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Research Article

On Convergence of Fixed Points in Fuzzy Metric Spaces

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We mainly focus on the convergence of the sequence of fixed points for some different sequences of contraction mappings or fuzzy metrics in fuzzy metric spaces. Our results provide a novel research direction for fixed point theory in fuzzy metric spaces as well as a substantial extension of several important results from classical metric spaces.

1. Introduction

Fixed point theory of classical metric spaces plays an important role in general topology. In 1988, Grabiec [1] first extended fixed point theorems of Banach and Edelstein to fuzzy metric spaces in the sense of Kramosil and Michalek. Since then, many authors had dedicated themselves to the study of fixed point theory in fuzzy metric spaces [2–18]. Besides, some authors extended fixed point theory to other types of fuzzy metric spaces in recent years. For instance, Alaca et al. [19] extended the well-known fixed point theorems of Banach and Edelstein to intuitionistic fuzzy metric spaces with the help of Grabiec's work. Simultaneously, Mohamad [20] and Razani [21] proved the existence of fixed point for a nonexpansive mapping of intuitionistic fuzzy metric spaces and the intuitionistic Banach fixed point theorem in complete intuitionistic fuzzy metric spaces, respectively. Later, Ćirić et al. [22] investigated the existence of fixed points for a class of asymptotically nonexpansive mappings in an arbitrary intuitionistic fuzzy metric space. On the other hand, Adibi et al. [23] extended a common fixed point theorem to L-fuzzy metric spaces and proved a coincidence point theorem and a fixed point theorem for compatible mappings of type (P) in these spaces. In 2008, Ješić and Babačev [24] further studied some common fixed point theorems for a pair of R-weakly commuting mappings with nonlinear contractive condition in intuitionistic fuzzy metric spaces and L-fuzzy metric spaces. In the same year, Park et al. [25] extended some common fixed point theorems for five mappings to M-fuzzy metric spaces. Up to now, one can see that the majority of papers mainly focus on the existence of fixed points for different mappings in different fuzzy metric spaces. However, the aim of this paper is to show that the convergence of the sequence of fixed points to some sequences of contraction mappings or fuzzy metrics satisfies certain conditions in fuzzy metric spaces.

2. Preliminaries

Now, we begin with some basic concepts and lemmas. Let $\mathbb{N}$ denote the set of all positive integers.

Definition 1 (Schweizer and Sklar [26]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a *continuous triangular norm* (shortly, continuous t-norm) if it satisfies the following conditions:

(TN-1) $*$ is commutative and associative;

(TN-2) $*$ is continuous;

(TN-3) $a * 1 = a$ for every $a \in [0,1]$;

(TN-4) $a * b \leq c * d$ whenever $a \leq c$, $b \leq d$, and $a, b, c, d \in [0,1]$.

In particular, a t-norm $*$ is said to be *positive* [27] if $a*b>0$ whenever $a,b\in(0,1]$.

We redefine the notion of a fuzzy metric space by appending the following condition (FM-6) based on the one in the sense of George and Veeramani [2].

Definition 2. A *fuzzy metric space* is an ordered triple $(X, M, *)$ such that X is a (nonempty) set, $*$ is a continuous t-norm, and M is a fuzzy set on $X\times X\times(0,\infty)$ satisfying the following conditions, for all $x,y,z\in X$, $s,t>0$:

(FM-1) $M(x,y,t)>0$;

(FM-2) $M(x,y,t)=1$ if and only if $x=y$;

(FM-3) $M(x,y,t)=M(y,x,t)$;

(FM-4) $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s)$;

(FM-5) $M(x,y,\cdot):(0,\infty)\to(0,1]$ is continuous;

(FM-6) $\lim_{t\to\infty}M(x,y,t)=1$.

