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Research Article

Boundedness of Solutions for a Class of Sublinear Reversible Oscillators with Periodic Forcing

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We study the boundedness of all solutions for the following differential equation $x'' + f(x)x' + (B + \varepsilon e(t))|x|^{\alpha-1}x = p(t)$, where $f(x)$, $p(t)$ are odd functions, $e(t)$ is an even function, $e(t)$, $p(t)$ are smooth 1-periodic functions, B is a nonzero constant, and ε is a small parameter. A sufficient and necessary condition for the boundedness of all solutions of the above equation is established. Moreover, the existence of Aubry-Mather sets is obtained as well.

1. Introduction

It is well known that the longtime behavior for periodically forced planar systems can be very intricate. For example, there are equations having unbounded solutions but with infinitely many zeros and with nearby unbounded solutions having randomly prescribed number of zeros and also periodic solutions; see [1]. In contrast to such unbounded phenomenon Littlewood [2] suggested to study the boundedness of all the solutions of the following differential equation:

$$\ddot{x} + g(x) = h(t) \tag{1}$$

in the following two cases:

(i) superlinear case: $g(x)/x \to +\infty$ as $x \to \pm\infty$;

(ii) sublinear case: $\operatorname{sgn}(x) \cdot g(x) \to +\infty$ and $g(x)/x \to 0$ as $x \to \pm\infty$. Later, one calls this subject as Littlewood boundedness problem.

The first result in superlinear case is obtained by Morris [3], who showed that all solutions of

$$\ddot{x} + 2x^3 = e(t) \tag{2}$$

are bounded, where $e(t) \in C^0$. Later, a series results in superlinear case were obtained by several authors, see [4–13] and references therein. However, in general, it is harder to study the Lagrange stability of sublinear systems since smoothness of sublinear term is insufficient. There are only a few works in sublinear case so far. In 1999, Küpper and You [14] proved the first result in the study of the equation

$$\ddot{x} + |x|^{\alpha-1}x = p(t), \tag{3}$$

where $0 < \alpha < 1$ and $p(t) \in C^{\infty}(\mathbb{T})$. Later, Liu [15] proved the same result for more general equation

$$\ddot{x} + g(x) = e(t), \tag{4}$$

where $g(x) \in C^6$ satisfying the sublinear condition (ii) and some inequalities, and $e(t) \in C^5(\mathbb{T})$. In 2004, Ortega and Verzini [16] studied the boundedness of (4) in a special case with the variational method. In 2009, Wang [17] gave a sufficient and necessary condition for the boundedness of all solutions for sublinear equation

$$\ddot{x} + e(t)|x|^{\alpha-1}x = p(t), \tag{5}$$

where $e(t), p(t) \in C^5(\mathbb{T})$.

As is widely known, there is a deep similarity between reversible and Hamiltonian dynamics. Many fundamental results of the Hamiltonian systems possess reversible counterparts. On boundedness problem for sublinear reversible systems, the first results were obtained by Li [18], later, Yang [19], in the study of a sublinear reversible systems

$$\ddot{x} + f(x)\dot{x} + |x|^{\alpha-1}x = e(t). \tag{6}$$

Recently, Wang [20] gave a sufficient and necessary condition for the boundedness of all solutions of the differential equation

$$\ddot{x} + f(x) g(\dot{x}) + \gamma |x|^{\alpha-1} x = p(t) \tag{7}$$

with $0 < \alpha < 1$, $\gamma \neq 0$.

By the discussions about the sublinear Hamiltonian equation (1.3) in [17] motivations, we will study the boundedness of all solutions for a sublinear reversible system like

$$\ddot{x} + f(x)\dot{x} + (B + \varepsilon e(t)) |x|^{\alpha-1} x = p(t), \tag{8}$$

where $B \neq 0$ and $0 < \alpha < 1$. Furthermore, we also show that (8) has solutions of Mather type. The results obtained in [18–20] can be regarded as corollary of result of this paper.

Remark 1. Using the method of this paper we also can consider the more general equation

$$\ddot{x} + f(x) g(\dot{x}) + (B + \varepsilon e(t)) |x|^{\alpha-1} x = p(t) \tag{9}$$

provided of adding suitable conditions for $g(x)$. For convenience, we only consider the case $g(x) \equiv x$.

