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Research Article

Weak Solutions for a p-Laplacian Impulsive Differential Equation

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By the virtue of variational method and critical point theory, we give some existence results of weak solutions for a p-Laplacian impulsive differential equation with Dirichlet boundary conditions.

1. Introduction

In this paper, we shall consider the following problem:

$$\begin{gathered}
-\left(\left|u'(t)\right|^{p-2}u'(t)\right)' = f(t,u(t)), \quad \text{a.e. } t \in [0,T], \\
\Delta\Phi\left(u'\left(t_j\right)\right) = \left|u'\left(t_j^+\right)\right|^{p-2}u'\left(t_j^+\right) - \left|u'\left(t_j^-\right)\right|^{p-2}u'\left(t_j^-\right) \\
= I_j\left(u\left(t_j\right)\right), \quad j = 1,2,\ldots,m, \\
u(0) = u(T) = 0,
\end{gathered} \tag{1}$$

where $p > 1$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $I_j : \mathbb{R} \to \mathbb{R}$, $(j = 1,2,\ldots,m)$ are continuous.

Many evolution processes are characterized by the fact that, at certain moments of time, they experience a change of state abruptly. These processes are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. Thus impulsive differential equations appear as a natural description of observed evolution phenomena of several real world problems.

Recently, there have been many papers to study impulsive problems by variational method and critical point theory, such as [1–11] and the references therein.

In [7], Nieto and O'Regan studied the linear Dirichlet impulsive problem

$$\begin{gathered}
-u''(t) + \lambda u(t) = \sigma(t), \quad t \neq t_j,\ t \in [0,T], \\
\Delta u'\left(t_j\right) = d_j, \quad j = 1,2,\ldots,p, \\
u(0) = u(T) = 0
\end{gathered} \tag{2}$$

and the nonlinear Dirichlet impulsive problem

$$\begin{gathered}
-u''(t) + \lambda u(t) = f(t,u(t)), \quad t \neq t_j,\ t \in [0,T], \\
\Delta u'\left(t_j\right) = I_j\left(u\left(t_j^-\right)\right), \quad j = 1,2,\ldots,p, \\
u(0) = u(T) = 0.
\end{gathered} \tag{3}$$

In the paper, they have shown that the impulsive problem minimizes some (energy) functional, and the critical points of that functional are indeed solutions of the impulsive problem.

In [3, 4], Sun et al. utilized some variant fountain theorems by [12] to consider the existence of infinitely many solutions for the following two impulsive problems:

$$\begin{gathered}
-u''(t) + g(t)u(t) = f(t,u(t)), \quad \text{a.e. } t \in [0,T], \\
\Delta u'\left(t_j\right) = I_j\left(u\left(t_j\right)\right), \quad j = 1,2,\ldots,p, \\
u(0) = u(T) = 0,
\end{gathered}$$

$$
\begin{gathered}
-\ddot{u}+A(t)u=\nabla W(t,u), \quad \text{a.e. } t\in[0,T],\\
\Delta\dot{u}\left(t_j\right)=I_{ij}\left(u^i\left(t_j\right)\right), \quad i=1,2,\ldots,N, \ j=1,2,\ldots,l,\\
u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0.
\end{gathered} \tag{4}
$$

Admittedly, they obtained many perfect results.

We also note that some people begin to study p-Laplacian differential equations with impulsive effects; for example, see [1, 2, 8–11].

In [1], Chen and Tang considered the p-Laplacian impulsive problem

$$
\begin{gathered}
-\left(\left|u'(t)\right|^{p-2}u'(t)\right)'+g(t)|u(t)|^{p-2}u(t)\\
=f(t,u(t)), \quad \text{a.e. } t\in[0,T],\\
\Delta u'\left(t_j\right)=I_j\left(u\left(t_j\right)\right), \quad j=1,2,\ldots,m,\\
u(0)=u(T)=0.
\end{gathered} \tag{5}
$$

They established some existence theorems for one or infinitely many solutions under more relaxed assumptions on their nonlinearity f, which satisfies a kind of new superquadratic and subquadratic condition.

In [8], Bogun discussed the existence of weak solutions for the p-Laplacian problem with superlinear impulses by the virtue of mountain pass theorem and symmetric mountain pass theorem

$$
\begin{gathered}
-\left(\left|u'\right|^{p-2}u'\right)'=f(t,u), \quad \text{in } \Omega,\\
u(0)=u(1)=0,\\
u\left(t_j^+\right)-u\left(t_j^-\right)=0, \quad j=1,2,\ldots,n,\\
\Delta\left|u'\left(t_j\right)\right|^{p-2}u'\left(t_j\right)=I_j\left(u\left(t_j\right)\right), \quad j=1,2,\ldots,n.
\end{gathered} \tag{6}
$$

In [9], Xu et al. reconsidered the previous problem by topological degree theory and Fountain theorem under Cerami condition.

In [11], by the virtue of three critical points theorem obtained by Bai and Dai is studied the existence of at least three solutions for the following p-Laplacian boundary value problem:

$$
\begin{gathered}
\left(\rho(t)\phi_p\left(u'(t)\right)\right)'-s(t)\phi_p(u(t))+\lambda f(t,u(t))=0,\\
\text{a.e. } t\in(a,b),\\
\alpha_1 u'\left(a^+\right)-\alpha_2 u(a)=0, \qquad \beta u'\left(b^-\right)+\beta_2 u(b)=0,\\
\Delta\left(\rho\left(t_j\right)\phi_p\left(u'\left(t_j\right)\right)\right)=I_j\left(u\left(t_j\right)\right), \quad j=1,2,\ldots,l.
\end{gathered} \tag{7}
$$

Motivated by the previous facts, in this paper, our aim is to study the existence and multiplicity of weak solutions for impulsive problem (1) by using variational method and critical point theory. It is well known that the Ambrosetti-Rabinowitz type condition is to ensure the boundedness of all (PS) sequences of the corresponding functional. However, without it, it will become more complicated. Therefore, we will use new variant fountain theorems due to Zou [12] to overcome this difficulty and obtain infinitely many weak solutions for (1). On the other hand, for the superlinear at $+\infty$ and asymptotically linear at $-\infty$, we obtain a weak solution for (1) by the mountain pass theorem. The results obtained here improve some existing results in the literature.