Definition 3 (Grabiec [1] and Vasuki and Veeramani [17]). Let $(X,M,*)$ be a fuzzy metric space. Then

(a) a sequence $\{x_n\}$ is said to *converge to* x in X, denoted by $x_n\to x$, if and only if $\lim_{n\to\infty}M(x_n,x,t)=1$ for all $t>0$; that is, for each $r\in(0,1)$ and $t>0$, there exists an $n_0\in\mathbb{N}$ such that $M(x_n,x,t)>1-r$ for all $n\geq n_0$;

(b) a sequence $\{x_n\}$ in X is a *G-Cauchy sequence* if and only if $\lim_{n\to\infty}M(x_{n+p},x_n,t)=1$ for any $p>0$ and $t>0$;

(c) the fuzzy metric space $(X,M,*)$ is called *G-complete* if every G-Cauchy sequence is convergent.

Definition 4 (Grabiec [1]). Let $(X,M,*)$ be a fuzzy metric space. A mapping $T:X\to X$ is called a *contraction mapping* if there exists $k\in(0,1)$ such that

$$M(Tx,Ty,kt)\geq M(x,y,t) \tag{1}$$

for every $x,y\in X$ and $t>0$.

According to fuzzy Banach contraction theorem of complete fuzzy metric space in the sense of Grabiec [1], we can obtain the following lemma.

Lemma 5. *Let $(X,M,*)$ be a G-complete fuzzy metric space. If $T:X\to X$ is a contraction mapping, then T has a unique fixed point.*

Definition 6. Let $(X,M,*)$ be a fuzzy metric space and let $\{T_n\}$ be a sequence of self-mappings on X. $T_0:X\to X$ is a given mapping. The sequence $\{T_n\}$ is said to *converge pointwise to* T_0 if for each $r\in(0,1)$ and $x_0\in X$, there exists an $n_0\in\mathbb{N}$ such that

$$M(T_nx_0,T_0x_0,t)>1-r \tag{2}$$

for all $n\geq n_0$ and $t>0$.

Definition 7. Let $(X,M,*)$ be a fuzzy metric space and let $\{T_n\}$ be a sequence of self-mappings on X. $T_0:X\to X$ is a given mapping. The sequence $\{T_n\}$ is said to *converge uniformly to* T_0 if for each $r\in(0,1)$ and $t>0$, there exists an $n_0\in\mathbb{N}$ such that

$$M(T_nx,T_0x,t)>1-r \tag{3}$$

for all $n\geq n_0$ and $x\in X$.

Definition 8. Let $(X,M,*)$ be a fuzzy metric space. A sequence of self-mappings $\{T_n\}$ is *uniformly equicontinuous* if for each $r\in(0,1)$, there exists an $\epsilon\in(0,1)$ such that $M(x,y,s)>1-\epsilon$ implies $M(T_nx,T_ny,t)>1-r$ for every $x,y\in X$, $n\in\mathbb{N}$, and $s,t>0$.

Definition 9 (George and Veeramani [2]). Let $(X,M,*)$ be a fuzzy metric space. The *open ball* $B(x,r,t)$ and *closed ball* $B[x,r,t]$ with center $x\in X$ and radius r, $0<r<1$, $t>0$, respectively, are defined as follows:

$$\begin{aligned}B(x,r,t)&=\{y\in X:M(x,y,t)>1-r\},\\ B[x,r,t]&=\{y\in X:M(x,y,t)\geq 1-r\}.\end{aligned} \tag{4}$$

Lemma 10 (George and Veeramani [2]). *Every open (closed) ball is an open (a closed) set.*

Definition 11 (Gregori and Romaguera [3]). A fuzzy metric space $(X,M,*)$ is a *compact space* if (X,τ_M) is a compact topological space, where τ_M is a topology induced by the fuzzy metric M.

Based on the corresponding conclusions stated in [2], we can easily obtain the following lemma.

Lemma 12. *Every closed subset A of a compact fuzzy metric space $(X,M,*)$ is compact.*

Lemma 13. *Let $(X,M,*)$ be a fuzzy metric space and let $\{T_n\}$ be a sequence of self-mappings on X. $T_0:X\to X$ is a contraction mapping of X; that is, there exists $k\in(0,1)$ such that $M(T_0x,T_0y,kt)\geq M(x,y,t)$ for every $x,y\in X$. A is a compact subset of X. If $\{T_n\}$ converges pointwise to T_0 in A and it is a uniformly equicontinuous sequence, then the sequence $\{T_n\}$ converges uniformly to T_0 in A.*