Remark 2. Adding the perturbation term $\varepsilon e(t)|x|^{\alpha-1}x$ will lead to a new difficulty for estimating $|S(\theta T_0)|^{\alpha-1}C(\theta T_0)$ appeared in (86). Fortunately, we can easily verify that $\int_0^1 |S(\theta T_0)|^{\alpha-1} C(\theta T_0) d\theta$ is bounded by a constant (see in the proof of Lemma 12).

Throughout this paper, we denote two universal positive constants without regarding their values by $c < 1$ and $C \geq 1$, and suppose that the following conditions hold:

(A1) $f(x) \in C^4(\mathbb{R})$, $p(t) \in C^3(\mathbb{T})$ and $e(t) \in C^3(\mathbb{T})$, $f(x)$ and $p(t)$ are odd, $e(t)$ is even, and $e(t)$, $p(t)$ are both 1-periodic functions, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$;

(A2) there is some positive constant μ such that the inequalities

$$\left| x^{i+1} f^{(i)}(x) \right| \leq C |x|^{\alpha/2-\beta} \tag{10}$$

are satisfied for $0 \leq i \leq 4$ and all $|x| \geq \mu$, where $0 < \beta < \alpha/2$.

We decompose $e(t)$ as $e(t) = \bar{e} + \widehat{e}(t)$, where $\bar{e}$ is the average of $e(t)$ and $\widehat{e}(t)$ has zero mean value. That is $\bar{e} = \int_0^1 e(s)ds$ and $\int_0^1 \widehat{e}(s)ds = 0$. If we write that $A = B + \varepsilon\bar{e}$, then it is easy to see that A and B have the same sign when $0 < \varepsilon < \varepsilon^*$ with $0 < \varepsilon^* < |B/\bar{e}|$.

Now we state the main results of this paper.

Theorem 3. *Assume that $B \neq 0$ and (A1)-(A2) hold. Then there exists an $0 < \varepsilon^{**} < \varepsilon^*$ such that for any $0 < \varepsilon < \varepsilon^{**}$, every solution of (8) is bounded if and only if $B > 0$.*

Theorem 4. *Under the conditions of Theorem 3, there is an $\varepsilon_0 > 0$ such that, for any $\omega \in (n, n + \varepsilon_0)$, (8) has a solution $(x_\omega(t), x'_\omega(t))$ of Mather type with rotation number ω. More precisely:*

(i) *if $\omega = p/q$ is rational, the solutions $(x_\omega(t+i), x'_\omega(t+i))$, $1 \leq i \leq q-1$, are periodic solutions of period q; moreover, in this case*

$$\lim_{\omega \to n} \min_{t \in \mathbb{R}} \left(\left| x_\omega(t) \right| + \left| x'_\omega(t) \right| \right) = +\infty; \tag{11}$$

(ii) *if ω is irrational, the solution $(x_\omega(t), x'_\omega(t))$ is either a usual quasi-periodic solution or a generalized one.*

We recall that a solution is called generalized quasi-periodic if the closed set

$$\overline{\{x(i), x'(i), i \in \mathbb{Z}\}} \tag{12}$$

is a Denjoys minimal set.

2. Reversible Systems and Action-Angle Variables

In this section, we will assume that $B > 0$ and $A > 0$. Firstly, we consider (8) which is equivalent to the following system:

$$\begin{aligned} \dot{x} &= z + P(t), \\ \dot{z} &= -A|x|^{\alpha-1}x - \varepsilon\widehat{e}(t)|x|^{\alpha-1}x - f(x)(z + P(t)), \end{aligned} \tag{13}$$

where $P(t) = \int_0^t p(s)ds$. Then we can obtain that (13) is reversible with respect to the transformation $(x, z) \mapsto (-x, z)$ by (A1).

Lemma 5. *There exists a G-invariant diffeomorphism $(x, y) \to (x, z)$ such that (13) is transformed into the following system:*

$$\begin{aligned} \dot{x} &= y + \varepsilon E(t)|x|^{\alpha-1}x + P(t), \\ \dot{y} &= -A|x|^{\alpha-1}x \\ &\quad - \left[\alpha\varepsilon E(t)|x|^{\alpha-1} + f(x)\right]\left[y + \varepsilon E(t)|x|^{\alpha-1}x + P(t)\right], \end{aligned} \tag{14}$$

where $E(t) = -\int_0^t \widehat{e}(s)ds$.