2. Preliminaries

In this section, we recall some fundamental facts of critical point theory which will be used in the proofs of our main results. Let W be the Sobolev space $W_0^{1,p}(0,T)$ with the usual norm

$$
\|u\| = \left(\int_0^T\left|u'(t)\right|^p \mathrm{d}t\right)^{1/p}, \quad \forall u\in W_0^{1,p}(0,T). \tag{8}
$$

It is clear that $W_0^{1,p}(0,T)$ is a reflexive Banach space. Next, we make a finite dimensional decomposition for W. In order to do this, we first need to consider the eigenvalue problem

$$
\begin{gathered}
\left(\left|u'(t)\right|^{p-2}u'(t)\right)'+\lambda|u(t)|^{p-2}u(t)=0, \quad t\in[0,T],\\
u(0)=u(T)=0.
\end{gathered} \tag{9}
$$

It is well known that the set of all eigenvalues of the problem (9) is given by the sequence of positive numbers (see [1, 13–15])

$$
\begin{gathered}
\lambda_k := (p-1)\left(\frac{k\pi_p}{T}\right)^p, \quad \text{for } k=1,2,\ldots,\\
\text{where } \pi_p := \frac{2\pi}{p\sin(\pi/p)}.
\end{gathered} \tag{10}
$$

We denote by φ_k the corresponding eigenfunctions associated with λ_k for all k, and $\varphi_k\in W$. Moreover, the first eigenvalue λ_1 is simple and isolated, and φ_1 is positive in $[0,T]$. Furthermore, the Poincaré inequality

$$
\int_0^T\left|u'(t)\right|^p\mathrm{d}t\ge\lambda_1\int_0^T|u(t)|^p\mathrm{d}t, \quad \forall u\in W_0^{1,p}(0,T) \tag{11}
$$

holds. Note that we can normalize φ_k such that

$$
\int_0^T\left|\varphi_k(t)\right|^p\mathrm{d}t=1, \quad \forall k. \tag{12}
$$

Fix any $k\ge 1$ define $Y_k=\operatorname{span}\{\varphi_1,\ldots,\varphi_k\}$ and

$$
Z_k=\bigcap_{j=1}^{k}\ker\left(\mathscr{L}_j\right)=\left\{u\in W:\mathscr{L}_1(u)=\cdots=\mathscr{L}_k(u)=0\right\}, \tag{13}
$$

where

$$
\mathscr{L}_j u=\int_0^T\left|\varphi_j(t)\right|^{p-2}\varphi_j(t)u(t)\,\mathrm{d}t. \tag{14}
$$

By [16, Section 5], the conclusions are

$$W = Y_k \oplus Z_k, \qquad \dim Y_k = k,$$
$$\lambda_{k+1} \int_0^T |u(t)|^p dt \le \int_0^T |u'(t)|^p dt, \quad \forall u \in Z_k,\ k \ge 1. \tag{15}$$

Note the definitions of λ_k and φ_k; by (9), we have

$$\left(\left|\varphi_k'(t)\right|^{p-2}\varphi_k'(t)\right)' + \lambda_k \left|\varphi_k(t)\right|^{p-2}\varphi_k(t) = 0 \quad \text{for } t \in [0,T]. \tag{16}$$

For each $v \in W$, multiply by v on both sides of (16) to obtain

$$\int_0^T \left|\varphi_k'(t)\right|^{p-2}\varphi_k'(t)\, v'(t)\, dt = \lambda_k \int_0^T \left|\varphi_k(t)\right|^{p-2}\varphi_k(t)\, v(t)\, dt. \tag{17}$$

In particular, choosing $v = \varphi_k$, we see $\int_0^T |\varphi_k'(t)|^p dt = \lambda_k$ for all k.

We denote the norms in $L^r(0,T)$ $(1 < r < \infty)$ and $C[0,T]$ as follows:

$$\|u\|_r := \left(\int_0^T |u(t)|^r dt\right)^{1/r}, \quad \forall u \in L^r(0,T),$$
$$\|u\|_\infty := \max_{t\in[0,T]} |u(t)|, \quad \forall u \in C[0,T]. \tag{18}$$

By the Sobolev embedding theorem, the embeddings $W \hookrightarrow L^r(0,T)$ and $W \hookrightarrow C[0,T]$ are compact. Consequently, we also find that there are two constants $C_{\text{emb1}} > 0$ and $C_{\text{emb2}} > 0$ such that

$$\|u\|_r \le C_{\text{emb1}} \|u\|, \quad \|u\|_\infty \le C_{\text{emb2}} \|u\|, \quad \forall u \in W. \tag{19}$$

For $u \in W^{1,p}(0,T)$, we have that u and u' are both absolutely continuous. Hence, $\Delta u'(t) = u'(t^+) - u'(t^-) = 0$ for any $t \in [0,T]$. If $u \in W_0^{1,p}(0,T)$, then u is absolutely continuous. In this case, the one-sided derivatives $u'(t^+)$, $u'(t^-)$ may not exist. It leads to the impulsive effects. As a result, we need to introduce a different concept of solution. Suppose that $u \in C[0,T]$ satisfies the Dirichlet condition $u(0) = u(T) = 0$. Assume that, for every $j = 1,2,\ldots,m$, $u_j = u|_{(t_j,t_{j+1})}$ and $u_j \in W^{1,p}(0,T)$. Let $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$.

Take $v \in W$ and multiply the two sides of the equality

$$-\left(\left|u'(t)\right|^{p-2}u'(t)\right)' = f(t,u(t)) \tag{20}$$

by v and integrate from 0 to T:

$$\int_0^T -\left(\left|u'(t)\right|^{p-2}u'(t)\right)' v(t)\, dt = \int_0^T f(t,u(t))\, v(t)\, dt. \tag{21}$$

For the left term, in view of impulsive effects, we find

$$\begin{aligned}
&\int_0^T -\left(\left|u'(t)\right|^{p-2}u'(t)\right)' v(t)\, dt \\
&= \sum_{j=0}^{m} \int_{t_j}^{t_{j+1}} -\left(\left|u'(t)\right|^{p-2}u'(t)\right)' v(t)\, dt \\
&= \sum_{j=1}^{m} I_j\left(u\left(t_j\right)\right) v\left(t_j\right) - v(T)\left|u'(T)\right|^{p-2}u'(T) \\
&\quad + v(0)\left|u'(0)\right|^{p-2}u'(0) + \int_0^T \left|u'(t)\right|^{p-2}u'(t)\, v'(t)\, dt \\
&= \sum_{j=1}^{m} I_j\left(u\left(t_j\right)\right) v\left(t_j\right) + \int_0^T \left|u'(t)\right|^{p-2}u'(t)\, v'(t)\, dt.
\end{aligned} \tag{22}$$

Consequently,

$$\int_0^T \left|u'(t)\right|^{p-2}u'(t)\, v'(t)\, dt + \sum_{j=1}^{m} I_j\left(u\left(t_j\right)\right) v\left(t_j\right) = \int_0^T f(t,u(t))\, v(t)\, dt. \tag{23}$$

Considering the previous, we introduce the following concept for the solution for (1).

