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Research Article

Commuting Quasihomogeneous Toeplitz Operator and Hankel Operator on Weighted Bergman Space

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We characterize the commuting Toeplitz operator and Hankel operator with quasihomogeneous symbols. Also, we use it to show the necessary and sufficient conditions for commuting Toeplitz operator and Hankel operator with ordinary functions.

1. Introduction

Let dA denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. For $\alpha > -1$, we denote by dA_α the measure $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$. For $1 \le p < +\infty$, the space $L^p(\mathbb{D}, dA_\alpha)$ is a Banach space. The weighted Bergman space $A^2_\alpha(\mathbb{D})$ is the closed subspace of analytic functions in the Hilbert space $L^2(\mathbb{D}, dA_\alpha)$. For each $z \in \mathbb{D}$, the application $L_z : A^2_\alpha(\mathbb{D}) \to \mathbb{C}$ is continuous and can be represented as $L_z(f) = f(z) = \langle f, K_z^{(\alpha)}\rangle_\alpha$, where

$$K_z^{(\alpha)}(w) = \frac{1}{(1-w\overline{z})^{2+\alpha}} = \sum_{n=0}^{\infty} \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)} (w\overline{z})^n, \quad z, w \in \mathbb{D}. \tag{1}$$

This means that, if P_α is the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ onto $A^2_\alpha(\mathbb{D})$, then P_α can be defined by

$$(P_\alpha f)(w) = \left\langle f, K_w^{(\alpha)} \right\rangle_\alpha = \int_{\mathbb{D}} f(z) \frac{1}{(1-\overline{w}z)^{2+\alpha}} dA_\alpha(z). \tag{2}$$

For a function $f \in L^\infty(\mathbb{D}, dA_\alpha(z))$, we define the Toeplitz operator $T_f : A^2_\alpha(\mathbb{D}) \to A^2_\alpha(\mathbb{D})$ with symbol f by

$$T_f(h) = P_\alpha(fh). \tag{3}$$

It is well known that

$$T_f(h)(z) = \int_{\mathbb{D}} f(w) h(w) \overline{K_z^{(\alpha)}(w)} dA_\alpha(w), \quad z \in \mathbb{D}. \tag{4}$$

Let $U^{(\alpha)} : L^2(\mathbb{D}, dA_\alpha(z)) \to L^2(\mathbb{D}, dA_\alpha(z))$ be the unitary operator defined by $U^{(\alpha)} f(z) = \widetilde{f(z)} = f(\overline{z})$, where f belongs to $L^2(\mathbb{D}, dA_\alpha(z))$. Let g be in $L^2(\mathbb{D}, dA_\alpha(z))$; we define a bounded linear operator M_g on $L^2(\mathbb{D}, dA_\alpha(z))$ as follows:

$$M_g(f) = gf. \tag{5}$$

Then we can define the small Hankel operator as follows:

$$H_g : A^2_\alpha(\mathbb{D}) \longrightarrow A^2_\alpha(\mathbb{D}) \tag{6}$$

as $H_g = P_\alpha U^{(\alpha)} M_g$.

The study of commuting Toeplitz operators on the Bergman and Hardy spaces over various domains and related operator algebras has a long lasting history. On the Hardy space of the unit disk, Brown and Halmos [1] first showed that two Toeplitz operators are commuting if and only if either both symbols of these operators are analytic, or both symbols of these operators are coanalytic or a nontrivial linear combination of the symbols of these operators is constant. On the Bergman space, the situation is more complicated. Axler and Čučković obtained the analogous result for Toeplitz operators with bounded harmonic symbols on the Bergman space of the unit disk [2]. The problem of characterizing commuting Toeplitz operators with arbitrary bounded symbols seems quite challenging and is not fully understood until now. In [3], Čučković and Rao used the Mellin transform to characterize all Toeplitz operators on L_a^2 which commute with $T_{e^{ip\theta}r^m}$ for $(m, p) \in N \times N$. Later in [4] Louhichi and Zakariasy gave a partial characterization of commuting Toeplitz operators on L_a^2 with quasihomogeneous symbols. Recently, Lu and Zhang [5, 6] characterized the commuting Toeplitz operators and Hankel operators with quasihomogeneous symbols. There are also many other important results [7–13]. Motivated by those works, we study commuting Toeplitz operator and Hankel operator on the weighted Bergman space. In this paper, we obtain the necessary and sufficient conditions for commuting Toeplitz operator and Hankel operator.

An operator that will arise in our study of Toeplitz operators is the Mellin transform, defined for any function $\varphi \in L^1([0,1], r\,dr)$; from the formula, $\widehat{\varphi_\alpha}(z)$ is the Mellin transform as follows:

$$\widehat{\varphi_\alpha}(z) = \int_0^1 \varphi(r)\left(1-r^2\right)^\alpha r^{z-1} dr, \tag{7}$$

which is a bounded holomorphic function in the half plane $\{z : \operatorname{Re} z > 2\}$.