Proof. Introduce a transformation Ψ:

$$x = x, \qquad z = y + U(x, t), \tag{15}$$

where $U(x, t)$ will be determined later. Under this transformation, the system (13) is transformed into a new system as follows:

$$\begin{aligned} \dot{x} &= y + U(x, t) + P(t), \\ \dot{y} &= -A|x|^{\alpha-1}x - \varepsilon\widehat{e}(t)|x|^{\alpha-1}x \\ &\quad - \left(f(x) + \frac{\partial U(x,t)}{\partial x}\right)\left[y + U(x,t) + P(t)\right] \\ &\quad - \frac{\partial U(x,t)}{\partial t}. \end{aligned} \tag{16}$$

Now, we define the function $U(x,t)$ by

$$-\varepsilon\widehat{e}(t)|x|^{\alpha-1}x-\frac{\partial U(x,t)}{\partial t}=0. \tag{17}$$

Since $\int_0^1\widehat{e}(t)dt=0$, so we can obtain $U(x,t)=\varepsilon E(t)|x|^{\alpha-1}x$. Then the new system can be expressed as in (14) by direct computation.

It is easy to know that $U(-x,-t)=U(x,t)$ by (A1), then we can obtain that the transformation Ψ is a G-invariant diffeomorphism. □

Let us consider the auxiliary system

$$\begin{aligned}\dot{x}&=y,\\ \dot{y}&=-A|x|^{\alpha-1}x,\end{aligned} \tag{18}$$

which is a time-independent Hamiltonian system with Hamiltonian

$$H_0(x,y)=\frac{y^2}{2}+\frac{A}{\alpha+1}|x|^{\alpha+1}. \tag{19}$$

It is easy to see that $H_0(x,y)>0$, $(x,y)\in R^2\setminus\{0\}$, $H_0(0,0)=0$. Note that each level line $H_0(x,y)=h>0$ is a close orbit of system (18), hence, all the solutions of (18) are periodic with period tending to zero as h tends to infinity.

Assume that $(S(t),C(t))$ is the solution of (18) with initial conditions $(S(0),C(0))=(0,1)$, and let $T_0>0$ be the minimal period. We can find that $S(t)$ and $C(t)$ satisfy

(i) $S(t)\in C^2(R)$, $C(t)\in C^1(R)$;

(ii) $(S(-t),C(-t))=(-S(t),C(t))$, $(S(t+T_0),C(t+T_0))=(S(t),C(t))$;

(iii) $\dot{S}(t)=C(t)$, $\dot{C}(t)=-A|S(t)|^{\alpha-1}S(t)$;

(iv) $(1/2)C^2(t)+(A/(\alpha+1))|S(t)|^{\alpha+1}=1/2$;

(v) $C(T_0t)=0\Leftrightarrow t(\mathrm{mod}(1/4))=0$;

(vi) $(S(T_0(1/2-t)),C(T_0(1/2-t)))=(S(T_0t),-C(T_0t))$;

(vii) $S(T_0t)=0\Leftrightarrow t(\mathrm{mod}(1/2))=0$.

Then we introduce the transformation

$$\begin{aligned}\Phi:\ \mathbb{R}^+\times\mathbb{T}&\longrightarrow\mathbb{R}^2\setminus\{0\},\\ (\rho,\varphi)&\longmapsto(x,y)\end{aligned} \tag{20}$$

which is

$$\begin{aligned}x&=\rho^bS(\varphi T_0),\\ y&=\rho^{1-b}C(\varphi T_0),\end{aligned} \tag{21}$$

where $b=2/(3+\alpha)$. It is easy to see that $1/2<b<2/3$ by $0<\alpha<1$. Since $(S(-t),C(-t))=(-S(t),C(t))$, this transformation is invariant with respect to the involutions $(\rho,\varphi)\mapsto(\rho,-\varphi)$ and $(x,y)\mapsto(-x,y)$, and we can find that the mapping Φ is a generalized canonical transformation by (iv). In fact,

$$\begin{aligned}&\left|\frac{\partial(x,y)}{\partial(\rho,\varphi)}\right|\\ &\quad=\left|AbT_0|S(\varphi T_0)|^{\alpha+1}+(1-b)T_0C^2(\varphi T_0)\right|\\ &\quad=\left|(1-b)T_0-\frac{\alpha+1}{2}bT_0C^2(\varphi T_0)+(1-b)T_0C^2(\varphi T_0)\right|\\ &\quad=(1-b)T_0,\\ &\begin{pmatrix}\dot{\rho}\\ \dot{\varphi}\end{pmatrix}=\begin{pmatrix}-dy_\varphi & dx_\varphi\\ dy_\rho & -dx_\rho\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix},\end{aligned} \tag{22}$$

where $d=((1-b)T_0)^{-1}$.