Definition 1. One says that a function $u \in W$ is a weak solution for (1) if the identity

$$\int_0^T \left|u'(t)\right|^{p-2}u'(t)\, v'(t)\, dt + \sum_{j=1}^{m} I_j\left(u\left(t_j\right)\right) v\left(t_j\right) = \int_0^T f(t,u(t))\, v(t)\, dt, \quad \forall v \in W. \tag{24}$$

Consider the functional $J : W \to \mathbb{R}$ defined by

$$J(u) = \frac{1}{p}\|u\|^p + \sum_{j=1}^{m} \int_0^{u(t_j)} I_j(t)\, dt - \int_0^T F(t,u(t))\, dt, \tag{25}$$

where $F(t,u) = \int_0^u f(t,s) ds$. Note that for the continuity of f and I_j $(j = 1,2,\ldots,m)$, we see $J \in C^1(W,\mathbb{R})$. Furthermore, the derivative of J is

$$\begin{aligned}
\left(J'(u), v\right) = &\int_0^T \left|u'(t)\right|^{p-2}u'(t)\, v'(t)\, dt \\
&+ \sum_{j=1}^{m} I_j\left(u\left(t_j\right)\right) v\left(t_j\right) \\
&- \int_0^T f(t,u(t))\, v(t)\, dt, \quad \forall u, v \in W.
\end{aligned} \tag{26}$$

Thus, we easily know that weak solutions of (1) coincide with the critical points of the C^1-functional J.

For the reader's convenience, we now present some critical point theorems; one can refer to [12, 17–22] for more details.

Definition 2. Let X be a real Banach space and $J \in C^1(X, \mathbb{R})$. For any sequence $\{u_n\} \subset X$, if $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence, then we say that J satisfies the Palais-Smale condition (PS condition for short).

Definition 3. One says that J satisfies $(\text{PS})_c$ condition if the existence of a sequence $\{u_n\} \subset X$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$ as $n \to \infty$ implies that $\{u_n\}$ has a convergent subsequence.

Lemma 4 (see [17]). *Let $J \in C^1(X, \mathbb{R})$ satisfy (PS) condition. Suppose that*

(i) $J(0) = 0$,

(ii) *there exist $\rho > 0$ and $\alpha > 0$ such that $J(u) \ge \alpha$ for all $u \in X$, $\|u\| = \rho$,*

(iii) *there exists $u_1 \in X$ with $\|u_1\| > \rho$ such that $J(u_1) < \alpha$.*

Then J has a critical value $c \ge \alpha$. Moreover, c can be characterized as $\inf_{g \in \Gamma} \max_{u \in g([0,1])} J(u)$, where $\Gamma := \{g \in C([0,1], X) : g(0) = 0, g(1) = u_1\}$.

Let X be a Banach space equipped with the norm $\|\cdot\|$ and $X = \overline{\bigoplus_{j \in \mathbb{N}} X_j}$, where $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \bigoplus_{j=1}^{k} X_j$ and $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$. In the following, one will introduce variant fountain theorems by Zou [12]. Let X and the subspaces Y_k and Z_k be defined as previously. Consider the following C^1-functional $J_\lambda : X \to \mathbb{R}$ defined by

$$J_\lambda(u) := A(u) - \lambda B(u), \quad \lambda \in [1, 2]. \tag{27}$$

Lemma 5. *If the functional J_λ satisfies the following:*

(T1) *J_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. Moreover, $J_\lambda(-u) = J_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times X$,*

(T2) *$B(u) \ge 0$ for all $u \in X$; moreover, $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$,*

(T3) *there exists $r_k > \rho_k > 0$ such that*

$$\begin{aligned} a_k(\lambda) &:= \inf_{u \in Z_k, \|u\| = \rho_k} J_\lambda(u) > b_k(\lambda) \\ &:= \max_{u \in Y_k, \|u\| = r_k} J_\lambda(u), \quad \forall \lambda \in [1, 2], \end{aligned} \tag{28}$$

then

$$a_k(\lambda) \le \zeta_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} J_\lambda(\gamma(u)), \quad \forall \lambda \in [1, 2], \tag{29}$$

where $B_k = \{u \in Y_k : \|u\| \le r_k\}$ and $\Gamma_k = \{\gamma \in C(B_k, X) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = \text{id}\}$. Moreover, for a.e. $\lambda \in [1, 2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^{\infty}$ such that

$$\begin{gathered} \sup_n \left\|u_n^k(\lambda)\right\| < \infty, \qquad J'_\lambda\left(u_n^k(\lambda)\right) \longrightarrow 0, \\ J_\lambda\left(u_n^k(\lambda)\right) \to \zeta_k(\lambda) \quad \text{as } n \longrightarrow \infty. \end{gathered} \tag{30}$$

Now, we list our assumptions on f and I_j ($j = 1, 2, \ldots, m$).

(H1) There exist $\mu_1, \mu_2 > 0$ such that $|f(t,u)| \le \mu_1 + \mu_2 |u|^{q-1}$ $(1 < q < \infty)$, $\forall u \in \mathbb{R}, t \in [0, T]$,

(H2) $\lim_{u \to 0}(f(t,u)/|u|^{p-2}u) = f_0 \in [0, +\infty)$ uniformly for $t \in [0, T]$,

(H3) $\lim_{u \to -\infty}(f(t,u)/|u|^{p-2}u) = f_\infty \in [0, +\infty)$ uniformly for $t \in [0, T]$,

(H4) There exist $\delta > 0$ and $\theta \in (0, 1/p)$ such that

$$0 < F(t,u) \le \theta u f(t,u), \quad t \in [0, T], \ |u| \ge \delta, \tag{31}$$

(H5) $F(t,u) \ge 0$ for all $(t,u) \in [0, T] \times \mathbb{R}$ and $\lim_{|u| \to \infty}(F(t,u)/|u|^p) = +\infty$, uniformly on $t \in [0, T]$,

(H6) there is a positive constant $b > 0$ such that $\lim_{|u| \to \infty}((-pF(t,u) + f(t,u)u)/|u|^q) \ge b$, uniformly on $t \in [0, T]$,

(H7) $\int_0^u I_j(s)ds \ge 0$, $\forall u \in \mathbb{R}$, $j = 1, 2, \ldots, m$,

(H8) $p\int_0^u I_j(s)ds - I_j(u)u \ge 0$, $\forall u \in \mathbb{R}$, $j = 1, 2, \ldots, m$,

(H9) There exist $\sigma_j, \tau_j > 0$ and $\gamma_j \in [1, p)$ such that $|I_j(u)| \le \sigma_j + \tau_j |u|^{\gamma_j - 1}$, $\forall u \in \mathbb{R}$ and $j = 1, 2, \ldots, m$,

(H10) $f(t,u)$ and $I_j(u)$ $(j = 1, 2, \ldots, m)$ are odd functions about u, for all $t \in [0, T]$.

