate general and explicit solutions for nonlinear Schrödinger equations," *Numerical Methods for Partial Differential Equations*, vol. 27, no. 5, pp. 1016–1025, 2011.

[21] R. Shanmugam, "Generalized exponential and logarithmic polynomials with statistical applications," *International Journal of Mathematical Education in Science and Technology*, vol. 19, no. 5, pp. 659–669, 1988.

[22] M. Xu, L. Chen, Z. Zeng, and Z.-B. Li, "Reachability analysis of rational eigenvalue linear systems," *International Journal of Systems Science*, vol. 41, no. 12, pp. 1411–1419, 2010.

[23] H. Ouerdiane and M. Ounaies, "Expansion in series of exponential polynomials of mean-periodic functions," *Complex Variables and Eliptic Equations*, vol. 57, no. 5, pp. 469–487, 2012.

[24] L. Székelyhidi, "On the extension of exponential polynomials," *Mathematica Bohemica*, vol. 125, no. 3, pp. 365–370, 2000.

[25] K. Ross, *Abstract Harmonic Analysis I, II*, Springer, Berlin, Germany, 1963.

[26] Ş. Yüzbaşı and M. Sezer, "An exponential matrix method for solving systems of linear differential equations," *Mathematical Methods in the Applied Sciences*, vol. 36, no. 3, pp. 336–348, 2013.

[27] S. Shahmorad, "Numerical solution of the general form linear Fredholm-Volterra integro-differential equations by the Tau method with an error estimation," *Applied Mathematics and Computation*, vol. 167, no. 2, pp. 1418–1429, 2005.

[28] Ş. Yüzbaşı, "An efficient algorithm for solving multi-pantograph equation systems," *Computers & Mathematics with Applications*, vol. 64, no. 4, pp. 589–603, 2012.

[29] Ş. Yüzbaşı and M. Sezer, "An improved Bessel collocation method with a residual error function to solve a class of Lane-Emden differential equations," *Mathematical and Computer Modelling*, vol. 57, no. 5-6, pp. 1298–1311, 2013.

[30] İ. Çelik, "Collocation method and residual correction using Chebyshev series," *Applied Mathematics and Computation*, vol. 174, no. 2, pp. 910–920, 2006.

[31] F. A. Oliveira, "Collocation and residual correction," *Numerische Mathematik*, vol. 36, no. 1, pp. 27–31, 1980.