Under the transformation Φ, the system (18) is transformed into the simpler form

$$\dot{\rho}=-\frac{\partial h_0}{\partial\varphi}=0,\qquad \dot{\varphi}=\frac{\partial h_0}{\partial\rho}=\frac{1}{T_0}\cdot\rho^{1-2b}, \tag{23}$$

where $h_0(\rho)=((2-2b)T_0)^{-1}\cdot\rho^{2(1-b)}$.

The original system (13) is transformed into the system

$$\begin{aligned}\frac{d\rho}{dt}&=l_1(\rho,\varphi)+l_2(\rho,\varphi,t)+\varepsilon l_3(\rho,\varphi,t)\\ &\quad+\alpha T_0|S(\varphi T_0)|^{\alpha-1}C(\varphi T_0)\,\varepsilon l_4(\rho,\varphi,t),\\ \frac{d\varphi}{dt}&=h_0'(\rho)+h_1(\rho,\varphi)+h_2(\rho,\varphi,t)+\varepsilon h_3(\rho,\varphi,t),\end{aligned} \tag{24}$$

where

$$\begin{aligned}l_1(\rho,\varphi)&=-dT_0\rho f\left(\rho^bS(\varphi T_0)\right)C^2(\varphi T_0)\\ &=:-dx_\varphi f(x)\,y,\\ l_2&(\rho,\varphi,t)\\ &=-dT_0\varepsilon\rho^{2-2b}f\left(\rho^bS(\varphi T_0)\right)\\ &\quad\times|S(\varphi T_0)|^{\alpha-1}S(\varphi T_0)C(\varphi T_0)E(t)\\ &\quad+AdT_0\rho^{1-b}|S(\varphi T_0)|^{\alpha-1}S(\varphi T_0)P(t)\\ &\quad-dT_0\rho^bf\left(\rho^bS(\varphi T_0)\right)C(\varphi T_0)P(t)\\ &=:-d\varepsilon x_\varphi|x|^{\alpha-1}xf(x)E(t)-dy_\varphi P(t)-dx_\varphi f(x)P(t),\\ l_3(\rho,\varphi,t)&=AdT_0\rho^{3-4b}|S(\varphi T_0)|^{2\alpha}E(t)\\ &=:-dy_\varphi|x|^{\alpha-1}xE(t),\\ l_4&(\rho,\varphi,t)\\ &=-d\rho^{3-4b}C(\varphi T_0)E(t)\\ &\quad-d\varepsilon\rho^{4-6b}|S(\varphi T_0)|^{\alpha-1}S(\varphi T_0)E^2(t)\\ &\quad-d\rho^{2-3b}P(t)E(t),\end{aligned}$$

$$
\begin{aligned}
h_1(\rho,\varphi) &= dbf\left(\rho^b S(\varphi T_0)\right) C(\varphi T_0) S(\varphi T_0) =: dx_\rho f(x)\, y,\\
h_2&(\rho,\varphi,t)\\
&= db\varepsilon\rho^{1-2b} f\left(\rho^b S(\varphi T_0)\right) \left|S(\varphi T_0)\right|^{\alpha+1} E(t)\\
&\quad + \alpha db\varepsilon^2 \rho^{3-6b} \left|S(\varphi T_0)\right|^{2\alpha} E^2(t)\\
&\quad + d(1-b)\rho^{-b} C(\varphi T_0) P(t)\\
&\quad + db\rho^{b-1} f\left(\rho^b S(\varphi T_0)\right) S(\varphi T_0) P(t)\\
&\quad + \alpha db\varepsilon\rho^{1-3b} \left|S(\varphi T_0)\right|^{\alpha-1} S(\varphi T_0) E(t) P(t)\\
&=: d\varepsilon x_\rho |x|^{\alpha-1} x f(x) E(t) + \alpha d\varepsilon^2 x_\rho |x|^{2\alpha-2} x E^2(t)\\
&\quad + dy_\rho P(t) + dx_\rho f(x) P(t) + \alpha d\varepsilon x_\rho |x|^{\alpha-1} E(t) P(t),\\
h_3&(\rho,\varphi,t)\\
&= d(1-b+\alpha b)\rho^{2-4b} \left|S(\varphi T_0)\right|^{\alpha-1} S(\varphi T_0) C(\varphi T_0) E(t)\\
&=: dy_\rho |x|^{\alpha-1} x E(t) + \alpha dx_\rho |x|^{\alpha-1} y E(t).
\end{aligned}
\tag{25}
$$