Proof. For each $\overline{r}\in(0,1)$, we may choose an appropriate r such that $(1-r)*(1-r)*(1-r)>1-\overline{r}$. Since $\{T_n\}$ is uniformly equicontinuous, there exists $\epsilon\in(0,1)$ $(\epsilon\leq r)$ such that $M(x,y,s)>1-\epsilon\Rightarrow M(T_nx,T_ny,t)>1-r$ for every $x,y\in X$, $s,t>0$, and $n\in\mathbb{N}$. For the foregoing ϵ, we fix $s>0$. Define $\mathscr{C}=\{B(x,\epsilon,s):x\in A\}$. By Lemma 10, $\mathscr{C}$ is a family of open sets of A. Obviously, $\mathscr{C}$ constitutes an open covering of A; that is, $A\subset\bigcup B(x,\epsilon,s)$. Since A is compact, there exist $x_1,x_2,\ldots,x_m\in A$ such that $A\subset\bigcup_{i=1}^{m}B(x_i,\epsilon,s)$. For every $x_i\in A$ $(i=1,2,\ldots,m)$, since $\{T_n\}$ converges pointwise to T_0 in A, for $r\in(0,1)$, there exist $n_i\in\mathbb{N}$ $(i=1,2,\ldots,m)$ such that $M(T_nx_i,T_0x_i,t)>1-r$ for all $n\geq n_i$. Set $n^*=\max\{n_i:i=1,2,\ldots,m\}$. Clearly, n^* depends only on r. For every $x\in X$, there is an $i_0\in\{1,2,\ldots,m\}$ such that $x\in B(x_{i_0},\epsilon,s)$. Then we have $M(x, x_{i_0}, s) > 1-\epsilon \Rightarrow M(T_n x, T_n x_{i_0}, t) > 1-r$ for all $n \in \mathbb{N}$. Thus, for all $n \geq n^*$,

$$\begin{aligned} M&(T_n x, T_0 x, (2t+ks)) \\ &\geq M(T_n x, T_n x_{i_0}, t) * M(T_n x_{i_0}, T_0 x, t+ks) \\ &\geq M(T_n x, T_n x_{i_0}, t) * M(T_n x_{i_0}, T_0 x_{i_0}, t) \\ &\quad * M(T_0 x_{i_0}, T_0 x, ks) \\ &\geq M(T_n x, T_n x_{i_0}, t) * M(T_n x_{i_0}, T_0 x_{i_0}, t) * M(x_{i_0}, x, s) \\ &\geq (1-r) * (1-r) * (1-\epsilon) \\ &\geq (1-r) * (1-r) * (1-r) > 1-\overline{r}. \end{aligned} \tag{5}$$

Hence, the sequence $\{T_n\}$ converges uniformly to T_0 in A. □

Definition 14. A fuzzy metric space $(X, M, *)$ in which every point has a compact neighborhood is called *locally compact*.

Definition 15. Let $(X, M_0, *)$ be a fuzzy metric space and let $\{M_n\}$ be a sequence of fuzzy metrics on X. The sequence $\{M_n\}$ is said to *upper semiconverge uniformly to* M_0 if for each $r \in (0,1)$ and $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that $M_n(x, y, t) \geq M_0(x, y, t)$ and $M_0(x, y, t)/M_n(x, y, t) > 1-r$ for all $n \geq n_0$, $x, y \in X$.

3. Main Results

Theorem 16. *Let $(X, M, *)$ be a G-complete fuzzy metric space and let $\{T_n\}$ be a sequence of self-mappings on X where t-norm $a * b = \min\{a, b\}$. T_0 is a contraction mapping of X; that is, there exists $k_0 \in (0,1)$ such that $M(T_0 x, T_0 y, k_0 t) \geq M(x, y, t)$ for all $x, y \in X$, $t > 0$, and satisfying $T_0 x_0 = x_0$. If there exists at least a fixed point x_n for each T_n ($n \in \mathbb{N}$) and the sequence $\{T_n\}$ converges uniformly to T_0, then $x_n \rightarrow x_0$.*