Let $\varphi \in L^1(\mathbb{D}, dA_\alpha)$ be a radial function; that is, suppose that $\varphi(z) = \varphi(|z|)$, $z \in \mathbb{D}$. In fact, if we define the function φ_r on $[0,1]$ by $\varphi_r(s) = \varphi(s)$, then a direct calculation shows that

$$\left\langle T_\varphi\left(z^k\right), z^l\right\rangle_\alpha = \begin{cases} 0 & \text{for } k \neq l, \\ 2(\alpha+1)\widehat{\varphi_\alpha}(2k+2) & \text{for } k = l, \end{cases} \tag{8}$$

so that if $k \in \mathbb{N}$,

$$\begin{aligned} T_\varphi\left(z^k\right) &= 2(1+\alpha)\frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)} \\ &\quad \times \int_0^1 \varphi(r) r^{2k+1}\left(1-|r|^2\right)^\alpha dr z^k \\ &= 2(1+\alpha)\frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)}\widehat{\varphi_\alpha}(2k+2) z^k. \end{aligned} \tag{9}$$

Thus, T_φ is a diagonal operator on $A_\alpha^2(\mathbb{D})$ with coefficient sequence as follows:

$$\left(2(1+\alpha)\frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)}\widehat{\varphi_\alpha}(2k+2)\right)_{k=0}^{\infty}. \tag{10}$$

This makes it relatively simple to work with the product of two operators with such radial symbols.

Now, we define the "radialization" of a function $f \in L^1(\mathbb{D}, dA_\alpha)$ by the following:

$$\operatorname{rad}(f)(z) = \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{it}z\right) dt. \tag{11}$$

It is clear that a function f is a radial if and only if $\operatorname{rad}(f) = f$.

Let $\mathfrak{R}^\alpha$ be the space of weighted square integrable radial functions on $\mathbb{D}$. By using that, trigonometric polynomials are dense in $L^2(\mathbb{D}, dA_\alpha)$ and that, for $k_1 \neq k_2$, $e^{ik_1\theta}\mathfrak{R}^\alpha$ is orthogonal to $e^{ik_2\theta}\mathfrak{R}^\alpha$, we see that is

$$L^2(\mathbb{D}, dA_\alpha) = \oplus_{k\in\mathbb{Z}} e^{ik\theta}\mathfrak{R}^\alpha. \tag{12}$$

Definition 1. Let φ be a function in $L^1(\mathbb{D}, dA_\alpha)$ which is of the form $e^{ik\theta}f$, where f is a radial function. Then one says that φ is a quasihomogeneous function of quasihomogeneous degree k.

A direct calculation gives the following lemmas which we will use often.

Lemma 2. *Let $p \in \mathbb{N}$ and let φ be an integrable radial function. Then,*

$$\begin{aligned} T_{e^{ip\theta}\varphi}\left(z^k\right) &= 2(\alpha+1)\frac{\Gamma(k+p+2+\alpha)}{(k+p)!\Gamma(2+\alpha)} \\ &\quad \times \widehat{\varphi_\alpha}(2k+p+2) z^{k+p}, \quad k \geq 0, \end{aligned}$$

$$T_{e^{-ip\theta}\varphi}\left(z^k\right) = \begin{cases} 0 & if\ 0 \leq k < p, \\ 2(\alpha+1)\dfrac{\Gamma(k-p+2+\alpha)}{(k-p)!\Gamma(2+\alpha)} & \\ \quad\times\widehat{\varphi_\alpha}(2k-p+2) z^{k-p} & if\ k \geq p. \end{cases} \tag{13}$$

Lemma 3. *Let φ be an integrable radial function. Then, for $p \in \mathbb{Z}_+$,*

$$H_{e^{ip\theta}\varphi}\left(z^k\right) = 0, \quad k \geq 0, \tag{14}$$

and for $p \in \mathbb{N}$,

$$H_{e^{-ip\theta}\varphi}\left(z^k\right) = \begin{cases} 0 & if\ k > p, \\ 2(\alpha+1)\dfrac{\Gamma(-k+p+\alpha+2)}{(-k+p)!\Gamma(2+\alpha)} & \\ \quad\times\widehat{\varphi_\alpha}(p+2) z^{-k+p} & if\ 0 \leq k \leq p. \end{cases} \tag{15}$$

2. Commuting of Toeplitz Operator and Hankel Operator

Theorem 4. *Let $e^{ip\theta}f$ be a bounded function of quasihomogeneous degree $p \geq 0$ and $g = \sum_{k\in Z} e^{ik\theta} g_{k,\alpha}(r) \in L^\infty(\mathbb{D}, dA_\alpha)$. Then $T_{e^{ip\theta}f}H_g = H_g T_{e^{ip\theta}f}$ if and only if the following conditions holds*

(1) $\widehat{f}_\alpha(2j+p+2)\widehat{g}_{-j-k-p,\alpha}(k+j+p+2) = 0$, *if* $0 \leq k \leq p-1$ *and* $j \geq 0$;

(2) $(\Gamma(k+2+\alpha)/k!)\widehat{f}_\alpha(2k+p+2)\widehat{g}_{-j-k,\alpha}(k+j+2) = (\Gamma(j+p+\alpha+2)/(j+p)!)\widehat{f}_\alpha(2j+p+2)\widehat{g}_{-j-k-2p,\alpha}(k+j+2p+2)$, *if* $k \geq 0$ *and* $j \geq 0$.