Let

$$
\begin{aligned}
L_2(\rho,\varphi,t) &= l_2(\rho,\varphi,t) + \varepsilon l_3(\rho,\varphi,t)\\
&\quad + \alpha T_0 \left|S(\varphi T_0)\right|^{\alpha-1} C(\varphi T_0)\, \varepsilon l_4(\rho,\varphi,t),\\
H_2(\rho,\varphi,t) &= h_2(\rho,\varphi,t) + \varepsilon h_3(\rho,\varphi,t).
\end{aligned}
\tag{26}
$$

Clearly, x is odd in φ and y is even in φ by the definitions of $S(t)$ and $C(t)$. Thus, by the evenness of $P(t)$ and the oddness of $f(x)$ and $E(t)$ we have

$$
\begin{aligned}
l_1(\rho,-\varphi) &= -l_1(\rho,\varphi), \qquad & L_2(\rho,-\varphi,-t) &= -L_2(\rho,\varphi,t),\\
h_1(\rho,-\varphi) &= h_1(\rho,\varphi), \qquad & H_2(\rho,-\varphi,-t) &= H_2(\rho,\varphi,t).
\end{aligned}
\tag{27}
$$

This implies that system (24) is reversible with respect to the involutions $(\rho,\varphi) \mapsto (\rho,-\varphi)$.

Lemma 6. *For $0 \le k+m \le 4$, the following inequalities hold:*

(1) $|(\partial^k/\partial\rho^k) l_1(\rho,\varphi)| \le C\rho^{-k+2-\gamma-(5/2)b}$,

(2) $|(\partial^{k+m}/\partial\rho^k\partial t^m) l_2(\rho,\varphi,t)| \le C\rho^{-k+a}$,

(3) $|(\partial^{k+m}/\partial\rho^k\partial t^m) l_3(\rho,\varphi,t)| \le C\rho^{-k+3-4b}$,

(4) $|(\partial^{k+m}/\partial\rho^k\partial t^m) l_4(\rho,\varphi,t)| \le C\rho^{-k+3-4b}$,

(5) $|(\partial^k/\partial\rho^k) h_1(\rho,\varphi)| \le C\rho^{-k+1-\gamma-(5/2)b}$,

(6) $|(\partial^{k+m}/\partial\rho^k\partial t^m) h_2(\rho,\varphi,t)| \le C\rho^{-k+\tau}$,

(7) $|(\partial^{k+m}/\partial\rho^k\partial t^m) h_3(\rho,\varphi,t)| \le C\rho^{-k+2-4b}$,

where $\gamma = \beta b$, $a = \max(3-(9/2)b-\gamma, 1-b)$, and $\tau = \max(3-6b, -b)$.

Proof. (1) It is easy to know that $(\partial^k/\partial\rho^k) l_1(\rho,\varphi)$ is a sum of terms of the form

$$
d\frac{\partial^{i_1} x_\varphi}{\partial\rho^{i_1}} \cdot \frac{\partial^{i_2} f(x)}{\partial\rho^{i_2}} \cdot \frac{\partial^{i_3} y}{\partial\rho^{i_3}}, \quad i_1 + i_2 + i_3 = k,
\tag{28}
$$

where $0 \le i_1, i_2, i_3 \le k$. Meanwhile, $\partial^{i_2} f(x)/\partial\rho^{i_2}$ is a sum terms of the form

$$
f^{(s)}(x) \cdot \frac{\partial^{l_1} x}{\partial\rho^{l_1}} \frac{\partial^{l_2} x}{\partial\rho^{l_2}} \cdots \frac{\partial^{l_s} x}{\partial\rho^{l_s}}, \quad 0 \le s \le i_2,\ l_1 + \cdots + l_s = i_2.
\tag{29}
$$