Remark 6. (1) As known to all, (H4) implies that $\lim_{u \to \infty}(F(t,u)/|u|^p) = +\infty$; that is, $f(t,u)$ is superlinear at ∞ with respect to $|u|^{p-2}u$. In view of (H3) and (H4), we see that $f(t,u)$ is superlinear at $+\infty$ and asymptotically linear at $-\infty$; this is a new case. However, the nonlinearity f in [9] is asymptotically linear at $\pm\infty$.

(2) Condition (H6) is weaker than the well-known Ambrosetti-Rabinowitz condition (H4); also see condition (p_2) in [8]. Indeed, by (H6), there is a $\delta > 0$ such that

$$-pF(t,u) + uf(t,u) \ge b|u|^q \ge 0, \quad \forall |u| \ge \delta. \tag{32}$$

Consequently, $pF(t,u) \le uf(t,u)$ for all $t \in [0, T]$ and $|u| \ge \delta$, which is weaker than condition (H4).

3. Main Results

Theorem 7. *Suppose that (H1)–(H4), (H7), and (H9) hold, $q \in (p, +\infty)$, and $f_0 < \lambda_1 < f_\infty$ with $f_\infty \ne \lambda_k$ for all k. Then (1) has a weak solution.*

Proof. From (H1)–(H4), for all $\varepsilon > 0$, there exist $\mu_3 > 0, \mu_4 > 0$ such that

$$F(t,u) \le \frac{1}{p}(f_0 + \varepsilon)|u|^p + \mu_3 |u|^q, \quad \forall t \in [0, T], \ u \in \mathbb{R}, \tag{33}$$

$$F(t,u) \ge \frac{1}{p}(f_\infty - \varepsilon)|u|^p - \mu_4, \quad \forall t \in [0, T], \ u \in \mathbb{R}. \tag{34}$$

Choose $\varepsilon > 0$ such that $(f_0 + \varepsilon) < \lambda_1$, together with (33), (11), (19), and (H7); we obtain

$$\begin{aligned} J(u) &= \frac{1}{p}\|u\|^p + \sum_{j=1}^{m}\int_0^{u(t_j)} I_j(t)\,dt - \int_0^T F(t,u(t))\,dt \\ &\geq \frac{1}{p}\|u\|^p - \int_0^T F(t,u(t))\,dt \\ &\geq \frac{1}{p}\|u\|^p - \int_0^T \left[\frac{1}{p}(f_0+\varepsilon)|u|^p + \mu_3|u|^q\right]dt \\ &\geq \frac{1}{p}\|u\|^p - \frac{1}{p}\frac{f_0+\varepsilon}{\lambda_1}\|u\|^p - \mu_3 C_{\text{emb1}}^q\|u\|^q. \end{aligned} \tag{35}$$

If ρ is small enough, (ii) of Lemma 4 can be proved.

On the other hand, we can take $\varepsilon > 0$ such that $f_\infty - \varepsilon > \lambda_1$; by (34), (19), and (H9), noting that $\gamma_j \in [1,p)$, we find

$$\begin{aligned} J(s\varphi_1) &= \frac{1}{p}\|s\varphi_1\|^p + \sum_{j=1}^{m}\int_0^{s\varphi_1(t_j)} I_j(t)\,dt \\ &\quad - \int_0^T F(t,s\varphi_1(t))\,dt \\ &\leq \frac{1}{p}\|s\varphi_1\|^p + \sum_{j=1}^{m}\int_0^{s\varphi_1(t_j)}\left[\sigma_j + \tau_j|t|^{\gamma_j-1}\right]dt \\ &\quad - \int_0^T\left[\frac{1}{p}(f_\infty-\varepsilon)|s\varphi_1|^p - \mu_4\right]dt \\ &\leq \frac{1}{p}\|s\varphi_1\|^p - \frac{1}{p}\frac{f_\infty-\varepsilon}{\lambda_1}\|s\varphi_1\|^p \\ &\quad + \sum_{j=1}^{m}\left[\sigma_j\|s\varphi_1\|_\infty + \frac{\tau_j}{\gamma_j}\|s\varphi_1\|_\infty^{\gamma_j}\right] + \mu_4 T \\ &\leq \frac{1}{p}\|s\varphi_1\|^p - \frac{1}{p}\frac{f_\infty-\varepsilon}{\lambda_1}\|s\varphi_1\|^p \\ &\quad + \sum_{j=1}^{m}\left[\sigma_j C_{\text{emb2}}\|s\varphi_1\| + \frac{\tau_j}{\gamma_j}C_{\text{emb2}}^{\gamma_j}\|s\varphi_1\|^{\gamma_j}\right] + \mu_4 T \\ &\longrightarrow -\infty \quad \text{as } s \longrightarrow -\infty. \end{aligned} \tag{36}$$

Therefore, (iii) of Lemma 4 is also proved, as required.

Now, we only claim that J satisfies (PS) condition. Supposing that $\{u_n\} \subset W$ is a (PS) sequence, for all $n \in \mathbb{N}$, we have

$$\left|\frac{1}{p}\|u_n\|^p + \sum_{j=1}^{m}\int_0^{u_n(t_j)} I_j(t)\,dt - \int_0^T F(t,u_n(t))\,dt\right| \leq \zeta, \tag{37}$$

$$\begin{aligned} &\left|\int_0^T \left|u_n'(t)\right|^{p-2}u_n'(t)\,v'(t)\,dt \right. \\ &\left. + \sum_{j=1}^{m} I_j\left(u_n\left(t_j\right)\right)v\left(t_j\right) - \int_0^T f(t,u_n(t))\,v(t)\,dt\right| \\ &\leq \varepsilon_n\|v\|, \quad \forall v \in W, \end{aligned} \tag{38}$$

where $\zeta > 0$ is a constant and $\varepsilon_n \to 0^+$ as $n \to \infty$. Next, we will show that $\{u_n\}$ is a bounded sequence in W. If not, there is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$\|u_n\| \longrightarrow \infty, \quad n \longrightarrow \infty. \tag{39}$$

Define $z_n = u_n/\|u_n\|$, then $\|z_n\| = 1, \forall n \in \mathbb{N}$, and thus it has a subsequence, still denoted $\{z_n\}$, such that $z_n \rightharpoonup z_0$ weakly in W, $z_n \to z_0$ strongly in $L^p(0,T)$, $z_n(t) \to z_0(t)$ a.e. $t \in [0,T]$, and $|z_n(t)| \leq \eta(t)$, a.e. $t \in [0,T]$, where $z_0 \in W$, $\eta \in L^p(0,T)$.