Proof. For $j \geq 0$,

$$
\begin{aligned}
T_{e^{ip\theta}f} H_g\left(z^j\right) &= T_{e^{ip\theta}f} H_{\sum_{k\in Z} e^{ik\theta} g_{k,\alpha}(r)}\left(z^j\right) \\
&= T_{e^{ip\theta}f}\left(\sum_{k=-j}^{-\infty} 2(\alpha+1)\frac{\Gamma(-j-k+\alpha+2)}{(-j-k)!\Gamma(2+\alpha)} \times \widehat{g}_{k,\alpha}(-k+2) z^{-j-k}\right) \\
&= 2\sum_{k=-j}^{-\infty}(\alpha+1)\frac{\Gamma(-j-k+\alpha+2)}{(-j-k)!\Gamma(2+\alpha)} \times \widehat{g}_{k,\alpha}(-k+2) T_{e^{ip\theta}f}\left(z^{-j-k}\right) \\
&= 2\sum_{k=-j}^{-\infty}(\alpha+1)\frac{\Gamma(-j-k+\alpha+2)}{(-j-k)!\Gamma(2+\alpha)} \times \widehat{g}_{k,\alpha}(-k+2)\, 2(\alpha+1) \\
&\quad \cdot \frac{\Gamma(-j-k+p+\alpha+2)}{(-j-k+p)!\Gamma(2+\alpha)} \times \widehat{f}_\alpha(-2j-2k+p+2) z^{-j-k+p} \\
&= 4(\alpha+1)^2\sum_{k=0}^{+\infty}\frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \times \frac{\Gamma(k+p+\alpha+2)}{(k+p)!\Gamma(2+\alpha)} \times \widehat{f}_\alpha(2k+p+2) \\
&\quad \cdot \widehat{g}_{-j-k,\alpha}(j+k+2) z^{k+p}, \\
H_g T_{e^{ip\theta}f}\left(z^j\right) &= H_{\sum_{k\in Z} e^{ik\theta} g_{k,\alpha}(r)} T_{e^{ip\theta}f}\left(z^j\right) \\
&= 2(\alpha+1)\frac{\Gamma(j+p+\alpha+2)}{(j+p)!\Gamma(2+\alpha)}\widehat{f}_\alpha(2j+p+2) \times H_{\sum_{k\in Z} e^{ik\theta} g_{k,\alpha}(r)}\left(z^{j+p}\right) \\
&= 2(\alpha+1)\frac{\Gamma(j+p+\alpha+2)}{(j+p)!\Gamma(2+\alpha)}\widehat{f}_\alpha(2j+p+2) \times \sum_{k=-j-p}^{-\infty} 2(\alpha+1) \\
&\quad \cdot \frac{\Gamma(-j-k-p+\alpha+2)}{(-j-p-k)!\Gamma(2+\alpha)}\widehat{g}_{k,\alpha}(-k+2) z^{-j-p-k} \\
&= \sum_{k=-j-p}^{-\infty} 4(\alpha+1)^2\frac{\Gamma(j+p+\alpha+2)}{(j+p)!\Gamma(2+\alpha)} \times \frac{\Gamma(-j-k-p+\alpha+2)}{(-j-p-k)!\Gamma(2+\alpha)} \\
&\quad \cdot \widehat{f}_\alpha(2j+p+2)\widehat{g}_{k,\alpha}(-k+2) z^{-j-p-k} \\
&= 4(\alpha+1)^2\sum_{k=0}^{+\infty}\frac{\Gamma(j+p+\alpha+2)}{(j+p)!\Gamma(2+\alpha)} \times \frac{\Gamma(k+\alpha+2)}{(k)!\Gamma(2+\alpha)}\widehat{f}_\alpha(2j+p+2) \\
&\quad \cdot \widehat{g}_{-j-p-k,\alpha}(k+j+p+2) z^k.
\end{aligned} \tag{16}
$$

If $T_{e^{ip\theta}f} H_g = H_g T_{e^{ip\theta}f}$, then we have

$$
\begin{aligned}
&\sum_{k=0}^{+\infty}\frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)}\frac{\Gamma(k+p+\alpha+2)}{(k+p)!\Gamma(2+\alpha)} \times \widehat{f}_\alpha(2k+p+2)\widehat{g}_{-j-k,\alpha}(j+k+2) z^{k+p} \\
&\quad = \sum_{k=0}^{+\infty}\frac{\Gamma(j+p+\alpha+2)}{(j+p)!\Gamma(2+\alpha)}\frac{\Gamma(k+\alpha+2)}{(k)!\Gamma(2+\alpha)} \times \widehat{f}_\alpha(2j+p+2)\widehat{g}_{-j-p-k,\alpha}(k+j+p+2) z^k,
\end{aligned} \tag{17}
$$

which is equivalent to

$$
\begin{aligned}
&\sum_{k=0}^{+\infty}\frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)}\frac{\Gamma(k+p+\alpha+2)}{(k+p)!\Gamma(2+\alpha)} \times \widehat{f}_\alpha(2k+p+2)\widehat{g}_{-j-k,\alpha}(j+k+2) z^{k+p} \\
&\quad = \sum_{k=0}^{p-1}\frac{\Gamma(j+p+\alpha+2)}{(j+p)!\Gamma(2+\alpha)}\frac{\Gamma(k+\alpha+2)}{(k)!\Gamma(2+\alpha)} \times \widehat{f}_\alpha(2j+p+2)\widehat{g}_{-j-p-k,\alpha} \times (k+j+p+2) z^k \\
&\quad + \sum_{k=p}^{+\infty}\frac{\Gamma(j+p+\alpha+2)}{(j+p)!\Gamma(2+\alpha)}\frac{\Gamma(k+\alpha+2)}{(k)!\Gamma(2+\alpha)} \times \widehat{f}_\alpha(2j+p+2)\widehat{g}_{-j-p-k,\alpha} \times (k+j+p+2) z^k,
\end{aligned} \tag{18}
$$