Hence, we obtain

$$
\begin{aligned}
\left|\frac{\partial^k}{\partial\rho^k} l_1(\rho,\varphi)\right| &\le C\left|\rho^{-i_1} x \cdot \rho^{-i_2} f(x) \cdot \rho^{-i_3} y\right|\\
&\le C\rho^{-k} \cdot \left|x \cdot f(x) \cdot y\right| \le C\rho^{-k} |x|^{\alpha/2-\beta} |y|\\
&\le C\rho^{-k+2-\gamma-(5/2)b}
\end{aligned}
\tag{30}
$$

by the assumptions on $f(x)$ and the definitions of $x(\rho,\varphi)$ and $y(\rho,\varphi)$.

(2) From the expression of $l_2(\rho,\varphi,t)$, we have

$$
\begin{aligned}
&\left|\frac{\partial^{k+m}\left(-d\varepsilon x_\varphi |x|^{\alpha-1} x f(x) E(t)\right)}{\partial\rho^k \partial t^m}\right|\\
&\qquad \le C\left|\frac{\partial^k\left(-d\varepsilon x_\varphi |x|^{\alpha-1} x f(x)\right)}{\partial\rho^k}\right| \left|E^{(m)}(t)\right|\\
&\qquad \le C\varepsilon\left|\rho^{-i_1} x \cdot \rho^{-i_2} f(x) \cdot \rho^{\alpha b - i_3}\right|\\
&\qquad \le C\varepsilon\rho^{-k+\alpha b}\left|x \cdot f(x)\right|\\
&\qquad \le C\varepsilon\rho^{-k+3-\gamma-9b/2},\\
&\left|\frac{\partial^{k+m}\left(-dy_\varphi P(t)\right)}{\partial\rho^k \partial t^m}\right| \le C\left|\frac{\partial^k\left(dy_\varphi\right)}{\partial\rho^k}\right| \left|\frac{d^m P(t)}{dt^m}\right|\\
&\qquad \le C\rho^{-k+1-b},\\
&\left|\frac{\partial^{k+m}\left(-dx_\varphi f(x) P(t)\right)}{\partial\rho^k \partial t^m}\right| \le C\left|\frac{\partial^k\left(-dx_\varphi f(x)\right)}{\partial\rho^k}\right| \left|\frac{d^m P(t)}{dt^m}\right|\\
&\qquad \le C\rho^{-k+1-\gamma-3b/2}.
\end{aligned}
\tag{31}
$$

We can find that

$$
\left|\frac{\partial^{k+m}}{\partial\rho^k \partial t^m} l_2(\rho,\varphi,t)\right| \le C\rho^{-k+a},
\tag{32}
$$

where $a = \max(3 - 9b/2 - \gamma, 1 - b)$.

(3) From the expression of $l_3(\rho,\varphi,t)$, we have

$$\begin{aligned}&\left|\frac{\partial^{k+m}\left(dy_{\varphi}|x|^{\alpha-1}xE(t)\right)}{\partial\rho^k\partial t^m}\right|\\&\quad\leq C\left|\frac{\partial^k\left(dy_{\varphi}|x|^{\alpha-1}x\right)}{\partial\rho^k}\right|\left|\frac{d^m(E(t))}{dt^m}\right|\\&\quad\leq C\rho^{-k+3-4b}.\end{aligned}\tag{33}$$

(4) From the expression of $l_4(\rho,\varphi,t)$, we can obtain that

$$\begin{aligned}&\left|\frac{\partial^{k+m}l_4(\rho,\varphi,t)}{\partial\rho^k\partial t^m}\right|\\&\leq C\left|\frac{\partial^k\left(-d\rho^{3-4b}C(\varphi T_0)\right)}{\partial\rho^k}\right|\left|\frac{d^m(E(t))}{dt^m}\right|\\&\quad+C\left|\frac{\partial^k\left(-d\rho^{2-3b}\right)}{\partial\rho^k}\right|\left|\frac{d^m(P(t)E(t))}{dt^m}\right|\\&\quad+C\left|\frac{\partial^k\left(-d\varepsilon\rho^{4-6b}|S(\varphi T_0)|^{\alpha-1}S(\varphi T_0)\right)}{\partial\rho^k}\right|\left|\frac{d^m\left(E^2(t)\right)}{dt^m}\right|\\&\leq C\rho^{-k+3-4b}.\end{aligned}\tag{34}$$