Divide (38) by $\|u_n\|^{p-1}$ to get

$$\begin{aligned} &\left|\int_0^T \left|z_n'(t)\right|^{p-2}z_n'(t)\,v'(t)\,dt \right. \\ &\left. + \sum_{j=1}^{m}\frac{I_j\left(u_n\left(t_j\right)\right)}{\|u_n\|^{p-1}}v\left(t_j\right) - \int_0^T \frac{f(t,u_n(t))}{\|u_n\|^{p-1}}v(t)\,dt\right| \\ &\leq \frac{\varepsilon_n}{\|u_n\|^{p-1}}\|v\|, \quad \forall v \in W. \end{aligned} \tag{40}$$

Passing to the limit in (40), we see

$$\begin{aligned} &\lim_{n\to\infty}\int_0^T \frac{f(t,u_n(t))}{\|u_n\|^{p-1}}v(t)\,dt \\ &\quad = \int_0^T \left|z_0'(t)\right|^{p-2}z_0'(t)\,v'(t)\,dt, \quad \forall v \in W, \end{aligned} \tag{41}$$

with the fact that

$$\begin{aligned} \left|\sum_{j=1}^{m}\frac{I_j\left(u_n\left(t_j\right)\right)}{\|u_n\|^{p-1}}v\left(t_j\right)\right| &\leq \sum_{j=1}^{m}\frac{\left|I_j\left(u_n\left(t_j\right)\right)\right|}{\|u_n\|^{p-1}}\left|v\left(t_j\right)\right| \\ &\leq \sum_{j=1}^{m}\frac{\sigma_j + \tau_j C_{\text{emb2}}^{\gamma_j-1}\|u_n\|^{\gamma_j-1}}{\|u_n\|^{p-1}}C_{\text{emb2}}\|v\| \longrightarrow 0, \\ &\qquad \forall v \in W. \end{aligned} \tag{42}$$

Now, we claim that $z_0(t) \leq 0$, a.e. $t \in [0,T]$. Indeed, in (41), taking $v = z_0^+ = \max\{z_0, 0\}$, we arrive at

$$\lim_{n\to\infty}\int_{\Omega^+}\frac{f(t,u_n(t))}{\|u_n\|^{p-1}}z_0(t)\,dt = \int_{\Omega^+}\left|z_0'(t)\right|^p dt < +\infty, \tag{43}$$

where $\Omega^+ := \{t \in [0,T] : z_0(t) > 0\}$. However, by (H3) and (H4), there is $\mu_5 > 0$ such that

$$\begin{aligned} &\frac{f(t,u_n(t))}{\|u_n\|^{p-1}}z_0(t) \\ &\quad \geq \left(-f_\infty|\eta(t)|^{p-2}\eta(t) - \mu_5\right)z_0(t), \\ &\qquad \text{a.e. } t \in [0,T]. \end{aligned} \tag{44}$$

In addition, note that $\lim_{n\to\infty} u_n(t) = +\infty$ a.e. $t \in \Omega^+$ and the super-linearity of f we have

$$\begin{aligned}\lim_{n\to\infty} & \frac{f(t,u_n(t))}{\|u_n\|^{p-1}} z_0(t) \\ &= \lim_{n\to\infty} \frac{f(t,u_n(t))}{u_n^{p-1}} z_n^{p-1}(t) z_0(t) \\ &= +\infty, \quad \text{a.e. } t \in \Omega^+.\end{aligned} \tag{45}$$

Consequently, if $|\Omega^+| > 0$, by the Fatou theorem, we get

$$\lim_{n\to\infty} \int_{\Omega^+} \frac{f(t,u_n(t))}{\|u_n\|^{p-1}} z_0(t)\, dt = +\infty, \tag{46}$$

which contradicts the fact of (43).

Obviously, $z_0(t) \not\equiv 0$. From (H2) and (H3), there exists $\mu_6 > 0$ such that $|f(t,u_n)|/|u_n|^{p-1} \le \mu_6$ a.e. $t \in [0,T]$. By (41), Lebesgue's dominated convergence theorem enables us to see

$$\begin{aligned}\int_0^T & |z_0'(t)|^{p-2} z_0'(t) v'(t)\, dt \\ &= f_\infty \int_0^T |z_0(t)|^{p-2} z_0(t) v(t)\, dt, \\ & \qquad \forall v \in W.\end{aligned} \tag{47}$$

This contradicts $f_\infty \ne \lambda_k$ for all k. Therefore, $\{u_n\}$ is bounded, as required. Going, if necessary, to a subsequence, we can assume that $u_n \rightharpoonup u$ weakly in W; then

$$\begin{aligned}&\left(J'(u_n) - J'(u), u_n - u\right) \\ &\quad = \int_0^T \left(|u_n'|^{p-2}u_n' - |u'|^{p-2}u'\right)(u_n' - u')\, dt \\ &\qquad + \sum_{j=1}^m \left(I_j(u_n(t_j)) - I_j(u(t_j))\right)(u_n(t_j) - u(t_j)) \\ &\qquad - \int_0^T (f(t,u_n) - f(t,u))(u_n - u)\, dt.\end{aligned} \tag{48}$$

$W \hookrightarrow\hookrightarrow C[0,T]$ enables us to obtain that

$$\begin{aligned}&\sum_{j=1}^m \left(I_j(u_n(t_j)) - I_j(u(t_j))\right)(u_n(t_j) - u(t_j)) \longrightarrow 0, \\ &\int_0^T (f(t,u_n) - f(t,u))(u_n - u)\, dt \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{49}$$

It follows from $u_n \rightharpoonup u$ weakly in W and $(J'(u_n) - J'(u), u_n - u) \to 0$ that

$$\begin{aligned}\int_0^T \left(|u_n'|^{p-2}u_n' - |u'|^{p-2}u'\right)(u_n' - u')\, dt \longrightarrow 0 \\ \text{as } n \longrightarrow \infty.\end{aligned} \tag{50}$$

Note that

$$\begin{aligned}&\int_0^T \left(|u_n'|^{p-2}u_n' - |u'|^{p-2}u'\right)(u_n' - u')\, dt \\ &\quad \ge \left(\|u_n\|^{p-1} - \|u\|^{p-1}\right)(\|u_n\| - \|u\|),\end{aligned} \tag{51}$$

and thus $\|u_n - u\| \to 0$ as $n \to \infty$. So, J satisfies (PS) condition. This completes the proof. □

In what follows, we will utilize Lemma 5 to study (1). Now, we define a class of functionals on W by

$$\begin{aligned}J_\lambda(u) &= \frac{1}{p}\|u\|^p + \sum_{j=1}^m \int_0^{u(t_j)} I_j(t)\, dt - \lambda \int_0^T F(t,u(t))\, dt \\ &= A(u) - \lambda B(u), \quad \lambda \in [1,2].\end{aligned} \tag{52}$$

It is easy to know that $J_\lambda \in C^1(W,\mathbb{R})$ for all $\lambda \in [1,2]$ and the critical points of J_1 correspond to the weak solutions of problem (1). Note that $J_1 = J$, where J is the functional defined in (25).