that is,

$$
\begin{aligned}
&\sum_{k=0}^{+\infty}\frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)}\frac{\Gamma(k+p+\alpha+2)}{(k+p)!\Gamma(2+\alpha)} \times \widehat{f}_\alpha(2k+p+2)\widehat{g}_{-j-k,\alpha} \times (j+k+2) z^{k+p} \\
&\quad = \sum_{k=0}^{p-1}\frac{\Gamma(j+p+\alpha+2)}{(j+p)!\Gamma(2+\alpha)}\frac{\Gamma(k+\alpha+2)}{(k)!\Gamma(2+\alpha)} \times \widehat{f}_\alpha(2j+p+2)\widehat{g}_{-j-p-k,\alpha}
\end{aligned}
$$

$$\times \left(k+j+p+2\right) z^{k}$$
$$+\sum_{k=0}^{+\infty} \frac{\Gamma\left(j+p+\alpha+2\right)}{(j+p)!\Gamma\left(2+\alpha\right)} \frac{\Gamma\left(p+k+\alpha+2\right)}{(p+k)!\Gamma\left(2+\alpha\right)}$$
$$\times \widehat{f}_{\alpha}\left(2j+p+2\right) \widehat{g}_{-j-2p-k,\alpha}$$
$$\times \left(k+j+2p+2\right) z^{p+k}. \tag{19}$$

From the aforementioned we get the following.

Case 1. For $0 \le k \le p-1$ and $j \ge 0$,

$$\widehat{f}_{\alpha}\left(2j+p+2\right) \widehat{g}_{-j-p-k,\alpha}\left(k+j+p+2\right) = 0. \tag{20}$$

Case 2. For $k \ge 0$ and $j \ge 0$,

$$\begin{aligned} &\frac{\Gamma\left(k+\alpha+2\right)}{k!} \widehat{f}_{\alpha}\left(2k+p+2\right) \widehat{g}_{-j-k,\alpha}\left(j+k+2\right) \\ &\quad = \frac{\Gamma\left(j+p+\alpha+2\right)}{(j+p)!} \widehat{f}_{\alpha}\left(2j+p+2\right) \widehat{g}_{-j-2p-k,\alpha} \\ &\quad \times \left(k+j+2p+2\right). \end{aligned} \tag{21}$$

As a special case of Theorem 4, we can have the following corollary. □

Corollary 5. *Let f be a bounded radial function and $g = \sum_{k\in Z} e^{ik\theta} g_{k,\alpha}(r) \in L^{\infty}(\mathbb{D}, dA_{\alpha})$. Then $T_f H_g = H_g T_f$ if and only if*

$$\begin{aligned} &\frac{\Gamma\left(k+2+\alpha\right)}{k!} \widehat{f}_{\alpha}\left(2k+2\right) \widehat{g}_{-j-k,\alpha}\left(k+j+2\right) \\ &\quad = \frac{\Gamma\left(j+\alpha+2\right)}{j!} \widehat{f}_{\alpha}\left(2j+2\right) \widehat{g}_{-j-k,\alpha}\left(k+j+2\right), \end{aligned} \tag{22}$$

for $k \ge 0$ and $j \ge 0$.

Theorem 6. *Let $e^{-ip\theta} f$ be a bounded function of quasihomogeneous degree $-p < 0$ and $g = \sum_{k\in Z} e^{ik\theta} g_{k,\alpha}(r) \in L^{\infty}(\mathbb{D}, dA_{\alpha})$. Then $T_{e^{-ip\theta} f} H_g = H_g T_{e^{-ip\theta} f}$ if and only if the following conditions holds*

(1) *$\widehat{f}_{\alpha}(2k+p+2)\widehat{g}_{-j-k-p,\alpha}(k+j+p+2) = 0$, if $k \ge 0$ and $p > j \ge 0$;*

(2) *$(\Gamma(k+p+2+\alpha)/(k+p)!)\widehat{f}_{\alpha}(2k+p+2)\widehat{g}_{-j-k-p,\alpha}(k+j+p+2) = (\Gamma(j-p+\alpha+2)/(j-p)!)\widehat{f}_{\alpha}(2j-p+2)\widehat{g}_{-j-k+p,\alpha}(k+j-p+2)$, if $k \ge 0$ and $j \ge p$.*