(5) From the definition of $h_1(\rho,\varphi)$, we have

$$\begin{aligned}&\left|\frac{\partial^k}{\partial\rho^k}h_1(\rho,\varphi)\right|\\&\quad\leq C\left|\rho^{-i_1-1}x\cdot\rho^{-i_2}f(x)\cdot\rho^{-i_3}y\right|\\&\quad\leq C\rho^{-k}\cdot|x\cdot f(x)\cdot y|\leq C\rho^{-k-1}|x|^{\alpha/2-\beta}|y|\\&\quad\leq C\rho^{-k+1-\gamma-(5/2)b}.\end{aligned}\tag{35}$$

(6) From the definition of $h_2(\rho,\varphi,t)$, we can obtain

$$\begin{aligned}&\left|\frac{\partial^{k+m}\left(d\varepsilon x_{\rho}|x|^{\alpha-1}xf(x)E(t)\right)}{\partial\rho^k\partial t^m}\right|\\&\quad\leq C\left|\frac{\partial^k\left(dx_{\rho}|x|^{\alpha-1}xf(x)\right)}{\partial\rho^k}\right|\left|E^{(m)}(t)\right|\\&\quad\leq C\varepsilon\left|\rho^{-i_1-1}x\cdot\rho^{-i_2}f(x)\cdot\rho^{\alpha b-i_3}\right|\\&\quad\leq C\varepsilon\rho^{-k+\alpha b-1}|x\cdot f(x)|\\&\quad\leq C\varepsilon\rho^{-k+2-\gamma-9b/2},\\&\left|\frac{\partial^{k+m}\left(\alpha d\varepsilon^2x_{\rho}|x|^{2\alpha-2}xE^2(t)\right)}{\partial\rho^k\partial t^m}\right|\\&\quad\leq C\left|\frac{\partial^k\left(\alpha d\varepsilon^2x_{\rho}|x|^{2\alpha-2}x\right)}{\partial\rho^k}\right|\left|\frac{d^m\left(E^2(t)\right)}{dt^m}\right|\\&\quad\leq C\varepsilon\rho^{-k+3-6b},\\&\left|\frac{\partial^{k+m}\left(-dy_{\rho}P(t)\right)}{\partial\rho^k\partial t^m}\right|\leq C\left|\frac{\partial^k\left(dy_{\rho}\right)}{\partial\rho^k}\right|\left|\frac{d^mP(t)}{dt^m}\right|\leq C\rho^{-k-b},\\&\left|\frac{\partial^{k+m}\left(dx_{\rho}f(x)P(t)\right)}{\partial\rho^k\partial t^m}\right|\leq C\left|\frac{\partial^k\left(-dx_{\rho}f(x)\right)}{\partial\rho^k}\right|\left|\frac{d^mP(t)}{dt^m}\right|\\&\quad\leq C\rho^{-k-\gamma-3b/2},\\&\left|\frac{\partial^{k+m}\left(\alpha d\varepsilon x_{\rho}|x|^{\alpha-1}E(t)P(t)\right)}{\partial\rho^k\partial t^m}\right|\\&\quad\leq C\left|\frac{\partial^k\left(\alpha d\varepsilon^2x_{\rho}|x|^{2\alpha-2}x\right)}{\partial\rho^k}\right|\left|\frac{d^m(E(t)P(t))}{dt^m}\right|\\&\quad\leq C\varepsilon\rho^{-k+1-3b}.\end{aligned}\tag{36}$$

Hence, we can know that

$$\left|\frac{\partial^{k+m}}{\partial\rho^k\partial t^m}h_2(\rho,\varphi,t)\right|\leq C\rho^{-k+\tau},\tag{37}$$

where $\tau=\max(3-6b,-b)$.