Theorem 8. *Assume that (H1) and (H5)–(H10) hold. Then* (1) *possesses infinitely many weak solutions.*

Proof. We first prove that there is a positive integer k_1 and two sequences $r_k > \rho_k \to \infty$ as $k \to \infty$ such that

$$a_k(\lambda) = \inf_{u\in Z_k, \|u\|=\rho_k} J_\lambda(u) > 0, \quad \forall k \ge k_1, \tag{53}$$

$$b_k(\lambda) = \max_{u\in Y_k, \|u\|=r_k} J_\lambda(u) < 0, \quad \forall k \in \mathbb{N}, \tag{54}$$

where $Y_k = \operatorname{span}\{\varphi_1, \ldots, \varphi_k\}$ and $Z_k = \bigcap_{j=1}^k \ker(\mathscr{L}_j)$.

Step 1. We will show that (53) holds true.

From (H1), we see that there exist $\mu_7 > 0$, $\mu_8 > 0$ such that

$$F(t,u) \le \mu_7 |u| + \mu_8 |u|^q, \quad \forall (t,u) \in [0,T] \times \mathbb{R}. \tag{55}$$

Consequently, by (H5) and (H7), we obtain

$$\begin{aligned}J_\lambda(u) &= \frac{1}{p}\|u\|^p + \sum_{j=1}^m \int_0^{u(t_j)} I_j(t)\, dt - \lambda \int_0^T F(t,u(t))\, dt \\ &\ge \frac{1}{p}\|u\|^p - 2\int_0^T F(t,u(t))\, dt \\ &\ge \frac{1}{p}\|u\|^p - 2\int_0^T \left(\mu_7|u| + \mu_8|u|^q\right) dt \\ &\ge \frac{1}{p}\|u\|^p - 2\mu_7 T^{1-1/q}\|u\|_q - 2\mu_8\|u\|_q^q.\end{aligned} \tag{56}$$

Let $\nu_q(k) = \sup_{u\in Z_k, \|u\|=1} \|u\|_q$, $\forall k \in \mathbb{N}$. Then by [19, Lemma 3.8], $\nu_q(k) \to 0$ as $k \to \infty$. Since $W \hookrightarrow\hookrightarrow L^q(0,T)$, we find

$$J_\lambda(u) \ge \frac{1}{p}\|u\|^p - 2\mu_7 T^{1-1/q} \nu_q(k)\|u\| - 2\mu_8 \nu_q^q(k)\|u\|^q. \tag{57}$$

Let $\rho_k = 1/\nu_q(k) \to \infty$ as $k \to \infty$. Then there exists k_1 such that $(1/p)\rho_k^p - 2\mu_7 T^{1-1/q} - 2\mu_8 > 0$, $\forall k \geq k_1$. Therefore,

$$a_k(\lambda) = \inf_{u \in Z_k, \|u\|=\rho_k} J_\lambda(u) \geq \frac{1}{p}\rho_k^p - 2\mu_7 T^{1-1/q} - 2\mu_8 > 0, \quad \forall k \geq k_1. \tag{58}$$

Step 2. We will show that (54) holds true.

We first prove that there exists $\varepsilon > 0$ such that

$$\begin{gathered} \operatorname{meas}(t \in [0,T] : |u(t)| \geq \varepsilon \|u\|) \geq \varepsilon, \\ \forall u \in \mathscr{X} \setminus \{0\}, \quad \forall \mathscr{X} \subset W, \quad \dim \mathscr{X} < \infty. \end{gathered} \tag{59}$$

There exists otherwise a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{X} \setminus \{0\}$ such that

$$\operatorname{meas}\left(t \in [0,T] : |u_n(t)| \geq \frac{\|u_n\|}{n}\right) < \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{60}$$

For each $n \in \mathbb{N}$, let $v_n := u_n/\|u_n\| \in \mathscr{X} \Rightarrow \|v_n\| = 1, \forall n \in \mathbb{N}$ and

$$\operatorname{meas}\left(t \in [0,T] : |v_n(t)| \geq \frac{1}{n}\right) < \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{61}$$

Passing to a subsequence if necessary, we may assume $v_n \to v_0$ in W for some $v_0 \in \mathscr{X}$ since $\mathscr{X}$ is of finite dimension. We easily find $\|v_0\| = 1$. Consequently, there exists a constant $\sigma_0 > 0$ such that

$$\operatorname{meas}(t \in [0,T] : |v_0(t)| \geq \sigma_0) \geq \sigma_0. \tag{62}$$

Indeed, if not, then we have

$$\begin{gathered} \operatorname{meas}\left(t \in [0,T] : |v_0(t)| \geq \frac{1}{n}\right) = 0, \\ \text{i.e., } \operatorname{meas}\left(t \in [0,T] : |v_0(t)| < \frac{1}{n}\right) = T, \quad \forall n \in \mathbb{N}, \end{gathered} \tag{63}$$

which implies

$$0 < \int_0^T |v_0(t)|^p \mathrm{d}t \leq \frac{T}{n^p} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{64}$$

This leads to $v_0 = 0$, contradicting to $\|v_0\| = 1$. In view of $W \hookrightarrow\hookrightarrow L^p(0,T)$ and the equivalence of any two norms on $\mathscr{X}$, we have

$$\int_0^T |v_n - v_0|^p \mathrm{d}t \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{65}$$

For every $n \in \mathbb{N}$, denote

$$\begin{gathered} \mathscr{N} := \left\{t \in [0,T] : |v_n(t)| < \frac{1}{n}\right\}, \\ \mathscr{N}^c := \left\{t \in [0,T] : |v_n(t)| \geq \frac{1}{n}\right\}, \end{gathered} \tag{66}$$

and $\mathscr{N}_0 := \{t \in [0,T] : |v_0(t)| \geq \sigma_0\}$, where σ_0 is defined by (62). Then for n large enough, by (62), we see

$$\operatorname{meas}(\mathscr{N} \cap \mathscr{N}_0) \geq \operatorname{meas}(\mathscr{N}_0) - \operatorname{meas}(\mathscr{N}^c) \geq \sigma_0 - \frac{1}{n} \geq \frac{\sigma_0}{2}. \tag{67}$$