Proof. For $j \ge 0$, we have

$$\begin{aligned} T_{e^{-ip\theta} f} H_g\left(z^{j}\right) &= T_{e^{-ip\theta} f} H_{\sum_{k\in Z} e^{ik\theta} g_{k,\alpha}(r)}\left(z^{j}\right) \\ &= T_{e^{-ip\theta} f} 2 \sum_{k=-j}^{-\infty} (\alpha+1) \frac{\Gamma\left(-j-k+\alpha+2\right)}{(-j-k)!\Gamma\left(2+\alpha\right)} \\ &\quad \times \widehat{g}_{k,\alpha}\left(-k+2\right)\left(z^{-j-k}\right) \\ &= 2 \sum_{k=-j}^{-\infty} (\alpha+1) \frac{\Gamma\left(-j-k+\alpha+2\right)}{(-j-k)!\Gamma\left(2+\alpha\right)} \\ &\quad \times \widehat{g}_{k,\alpha}\left(-k+2\right) T_{e^{-ip\theta} f}\left(z^{-j-k}\right) \\ &= 4(\alpha+1)^{2} \sum_{k=-j-p}^{-\infty} \frac{\Gamma\left(-j-k+\alpha+2\right)}{(-j-k)!\Gamma\left(2+\alpha\right)} \\ &\quad \times \widehat{g}_{k,\alpha}\left(-k+2\right) \\ &\quad \cdot \frac{\Gamma\left(-j-k-p+\alpha+2\right)}{(-j-k-p)!\Gamma\left(2+\alpha\right)} \\ &\quad \times \widehat{f}_{\alpha}\left(-2j-2k-p+2\right) z^{-j-k-p} \\ &= 4(\alpha+1)^{2} \sum_{k=0}^{+\infty} \frac{\Gamma\left(k+p+\alpha+2\right)}{(k+p)!\Gamma\left(2+\alpha\right)} \\ &\quad \times \widehat{g}_{-k-j-p,\alpha}\left(k+j+p+2\right) \\ &\quad \cdot \frac{\Gamma\left(k+\alpha+2\right)}{k!\Gamma\left(2+\alpha\right)} \widehat{f}_{\alpha}\left(2k+p+2\right) z^{k}. \end{aligned} \tag{23}$$

Then one has the following.

Case 1. For $p > j \ge 0$,

$$H_g T_{e^{-ip\theta} f}\left(z^{j}\right) = H_{\sum_{k\in Z} e^{ik\theta} g_{k,\alpha}(r)} T_{e^{-ip\theta} f}\left(z^{j}\right) = 0. \tag{24}$$

Case 2. For $j \ge p$,

$$\begin{aligned} H_g T_{e^{-ip\theta} f}\left(z^{j}\right) &= H_{\sum_{k\in Z} e^{ik\theta} g_{k,\alpha}(r)} T_{e^{-ip\theta} f}\left(z^{j}\right) \\ &= 2\left(\alpha+1\right) \frac{\Gamma\left(j-p+\alpha+2\right)}{(j-p)!\Gamma\left(2+\alpha\right)} \\ &\quad \times \widehat{f}_{\alpha}\left(2j-p+2\right) H_{\sum_{k\in Z} e^{ik\theta} g_{k,\alpha}(r)}\left(z^{j-p}\right) \\ &= 4(\alpha+1)^{2} \sum_{k=p-j}^{-\infty} \frac{\Gamma\left(j-p+\alpha+2\right)}{(j-p)!\Gamma\left(2+\alpha\right)} \\ &\quad \times \widehat{f}_{\alpha}\left(2j-p+2\right) \\ &\quad \times \frac{\Gamma\left(-j+p-k+\alpha+2\right)}{(-j+p-k)!\Gamma\left(2+\alpha\right)} \\ &\quad \cdot \widehat{g}_{k,\alpha}\left(-k+2\right) z^{-j+p-k} \end{aligned}$$

$$= 4(\alpha+1)^2 \sum_{k=0}^{+\infty} \frac{\Gamma(j-p+\alpha+2)}{(j-p)!\Gamma(2+\alpha)} \times \widehat{f}_\alpha(2j-p+2) \times \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \times \widehat{g}_{-j-k+p,\alpha}(j+k-p+2) z^k. \quad (25)$$

If $T_{e^{-ip\theta}f} H_g = H_g T_{e^{-ip\theta}f}$, then we have the following.

Case 1. For $p > j \geq 0$,

$$\sum_{k=0}^{+\infty} \frac{\Gamma(k+p+\alpha+2)}{(k+p)!\Gamma(2+\alpha)} \widehat{g}_{-k-j-p,\alpha}(k+j+p+2) \times \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \widehat{f}_\alpha(2k+p+2) z^k = 0. \quad (26)$$

Case 2. For $j \geq p$,

$$\begin{aligned} &\sum_{k=0}^{+\infty} \frac{\Gamma(k+p+\alpha+2)}{(k+p)!\Gamma(2+\alpha)} \widehat{g}_{-k-j-p,\alpha}(k+j+p+2) \times \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \widehat{f}_\alpha(2k+p+2) z^k \\ &= \sum_{k=0}^{+\infty} \frac{\Gamma(j-p+\alpha+2)}{(j-p)!\Gamma(2+\alpha)} \widehat{f}_\alpha(2j-p+2) \times \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \widehat{g}_{-j-k+p,\alpha} \times (j+k-p+2) z^k. \end{aligned} \quad (27)$$

That is one has the following.