Proof. Suppose that $x_n \nrightarrow x_0$; namely, there exist $t_0 > 0$ and $r_0 \in (0,1)$ such that for any $n \in \mathbb{N}$ there is a $k(n) > n$ satisfying $M(x_{k(n)}, x_0, t_0) < 1-r_0$. Fix a number $h \in (k_0, 1)$. According to the condition (FM-6) of Definition 2, for $t_0 > 0$, we can find an appropriate $p \in \mathbb{N}$ such that $M(x_n, x_0, t_0(h/k_0)^p) > 1-r_0$ for any $n \in \mathbb{N}$. Since the sequence $\{T_n\}$ converges uniformly to T_0, we can make n_0 sufficiently large such that $M(T_n x_n, T_0 x, t) > 1-r_0$ for all $n \geq n_0$, $t > 0$. Now for $n \geq n_0$, we have

$$\begin{aligned} 1-r_0 &> M(x_{k(n)}, x_0, t_0) \\ &= M(T_{k(n)} x_{k(n)}, T_0 x_0, t_0) \\ &\geq M(T_{k(n)} x_{k(n)}, T_0 x_{k(n)}, (1-h) t_0) \\ &\quad * M(T_0 x_{k(n)}, T_0 x_0, h t_0) \\ &\geq M(T_{k(n)} x_{k(n)}, T_0 x_{k(n)}, (1-h) t_0) \\ &\quad * M\left(x_{k(n)}, x_0, \frac{t_0 h}{k_0}\right) \\ &\geq M(T_{k(n)} x_{k(n)}, T_0 x_{k(n)}, (1-h) t_0) \\ &\quad * M\left(T_{k(n)} x_{k(n)}, T_0 x_{k(n)}, \frac{t_0(1-h) h}{k_0}\right) \\ &\quad * M\left(x_{k(n)}, x_0, t_0\left(\frac{h}{k_0}\right)^2\right) \\ &\geq M(T_{k(n)} x_{k(n)}, T_0 x_{k(n)}, (1-h) t_0) \\ &\quad * M\left(T_{k(n)} x_{k(n)}, T_0 x_{k(n)}, \frac{t_0(1-h) h}{k_0}\right) \\ &\quad * \cdots * M\left(T_{k(n)} x_{k(n)}, T_0 x_{k(n)}, t_0(1-h)\left(\frac{h}{k_0}\right)^{p-1}\right) \\ &\quad * M\left(x_{k(n)}, x_0, t_0\left(\frac{h}{k_0}\right)^p\right) \\ &\geq \underbrace{(1-r_0) * (1-r_0) * \cdots * (1-r_0)}_{p} * (1-r_0) \\ &= 1-r_0. \end{aligned} \tag{6}$$

This leads to a contradiction. Hence, $x_n \rightarrow x_0$. □

Theorem 17. *Let $(X, M, *)$ be a G-complete fuzzy metric space where t-norm is positive. If $T_0 : X \rightarrow X$ is a self-mapping of X and T_0^m is a contraction mapping for a certain positive integer m, then T_0 has a unique fixed point.*

Proof. First of all, if $m = 1$, the theorem is evident. In addition, if $m \geq 2$, according to Lemma 5, we need only to prove that T_0 is a contraction mapping. Since T_0^m is a contraction mapping, there is $k_0 \in (0,1)$ such that $M(T_0^m x, T_0^m y, k_0^m t) \geq M(x, y, t)$ for every $x, y \in X$, and $t > 0$. Define another fuzzy metric $\widetilde{M}(x, y, t)$ on X using $M(x, y, t)$ as follows:

$$\begin{aligned} \widetilde{M}&(x, y, t) \\ &= M(x, y, t) * M(T_0 x, T_0 y, kt) \\ &\quad * M(T_0^2 x, T_0^2 y, k^2 t) * \cdots * M(T_0^{m-1} x, T_0^{m-1} y, k^{m-1} t). \end{aligned} \tag{7}$$