Consequently, for n large enough, we arrive immediately at

$$\begin{aligned} \int_0^T & |v_n - v_0|^p \mathrm{d}t \\ & \geq \int_{\mathscr{N} \cap \mathscr{N}_0} |v_n - v_0|^p \mathrm{d}t \\ & \geq \frac{1}{2^p} \int_{\mathscr{N} \cap \mathscr{N}_0} |v_0|^p \mathrm{d}t - \int_{\mathscr{N} \cap \mathscr{N}_0} |v_n|^p \mathrm{d}t \\ & \geq \left(\frac{\sigma_0^p}{2^p} - \frac{1}{n^p}\right) \operatorname{meas}(\mathscr{N} \cap \mathscr{N}_0) \geq \frac{\sigma_0^{p+1}}{2^{p+2}} > 0. \end{aligned} \tag{68}$$

This contradicts (65). Therefore, (59) holds. For the ε given in (59), we let

$$\mathscr{N}_u := \{t \in [0,T] : |u(t)| \geq \varepsilon \|u\|\}, \quad \forall u \in \mathscr{X} \setminus \{0\}. \tag{69}$$

Then by (59), we find

$$\operatorname{meas}(\mathscr{N}_u) \geq \varepsilon, \quad \forall u \in \mathscr{X} \setminus \{0\}. \tag{70}$$

By (H5), for any $k \in \mathbb{N}$, there is a constant $S_k > 0$ such that

$$F(t,u) \geq \frac{|u|^p}{\varepsilon^{p+1}}, \quad \forall |u| \geq S_k, \tag{71}$$

where ε is determined in (59). Therefore,

$$\begin{aligned} & J_\lambda(u) \\ & = \frac{1}{p}\|u\|^p + \sum_{j=1}^m \int_0^{u(t_j)} I_j(t)\,\mathrm{d}t - \lambda \int_0^T F(t,u(t))\,\mathrm{d}t \\ & \leq \frac{1}{p}\|u\|^p + \sum_{j=1}^m \int_0^{u(t_j)} \left[\sigma_j + \tau_j |t|^{\gamma_j - 1}\right] \mathrm{d}t - \int_{\mathscr{N}_u} \frac{|u|^p}{\varepsilon^{p+1}} \mathrm{d}t \\ & \leq \left(\frac{1}{p} - 1\right)\|u\|^p + \sum_{j=1}^m \left[\sigma_j C_{\mathrm{emb2}} \|u\| + \frac{\tau_j}{\gamma_j} C_{\mathrm{emb2}}^{\gamma_j} \|u\|^{\gamma_j}\right]. \end{aligned} \tag{72}$$

Now for any $k \in \mathbb{N}$, if we take $r_k > \max\{\rho_k, S_k/\varepsilon\}$, noting that $p > 1$ and $p > \gamma_j$ and $\|u\| = r_k$ large enough, we have

$$b_k(\lambda) = \max_{u \in Y_k, \|u\| = r_k} J_\lambda(u) < 0, \quad \forall k \in \mathbb{N}. \tag{73}$$

Step 3. The continuity f and I_j $(j = 1, 2, \ldots, m)$ and $J_\lambda \in C^1(W, \mathbb{R})$ imply that J_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. In view of (H10), $J_\lambda(-u) = J_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times W$. Thus condition (T1) of Lemma 5 holds. Besides, by (H7), we get

$$A(u) = \frac{1}{p}\|u\|^p + \sum_{j=1}^{m} \int_0^{u(t_j)} I_j(t)\,dt \geq \frac{1}{p}\|u\|^p \longrightarrow \infty \quad \text{as } \|u\| \longrightarrow \infty \tag{74}$$

and $B(u) \geq 0$ since $F(t,u) \geq 0$. Thus the condition (T2) of Lemma 5 holds. For step 1 and step 2, the condition (T3) of Lemma 5 also holds for all $k \geq k_1$. Consequently, Lemma 5 implies that, for any $k \geq k_1$ and a.e. $\lambda \in [1,2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^{\infty}$ such that

$$\sup_n \left\|u_n^k(\lambda)\right\| < \infty, \qquad J'_\lambda\left(u_n^k(\lambda)\right) \longrightarrow 0, \qquad J_\lambda\left(u_n^k(\lambda)\right) \longrightarrow \zeta_k(\lambda) \quad \text{as } n \longrightarrow \infty, \tag{75}$$

where $B_k = \{u \in Y_k : \|u\| \leq r_k\}$, $\Gamma_k = \{\gamma \in C(B_k, W) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = \text{id}\}$ and $\zeta_k(\lambda) = \inf_{\gamma\in\Gamma_k}\max_{u\in B_k} J_\lambda(\gamma(u))$, $\forall\lambda \in [1,2]$.

Furthermore, we easily have

$$\zeta_k(\lambda) \in \left[\overline{a}_k, \overline{\zeta}_k\right], \quad \forall k \geq k_1, \tag{76}$$

where $\overline{\zeta}_k := \max_{u\in B_k} J_\lambda(\gamma(u))$ and $\overline{a}_k := (1/p)\rho_k^p - 2\mu_7 T^{1-1/q} - 2\mu_8 \to \infty$ as $k \to \infty$.

Claim 1. $\{u_n^k(\lambda)\}_{n=1}^{\infty} \subset W$ possesses a strong convergent subsequence in W, for all $\lambda \in [1,2]$ and $k \geq k_1$. In fact, by the boundedness of $\{u_n^k(\lambda)\}_{n=1}^{\infty}$, passing to a subsequence, as $n \to \infty$, we may assume $u_n^k(\lambda) \rightharpoonup u^k(\lambda)$ in W. By the method of Theorem 7, we easily prove that $u_n^k(\lambda) \to u^k(\lambda)$ strongly in W.