Case 1. For $p > j \geq 0, k \geq 0$,

$$\widehat{g}_{-k-j-p,\alpha}(k+j+p+2) \widehat{f}_\alpha(2k+p+2) = 0. \quad (28)$$

Case 2. For $j \geq p, k \geq 0$,

$$\begin{aligned} &\frac{\Gamma(k+p+\alpha+2)}{(k+p)!} \widehat{g}_{-k-j-p,\alpha} \times (k+j+p+2) \widehat{f}_\alpha(2k+p+2) \\ &= \frac{\Gamma(j-p+\alpha+2)}{(j-p)!} \widehat{f}_\alpha(2j-p+2) \times \widehat{g}_{-j-k+p,\alpha}(j+k-p+2). \end{aligned} \quad (29)$$

□

Theorem 7. *Let $e^{-is\theta}g$ be a bounded function of quasihomogeneous degree $-s \leq 0$ and $f = \sum_{k\in Z} e^{ik\theta} f_{k,\alpha}(r) \in L^\infty(\mathbb{D}, dA_\alpha)$. Then $T_f H_{e^{-is\theta}g} = H_{e^{-is\theta}g} T_f$ if and only if $\widehat{g}_\alpha(s+2) = 0$ or*

(1) *$\widehat{f}_{k-j,\alpha}(k+j+2) = 0$, if $s \geq k \geq 0$ and $j > s$;*

(2) *$(\Gamma(-j+s+\alpha+2)/(-j+s)!) \widehat{f}_{k+j-s,\alpha}(-j+s+k+2) = (\Gamma(s-k+\alpha+2)/(s-k)!) \widehat{f}_{s-k-j,\alpha}(s-k+j+2)$, if $s \geq k \geq 0$ and $s \geq j \geq 0$;*

(3) *$\widehat{f}_{k+j-s,\alpha}(-j+s+k+2) = 0$, if $k > s$ and $s \geq j \geq 0$.*

Proof. For $j \geq 0$, we have the following.

Case 1. For $j > s$,

$$T_f H_{e^{-is\theta}g}(z^j) = T_{\sum_{k\in Z} e^{ik\theta} f_{k,\alpha}(r)} H_{e^{-is\theta}g}(z^j) = 0. \quad (30)$$

Case 2. For $s \geq j \geq 0$,

$$\begin{aligned} T_f H_{e^{-is\theta}g}(z^j) &= T_{\sum_{k\in Z} e^{ik\theta} f_{k,\alpha}(r)} H_{e^{-is\theta}g}(z^j) \\ &= 2(\alpha+1) \frac{\Gamma(-j+s+\alpha+2)}{(-j+s)!\Gamma(2+\alpha)} \widehat{g}_\alpha(s+2) \times T_{\sum_{k\in Z} e^{ik\theta} f_{k,\alpha}(r)}(z^{-j+s}) \\ &= 4(\alpha+1)^2 \frac{\Gamma(-j+s+\alpha+2)}{(-j+s)!\Gamma(2+\alpha)} \widehat{g}_\alpha(s+2) \times \sum_{k=j-s}^{+\infty} \frac{\Gamma(s-j+k+\alpha+2)}{(s-j+k)!\Gamma(2+\alpha)} \widehat{f}_{k,\alpha} \times (2s-2j+k+2) z^{-j+s+k}, \\ H_{e^{-is\theta}g} T_f(z^j) &= H_{e^{-is\theta}g} T_{\sum_{k\in Z} e^{ik\theta} f_{k,\alpha}(r)}(z^j) \\ &= H_{e^{-is\theta}g}\left(\sum_{k=-j}^{+\infty} 2(\alpha+1) \times \frac{\Gamma(j+k+\alpha+2)}{(j+k)!\Gamma(2+\alpha)} \times \widehat{f}_{k,\alpha}(2j+k+2) z^{j+k} \right) \\ &= 2\sum_{k=0}^{+\infty} (\alpha+1) \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \times \widehat{f}_{k-j,\alpha}(k+j+2) H_{e^{-is\theta}g}(z^k) \\ &= 4(\alpha+1)^2 \sum_{k=0}^{s} \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \times \frac{\Gamma(-k+s+\alpha+2)}{(-k+s)!\Gamma(2+\alpha)} \widehat{f}_{k-j,\alpha} \times (k+j+2) \widehat{g}_\alpha(s+2) z^{-k+s}. \end{aligned} \quad (31)$$

Applying $H_{e^{-is\theta}g}T_f = T_f H_{e^{-is\theta}g}$, we have the following.

Case 1. For $j > s$,

$$4(\alpha+1)^2 \sum_{k=0}^{s} \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \frac{\Gamma(-k+s+\alpha+2)}{(-k+s)!\Gamma(2+\alpha)} \times \widehat{f}_{k-j,\alpha}(k+j+2)\,\widehat{g}_\alpha(s+2)\,z^{-k+s} = 0. \tag{32}$$

Case 2. For $s \geq j \geq 0$,

$$\begin{aligned}&\sum_{k=j-s}^{+\infty} \frac{\Gamma(s-j+k+\alpha+2)}{(s-j+k)!\Gamma(2+\alpha)} \widehat{f}_{k,\alpha}(2s-2j+k+2) \\ &\quad\times \frac{\Gamma(-j+s+\alpha+2)}{(-j+s)!\Gamma(2+\alpha)} \widehat{g}_\alpha(s+2)\,z^{-j+s+k} \\ &= \sum_{k=0}^{s} \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \frac{\Gamma(-k+s+\alpha+2)}{(-k+s)!\Gamma(2+\alpha)} \\ &\quad\times \widehat{f}_{k-j,\alpha}(k+j+2)\,\widehat{g}_\alpha(s+2)\,z^{-k+s}.\end{aligned} \tag{33}$$

If $\widehat{g}_\alpha(s+2) = 0$, then the equation holds.