Actually, it is easy to verify that the foregoing two fuzzy metrics are equivalent. Meantime, we claim that T_0 is a contraction mapping with respect to the fuzzy metric $\widetilde{M}(x, y, t)$, since

$$\begin{aligned} \widetilde{M}&(T_0 x, T_0 y, k_0 t) \\ &= M(T_0 x, T_0 y, kt) * M(T_0^2 x, T_0^2 y, k^2 t) \\ &\quad * M(T_0^3 x, T_0^3 y, k^3 t) * \cdots * M(T_0^m x, T_0^m y, k^m t) \\ &\geq M(T_0 x, T_0 y, kt) * M(T_0^2 x, T_0^2 y, k^2 t) \\ &\quad * M(T_0^3 x, T_0^3 y, k^3 t) * \cdots * M(x, y, t) = \widetilde{M}(x, y, t). \end{aligned} \tag{8}$$

□

Corollary 18. *Let $(X, M, *)$ be a G-complete fuzzy metric space and let $\{T_n\}$ be a sequence of self-mappings on X where t-norm $a * b = \min\{a, b\}$. $T_0 : X \to X$ is a self-mapping of X, and T_0^m is a contraction mapping for a certain positive integer m. If there exists at least a fixed point x_n for each T_n $(n \in \mathbb{N})$ and the sequence $\{T_n\}$ converges uniformly to T_0, then $x_n \to x_0 = T_0x_0$.*

Proof. It follows from Theorems 16 and 17. □

Theorem 19. *Let $(X, M, *)$ be a locally compact fuzzy metric space and let $\{T_n\}$ be a sequence of self-mappings on X. $T_0 : X \to X$ is a contraction mapping; that is, there exists a $k_0 \in (0, 1)$ such that $M(T_0x, T_0y, k_0t) \geq M(x, y, t)$ for all $x, y \in X$, $t > 0$. If the following conditions are satisfied:*

(i) *T_n^m is a contraction mapping for a certain $m = m(n)$,*

(ii) *$\{T_n\}$ converges pointwise to T_0 and $\{T_n\}$ is a uniformly equicontinuous sequence,*

(iii) *$T_nx_n = x_n$, $n = 0, 1, 2, 3, \ldots$,*

then the sequence $\{x_n\}$ converges to x_0; that is, $x_n \to x_0$.

Proof. For each $\epsilon \in (0, 1)$, we choose $r \in (0, 1)$ such that $(1-r) * (1-r) \geq 1-\epsilon$. If given $x_0 \in X$, we may assume that r is sufficiently small such that $K(x_0, r) = \{x : M(x, x_0, t) \geq 1 - r\}$ is a compact subset of X. By Lemma 13, since $\{T_n\}$ is uniformly equicontinuous and pointwise convergent on $K(x_0, r)$, we know that $\{T_n\}$ converges uniformly to T_0 on the compact subset $K(x_0, r)$. Then, for the foregoing r, there exists $n_\epsilon \in \mathbb{N}$ such that $M(T_nx, T_0x, (1 - k_0)t) > 1 - r$ for all $n \geq n_\epsilon$, $t > 0$, and $x \in K(x_0, r)$. In addition, since T_0 is a contraction mapping, we have $M(T_0x, T_0y, k_0t) \geq M(x, y, t)$ for all $x, y \in K(x_0, r)$. Thus, for all $n \geq n_\epsilon$ and $x \in K(x_0, r)$, we can obtain

$$\begin{aligned} M(T_nx, x_0, t) &= M(T_nx, T_0x_0, t) \\ &\geq M(T_nx, T_0x, (1-k_0)t) * M(T_0x, T_0x_0, k_0t) \\ &\geq M(T_nx, T_0x, (1-k_0)t) * M(x, x_0, t) \\ &\geq (1-r) * (1-r) \geq 1-\epsilon. \end{aligned} \tag{9}$$

Therefore, for all $n \geq n_\epsilon$, $K(x_0, r)$ is an invariant set for T_n. Since T_n^m is a contraction mapping for a certain positive integer $m = m(n)$, it follows that the fixed point x_n of T_n is contained in the set $K(x_0, r)$, when $n \geq n_\epsilon$. By the definition of $K(x_0, r)$, we have $M(x_n, x_0, t) \geq 1-r$ for all $n \geq n_\epsilon$. In fact, although r should satisfy the foregoing condition, it may be sufficiently small. Hence, we can obtain $x_n \to x_0$. □

In addition, if t-norm $a * b = a \cdot b$, then we can obtain the following some important conclusions.