Thus, for each $k \geq k_1$, we can choose $\lambda_l \to 1$ such that the sequence $\{u_n^k(\lambda_l)\}_{n=1}^{\infty}$ obtained a convergent subsequence, and passing again to a subsequence, we may assume

$$\lim_{n\to\infty} u_n^k(\lambda_l) = u_l^k \text{ in } W, \quad \forall l \in \mathbb{N},\ k \geq k_1. \tag{77}$$

Thus we obtain

$$J'_{\lambda_l}\left(u_l^k\right) = 0, \quad J_{\lambda_l}\left(u_l^k\right) \in \left[\overline{a}_k, \overline{\zeta}_k\right], \quad \forall l \in \mathbb{N},\ k \geq k_1. \tag{78}$$

Claim 2. $\{u_l^k\}$ is bounded in W and has a convergent subsequence with the limit $u^k \in W$ for all $k \geq k_1$. For convenience, we set $u_l^k = u_l$ for all $l \in \mathbb{N}$. If not, $\{u_l\}$ is unbounded in W; that is, $\|u_l\| \to \infty$. By (H6), there is $\mu_9 > 0$ such that

$$-pF(t,u) + f(t,u)u \geq b|u|^q - \mu_9, \quad \forall (t,u) \in [0,T] \times \mathbb{R}. \tag{79}$$

Combining this and (H8), we have

$$\begin{aligned} &pJ_{\lambda_l}(u_l) - \left(J'_{\lambda_l}(u_l), u_l\right) \\ &\geq -\lambda_l p \int_0^T F(t, u_l(t))\,dt + \lambda_l \int_0^T f(t, u_l(t))\,u_l(t)\,dt \\ &\geq \lambda_l \int_0^T \left(b|u_l|^q - \mu_9\right)dt = \lambda_l b \int_0^T |u_l|^q dt - \lambda_l \mu_9 T. \end{aligned} \tag{80}$$

This implies that

$$\frac{\int_0^T |u_l|^q dt}{\|u_l\|^p} \longrightarrow 0 \quad \text{as } l \longrightarrow \infty. \tag{81}$$

On the other hand, by (H1) and (H9), we see

$$\begin{aligned} &\left(J'_{\lambda_l}(u_l), u_l\right) \\ &= \|u_l\|^p + \sum_{j=1}^{m} I_j\left(u_l\left(t_j\right)\right)u_l\left(t_j\right) - \lambda_l \int_0^T f(t, u_l(t))\,u_l(t)\,dt \\ &\geq \|u_l\|^p + \sum_{j=1}^{m}\left(-\sigma_j - \tau_j\left|u_l\left(t_j\right)\right|^{\gamma_j - 1}\right)\left|u_l\left(t_j\right)\right| - \lambda_l \int_0^T \left(\mu_1 + \mu_2 |u_l(t)|^{q-1}\right)|u_l(t)|\,dt \\ &\geq \|u_l\|^p - \sum_{j=1}^{m}\left[\sigma_j C_{\text{emb2}}\|u_l\| + \tau_j C_{\text{emb2}}^{\gamma_j}\|u_l\|^{\gamma_j}\right] - \lambda_l \mu_1 T^{1-1/q}\left(\int_0^T |u_l(t)|^q dt\right)^{1/q} - \lambda_l \mu_2 \int_0^T |u_l(t)|^q dt. \end{aligned} \tag{82}$$

Consequently, noting that $p > 1$ and $p > \gamma_j$, we have

$$\begin{aligned} 1 = \frac{\|u_l\|^p}{\|u_l\|^p} \leq{}& \frac{\left(J'_{\lambda_l}(u_l), u_l\right)}{\|u_l\|^p} + \frac{\sum_{j=1}^{m}\left[\sigma_j C_{\text{emb2}}\|u_l\| + \tau_j C_{\text{emb2}}^{\gamma_j}\|u_l\|^{\gamma_j}\right]}{\|u_l\|^p} \\ &+ \frac{\lambda_l \mu_1 T^{1-1/q}\left(\int_0^T |u_l(t)|^q dt\right)^{1/q}}{\|u_l\|^p} + \frac{\lambda_l \mu_2 \int_0^T |u_l(t)|^q dt}{\|u_l\|^p} \longrightarrow 0 \quad \text{as } l \longrightarrow \infty. \end{aligned} \tag{83}$$

This is a contradiction. Therefore, $\{u_l\}_{l=1}^{\infty}$ is bounded in W. By claim 1, we see that $\{u_l\}_{l=1}^{\infty}$ has a convergent subsequence, which converges to an element $u^k \in W$ for all $k \geq k_1$.

Hence, passing to the limit in (78), we see

$$J'_1\left(u^k\right) = 0, \quad J_1\left(u^k\right) \in \left[\overline{a}_k, \overline{\zeta}_k\right], \quad \forall l \in \mathbb{N},\ k \geq k_1. \tag{84}$$

Since $\overline{a}_k \to \infty$ as $k \to \infty$, we get infinitely many nontrivial critical points of $J_1 = J$. Therefore (1) possesses infinitely many nontrivial solutions by Lemma 5. This completes the proof. □

Remark 9. (1) Let

$$f(t,u)=\begin{cases} g(u)\,|u|^{p-2}u, & u\le 0,\\ g(u)\,|u|^{p-2}u+u^{q-1}, & u>0,\end{cases} \tag{85}$$

where $g\in C(\mathbb{R})$, $g(0)=0$, $g(-\infty)\in(\lambda_1,+\infty)$ with avoiding λ_k for all k. Then (H1)–(H4) hold true with $q\in(p,+\infty)$ and $f_0<\lambda_1<f_\infty$ with $f_\infty\ne\lambda_k$ for all k.

(2) Let $p=5$, $q=6$, $f(t,u)=u^5$, and $I_j(u)=\sqrt[3]{u}$ for all $t\in[0,T]$ and $u\in\mathbb{R}$. Then $F(t,u)=\int_0^u f(t,s)ds=(1/6)|u|^6$, $\int_0^u I_j(s)ds=(3/4)\sqrt[3]{u^4}$. Clearly, (H1), (H7), (H9), and (H10) are satisfied.

(1) $\lim_{|u|\to\infty}(F(t,u)/|u|^p)=\lim_{|u|\to\infty}(|u|^6/6|u|^5)=+\infty$, uniformly on $t\in[0,T]$, and $F(t,u)\ge 0$ for all $(t,u)\in[0,T]\times\mathbb{R}$. So, (H5) holds.

(2) $\lim_{|u|\to\infty}((-pF(t,u)+f(t,u)u)/|u|^q)=\lim_{|u|\to\infty}((-5\times(1/6)|u|^6+u^5u)/|u|^6)=1/6$, uniformly on $t\in[0,T]$, and hence (H6) holds.

(3) $p\int_0^u I_j(s)ds-I_j(u)u=5\times(3/4)\sqrt[3]{u^4}-\sqrt[3]{u}\times u=(11/4)\sqrt[3]{u^4}\ge 0$, $\forall u\in\mathbb{R}$, $j=1,2,\ldots,m$. Therefore, (H8) holds.

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