Otherwise $\widehat{g}_\alpha(s+2) \neq 0$, we have the following.

Case 1. For $j > s$,

$$\sum_{k=0}^{s} \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \frac{\Gamma(-k+s+\alpha+2)}{(-k+s)!\Gamma(2+\alpha)} \times \widehat{f}_{k-j,\alpha}(k+j+2)\,z^{-k+s} = 0. \tag{34}$$

Case 2. For $s \geq j \geq 0$,

$$\begin{aligned}&\sum_{k=j-s}^{+\infty} \frac{\Gamma(s-j+k+\alpha+2)}{(s-j+k)!\Gamma(2+\alpha)} \widehat{f}_{k,\alpha}(2s-2j+k+2) \\ &\quad\times \frac{\Gamma(-j+s+\alpha+2)}{(-j+s)!\Gamma(2+\alpha)} z^{-j+s+k} \\ &= \sum_{k=0}^{s} \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \frac{\Gamma(-k+s+\alpha+2)}{(-k+s)!\Gamma(2+\alpha)} \\ &\quad\times \widehat{f}_{k-j,\alpha}(k+j+2)\,z^{-k+s}.\end{aligned} \tag{35}$$

that is one has the following.

Case 1. For $s \geq k \geq 0$, $j > s$,

$$\widehat{f}_{k-j,\alpha}(k+j+2)\,z^{-k+s} = 0. \tag{36}$$

Case 2. For $s \geq j \geq 0$,

$$\begin{aligned}&\sum_{k=0}^{+\infty} \frac{\Gamma(s-j+\alpha+2)}{(s-j)!\Gamma(2+\alpha)} \widehat{f}_{k+j-s,\alpha} \\ &\quad\times (s-j+k+2) \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} z^k \\ &= \sum_{k=0}^{s} \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \frac{\Gamma(-k+s+\alpha+2)}{(-k+s)!\Gamma(2+\alpha)} \\ &\quad\times \widehat{f}_{s-k-j,\alpha}(s-k+j+2)\,z^k.\end{aligned} \tag{37}$$

Then we get the following.

Case 1. For $s \geq k \geq 0$, $j > s$, $\widehat{f}_{k-j,\alpha}(k+j+2) = 0$.

Case 2. For $s \geq j \geq 0$,

$$\begin{aligned}&\frac{\Gamma(s-j+\alpha+2)}{(s-j)!} \widehat{f}_{k+j-s,\alpha}(s-j+k+2) \\ &= \frac{\Gamma(-k+s+\alpha+2)}{(-k+s)!} \widehat{f}_{s-k-j,\alpha}(s-k+j+2).\end{aligned} \tag{38}$$

Case 3. For $k > s$, $s \geq j \geq 0$, $\widehat{f}_{k+j-s,\alpha}(-j+s+k+2) = 0$. □

Corollary 8. *Let g be a bounded radial function and $f = \sum_{k\in Z} e^{ik\theta} f_{k,\alpha}(r) \in L^\infty(\mathbb{D}, dA_\alpha)$. Then $T_f H_g = H_g T_f$ if and only if*

(1) $\widehat{g}_\alpha(s+2) = 0$;

(2) $\widehat{f}_{k,\alpha}(k+2) = 0$ *and* $\widehat{f}_{-k,\alpha}(k+2) = 0$, *for* $k > 0$.

Finally, we will investigate the situation that both functions are ordinary functions.

Theorem 9. *Let $f = \sum_{k\in Z} e^{ik\theta} f_{k,\alpha}(r) \in L^\infty(\mathbb{D}, dA_\alpha)$ and $g = \sum_{l\in Z} e^{il\theta} g_{l,\alpha}(r) \in L^\infty(\mathbb{D}, dA_\alpha)$. Then $T_f H_g = H_g T_f$ if and only if*

$$\begin{aligned}&\sum_{k=0}^{+\infty} \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \widehat{f}_{k-m,\alpha}(m+k+2) \\ &\quad\times \widehat{g}_{-n-k,\alpha}(n+k+2) \\ &= \sum_{k=0}^{+\infty} \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \widehat{f}_{n-k,\alpha}(n+k+2) \\ &\quad\times \widehat{g}_{-k-m,\alpha}(k+m+2),\end{aligned} \tag{39}$$

for $m \geq 0$ and $n \geq 0$.