Lemma 20. *Let $(X, M_0, *)$ be a G-complete fuzzy metric space and let A be a compact subset of X where t-norm $a * b = a \cdot b$. $\{M_n\}$ and $\{T_n\}$ are a sequence of fuzzy metrics and a sequence of self-mappings on X, respectively. If they satisfy the following conditions:*

(i) *$\{M_n\}$ upper semiconverges uniformly to M_0,*

(ii) *T_n is a contraction mapping for the fuzzy metric M_n, $n = 0, 1, 2, \ldots$,*

(iii) *$\{T_n\}$ converges pointwise to T_0,*

then $\{T_n\}$ converges uniformly to T_0 in A with regard to the fuzzy metric M_0.

Proof. For each $\epsilon \in (0, 1)$, choose $r \in (0, 1)$ such that $(1-r) * (1-r) > 1-\epsilon$. Since $\{M_n\}$ upper semiconverges uniformly to M_0, there exists $n_r \in \mathbb{N}$ such that $M_n(x, y, t) \geq M_0(x, y, t)$ and $M_0(x, y, t)/M_n(x, y, t) > 1 - r$ for all $n \geq n_r$, $t > 0$. Choose x, y in X such that $M_0(x, y, t) > 1 - r$ for each $t > 0$. Then, for all $n \geq n_r$, we have

$$\begin{aligned} M_0(T_nx, T_ny, t) &= \frac{M_0(T_nx, T_ny, t)}{M_n(T_nx, T_ny, t)} * M_n(T_nx, T_ny, t) \\ &\geq (1-r) * M_n(T_nx, T_ny, t) \\ &\geq (1-r) * M_n\left(x, y, \frac{t}{k_n}\right) \quad (k_n \in (0, 1)) \\ &\geq (1-r) * M_0\left(x, y, \frac{t}{k_n}\right) \\ &\geq (1-r) * (1-r) > 1-\epsilon. \end{aligned} \tag{10}$$

Therefore, the sequence $\{T_n\}$ $(n \geq n_r)$ is uniformly equicontinuous in A with regard to the fuzzy metric M_0. Since $\{T_n\}$ is pointwise convergent and A is a compact subset of X, according to Lemma 13, it follows that the subsequence $\{T_n\}$ $(n \geq n_r)$ converges uniformly to T_0 in A. Hence, $\{T_n\}$ converges uniformly to T_0 in A. □

Theorem 21. *Let $(X, M_0, *)$ be a locally compact fuzzy metric space where t-norm $a * b = a \cdot b$. If $\{M_n\}$ and $\{T_n\}$ satisfy the following conditions:*

(i) *$\{M_n\}$ upper semiconverges uniformly to M_0,*

(ii) *T_n is a contraction mapping for the fuzzy metric M_n, $n = 0, 1, 2, \ldots$,*

(iii) *$\{T_n\}$ converges pointwise to T_0,*

(iv) *$T_nx_n = x_n$, $n = 0, 1, 2, \ldots$,*

then the sequence of fixed points $\{x_n\}$ of $\{T_n\}$ converges to the fixed point x_0 of T_0; that is, $x_n \to x_0$.

Proof. For each $\epsilon \in (0, 1)$, choose $r \in (0, 1)$ such that $(1 - r) * (1 - r) \geq 1 - \epsilon$. Meantime, for $x_0 \in X$, we may make r sufficiently small such that $K(x_0, r) = \{x : M(x, x_0, t) \geq 1-r\}$ is compact in X for each $t > 0$. By Lemma 20, we know that $\{T_n\}$ converges uniformly to T_0 in $K(x_0, r)$ with respect to the fuzzy metric M_0. Then, for every $x \in X$, there exists an $n_r \in \mathbb{N}$

such that $M_0(T_n x, T_0 x, t) > 1 - r$ for all $n \geq n_r$, $t > 0$. Thus, when $n \geq n_r$, for all $x \in K(x_0, r)$, we have

$$\begin{aligned} M_0 (T_n x, x_0, (1 + k_0) t) &\geq M_0 (T_n x, T_0 x, t) * M_0 (T_0 x, x_0, k_0 t) \\ &\geq M_0 (T_n x, T_0 x, t) * M_0 (T_0 x, T_0 x_0, k_0 t) \\ &\geq M_0 (T_n x, T_0 x, t) * M_0 (x, x_0, t) \\ &\geq (1 - r) * (1 - r) \geq 1 - \epsilon. \end{aligned} \tag{11}$$