Proof. For $j \geq 0$, we have

$$\begin{aligned}
H_gT_f\left(z^j\right) &= H_{\sum_{l\in Z} e^{il\theta} g_{l,\alpha}(r)} T_{\sum_{k\in Z} e^{ik\theta} f_{k,\alpha}(r)} \left(z^j\right) \\
&= H_{\sum_{l\in Z} e^{il\theta} g_{l,\alpha}(r)} \left(\sum_{k=-j}^{+\infty} 2(\alpha+1) \times \frac{\Gamma(j+k+\alpha+2)}{(j+k)!\Gamma(2+\alpha)} \times \widehat{f}_{k,\alpha}(2j+k+2) z^{j+k} \right) \\
&= \sum_{k=-j}^{+\infty} 2(\alpha+1) \frac{\Gamma(j+k+\alpha+2)}{(j+k)!\Gamma(2+\alpha)} \times \widehat{f}_{k,\alpha}(2j+k+2) \\
&\quad \cdot \left(\sum_{l=-(j+k)}^{-\infty} 2(\alpha+1) \frac{\Gamma(-j-k-l+\alpha+2)}{(-j-k-l)!\Gamma(2+\alpha)} \times \widehat{g}_{l,\alpha}(-l+2) z^{-k-j-l} \right) \\
&= \sum_{k=0}^{+\infty} 2(\alpha+1) \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \times \widehat{f}_{k-j,\alpha}(j+k+2) \\
&\quad \cdot \left(\sum_{l=0}^{+\infty} 2(\alpha+1) \frac{\Gamma(l+\alpha+2)}{l!\Gamma(2+\alpha)} \times \widehat{g}_{-l-k,\alpha}(l+k+2) z^l \right), \\
T_fH_g\left(z^j\right) &= T_{\sum_{k\in Z} e^{ik\theta} f_{k,\alpha}(r)} H_{\sum_{l\in Z} e^{il\theta} g_{l,\alpha}(r)} \left(z^j\right) \\
&= T_{\sum_{k\in Z} e^{ik\theta} f_{k,\alpha}(r)} \left(\sum_{l=-j}^{-\infty} 2(\alpha+1) \times \frac{\Gamma(-j-l+\alpha+2)}{(-j-l)!\Gamma(2+\alpha)} \times \widehat{g}_{l,\alpha}(-l+2) z^{-j-l} \right) \\
&= \sum_{l=-j}^{-\infty} 2(\alpha+1) \frac{\Gamma(-j-l+\alpha+2)}{(-j-l)!\Gamma(2+\alpha)} \times \widehat{g}_{l,\alpha}(-l+2) \\
&\quad \cdot \left(\sum_{k=j+l}^{+\infty} 2(\alpha+1) \frac{\Gamma(-j-l+k+\alpha+2)}{(-j-l+k)!\Gamma(2+\alpha)} \times \widehat{f}_{k,\alpha}(-2j-2l+k+2) z^{-j-l+k} \right) \\
&= \sum_{k=0}^{+\infty} 2(\alpha+1) \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \times \widehat{g}_{-k-j,\alpha}(k+j+2) \\
&\quad \cdot \left(\sum_{l=0}^{+\infty} 2(\alpha+1) \frac{\Gamma(l+\alpha+2)}{l!\Gamma(2+\alpha)} \times \widehat{f}_{l-k,\alpha}(l+k+2) z^l \right).
\end{aligned} \tag{40}$$

From the aforementioned we get, for $m \geq 0$ and $n \geq 0$,

$$\begin{aligned}
\left(H_gT_f\left(z^m\right), z^n\right) &= \left(\sum_{k=0}^{+\infty} 2(\alpha+1) \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \widehat{f}_{k-m,\alpha} \times (m+k+2) \cdot \sum_{l=0}^{+\infty} 2(\alpha+1) \frac{\Gamma(l+\alpha+2)}{l!\Gamma(2+\alpha)} \widehat{g}_{-l-k,\alpha} \times (l+k+2) z^l, z^n \right) \\
&= \left(\sum_{k=0}^{+\infty} 2(\alpha+1) \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \times \widehat{f}_{k-m,\alpha}(m+k+2) \cdot 2(\alpha+1) \frac{\Gamma(n+\alpha+2)}{n!\Gamma(2+\alpha)} \widehat{g}_{-n-k,\alpha} \times (n+k+2) z^n, z^n \right) \\
&= 4(\alpha+1)^2 \sum_{k=0}^{+\infty} \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \widehat{f}_{k-m,\alpha} \times (m+k+2) \widehat{g}_{-n-k,\alpha} \times (n+k+2), \\
\left(T_fH_g\left(z^m\right), z^n\right) &= \left(\sum_{k=0}^{+\infty} 2(\alpha+1) \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \times \widehat{g}_{-k-m,\alpha}(k+m+2) \cdot \sum_{l=0}^{+\infty} 2(\alpha+1) \frac{\Gamma(l+\alpha+2)}{l!\Gamma(2+\alpha)} \times \widehat{f}_{l-k,\alpha}(l+k+2) z^l, z^n \right)
\end{aligned}$$

$$= \left(\sum_{k=0}^{+\infty} 2(\alpha+1) \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \times \widehat{g}_{-k-m,\alpha}(k+m+2) \cdot 2(\alpha+1) \frac{\Gamma(n+\alpha+2)}{n!\Gamma(2+\alpha)} \widehat{f}_{n-k,\alpha} \times (n+k+2) z^n, z^n \right)$$
$$= 4(\alpha+1)^2 \sum_{k=0}^{+\infty} \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \times \widehat{f}_{n-k,\alpha}(n+k+2) \times \widehat{g}_{-k-m,\alpha}(k+m+2). \quad (41)$$

If $H_g T_f = T_f H_g$, then we get

$$\sum_{k=0}^{+\infty} \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \widehat{f}_{k-m,\alpha}(m+k+2) \times \widehat{g}_{-n-k,\alpha}(n+k+2) = \sum_{k=0}^{+\infty} \frac{\Gamma(k+\alpha+2)}{k!\Gamma(2+\alpha)} \widehat{f}_{n-k,\alpha}(n+k+2) \times \widehat{g}_{-k-m,\alpha}(k+m+2), \quad (42)$$

for $m \geq 0$ and $n \geq 0$.

The converse is easy to get. □

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