Therefore, $K(x_0, r)$ is an invariant set in X with regard to M_0. Since T_n is still a contraction mapping restricted to $K(x_0, r)$ concerning on M_n, one can see that the fixed point is also included in $K(x_0, r)$. Apparently, for all $n \geq n_r$, we can obtain $M(x_n, x_0, t) \geq 1 - r$. Since r is sufficiently small, it can easily be shown that $\{x_n\}$ converges to x_0; that is, $x_n \rightarrow x_0$. This completes the proof. □

Theorem 22. *Let $(X, M_0, *)$ be a compact fuzzy metric space where t-norm $a * b = a \cdot b$. The sequences $\{M_n\}$ and $\{T_n\}$ satisfy the following conditions:*

(i) *$\{M_n\}$ upper semiconverges uniformly to M_0;*

(ii) *T_n is a contraction mapping for the fuzzy metric M_n, $n = 0, 1, 2, \ldots$;*

(iii) *$\{T_n\}$ converges pointwise to T_0.*

If every mapping T_n ($n \in \mathbb{N}$) has a fixed point x_n and there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to x_0, then $T_0 x_0 = x_0$.

Proof. Let K denote the closure of the set $\{x_{n_k}\}$. By Lemma 12, we can easily know that K is a compact set. According to Lemma 20, it follows that the subsequence $\{T_{n_k}\}$ converges uniformly to T_0 in K with regard to M_0. Obviously, $\{T_{n_k} x_{n_k}\}$ converges to $T_0 x_0$. Hence, $T_0 x_0 = x_0$. □

Theorem 23. *Let $(X, M, *)$ be a fuzzy metric space. $\{T_n\}$ is a sequence of contraction mappings and satisfying $T_n x_n = x_n$ ($n = 1, 2, 3, \ldots$). $T_0 : X \rightarrow X$ is a contraction mapping. If $\{T_n\}$ is a pointwise convergent sequence with respect to T_0 and the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x_0, then T_0 has a fixed point $x_0 = T_0 x_0$.*

Proof. For each $\epsilon \in (0, 1)$, choose $r \in (0, 1)$ such that $(1 - r) * (1 - r) \geq 1 - \epsilon$. Since $\{x_{n_k}\}$ is a convergent subsequence and $\{T_n\}$ is a pointwise convergent sequence, for a given x_0, we may choose $K_r \in \mathbb{N}$ such that $M(x_{n_k}, x_0, t) \geq 1 - r$ and $M(T_{n_k} x_0, T_0 x_0, t) \geq 1 - r$ for all $k \geq K_r$, $t > 0$. For every $n \in \mathbb{N} \cup \{0\}$, we denote by l_n ($l_n \in (0, 1)$) the contraction constant of T_n. Thus, for all $k \geq K_r$, we have

$$\begin{aligned} M \left(x_{n_k}, T_0 x_0, \left(l_{n_k} + 1 \right) t \right) &= M \left(T_{n_k} x_{n_k}, T_0 x_0, \left(l_{n_k} + 1 \right) t \right) \quad l_{n_k} \in (0, 1) \\ &\geq M \left(T_{n_k} x_{n_k}, T_{n_k} x_0, l_{n_k} t \right) * M \left(T_{n_k} x_0, T_0 x_0, t \right) \\ &\geq M \left(x_{n_k}, x_0, t \right) * M \left(T_{n_k} x_0, T_0 x_0, t \right) \\ &\geq (1 - r) * (1 - r) \geq 1 - \epsilon. \end{aligned} \tag{12}$$

Therefore, the subsequence $\{x_{n_k}\}$ converges to $T_0 x_0$. According to the uniqueness of limit, it follows that $x_0 = T_0 x_0$; that is, x_0 is a fixed point of T_0. □

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