(7) From the expression of $h_3(\rho,\varphi,t)$, we have

$$\begin{aligned}&\left|\frac{\partial^{k+m}}{\partial\rho^k\partial t^m}h_3(\rho,\varphi,t)\right|\\&\leq\left|\frac{\partial^{k+m}\left(d(1-b+\alpha b)\rho^{2-4b}|S(\varphi T_0)|^{\alpha-1}S(\varphi T_0)C(\varphi T_0)E(t)\right)}{\partial\rho^k\partial t^m}\right|\\&\leq C\left|\frac{\partial^k\left(d(1-b+\alpha b)\rho^{2-4b}|S(\varphi T_0)|^{\alpha-1}S(\varphi T_0)C(\varphi T_0)\right)}{\partial\rho^k}\right|\\&\quad\times\left|\frac{d^m(E(t))}{dt^m}\right|\\&\leq C\rho^{-k+2-4b}.\end{aligned}\tag{38}$$

□

For $\lambda_0>0$, we define the domain

$$\mathscr{A}_{\lambda_0}=\left\{(\lambda,\varphi,t):\lambda\geq\lambda_0,\ (\varphi,t)\in\mathbb{T}^2\right\}.\tag{39}$$

Lemma 7. *There exists a G-invariant diffeomorphism* Ψ_1*:*

$$\rho=I+U_1(I,\theta),\quad\varphi=\theta\tag{40}$$

such that $\mathscr{A}_{I^+}\subset\Psi_1(\mathscr{A}_{I_0})\subset\mathscr{A}_{I_-}$ *for some* $I_-<I_0<I_+$*. Under this transformation, (24) is transformed into the system*

$$\begin{aligned}\frac{dI}{dt}&=\tilde{l}_1(I,\theta)+\tilde{l}_2(I,\theta,t)+\varepsilon\tilde{l}_3(I,\theta,t)\\&\quad+\alpha T_0|S(\theta T_0)|^{\alpha-1}C(\theta T_0)\varepsilon\tilde{l}_4(I,\theta,t),\\\frac{d\theta}{dt}&=h_0'(I)+\tilde{h}_1(I,\theta)+\tilde{h}_2(I,\theta,t)+\varepsilon\tilde{h}_3(I,\theta,t),\end{aligned}\tag{41}$$

where

$$\begin{aligned}
\widetilde{l}_1(I,\theta) &= \frac{\partial V_1(\rho,\varphi)}{\partial\rho}\cdot l_1(\rho,\varphi) + \frac{\partial V_1(\rho,\varphi)}{\partial\varphi}\cdot h_1(\rho,\varphi),\\
\widetilde{l}_2(I,\theta,t) &= l_2(\rho,\varphi,t)\\
&\quad + \frac{\partial V_1(\rho,\varphi)}{\partial\rho}\cdot\left(l_2(\rho,\varphi,t)+\varepsilon l_3(\rho,\varphi,t)\right)\\
&\quad + \frac{\partial V_1(\rho,\varphi)}{\partial\varphi}\cdot\left(h_2(\rho,\varphi,t)+\varepsilon h_3(\rho,\varphi,t)\right)\\
&\quad + \varepsilon\left(l_3(\rho,\varphi,t)-l_3(I,\theta,t)\right),\\
\widetilde{l}_3(I,\theta,t) &= l_3(I,\theta,t),\\
\widetilde{l}_4(I,\theta,t) &= l_4(\rho,\varphi,t) + \frac{\partial V_1(\rho,\varphi)}{\partial\rho}\cdot l_4(\rho,\varphi,t),\\
\widetilde{h}_1(I,\theta) &= h_0'(\rho) - h_0'(I) + h_1(\rho,\varphi),\\
\widetilde{h}_2(I,\theta,t) &= h_2(\rho,\varphi,t) + \varepsilon\left(h_3(\rho,\varphi,t)-h_3(I,\theta,t)\right),\\
\widetilde{h}_3(I,\theta,t) &= h_3(I,\theta,t),
\end{aligned} \tag{42}$$

with

$$V_1(\rho,\varphi) = -\int_0^{\varphi}\frac{l_1(\rho,s)}{h_0'(\rho)}ds. \tag{43}$$

Proof. Define a transformation Φ_1 by

$$\Phi_1 : I = \rho + V_1(\rho,\varphi), \quad \theta = \varphi. \tag{44}$$

By

$$\begin{aligned}
l_1(\rho,-\varphi) &= -l_1(\rho,\varphi),\\
\left|\frac{\partial^k}{\partial\rho^k}l_1(\rho,\varphi)\right| &\le C\rho^{-k+2-\gamma-(5/2)b}, \quad 0\le k\le 4,
\end{aligned} \tag{45}$$

we get

$$V_1(\rho,-\varphi) = V_1(\rho,\varphi), \tag{46}$$

$$\left|\frac{\partial^k}{\partial\rho^k}V_1(\rho,\varphi)\right| \le C\rho^{-k+1-\gamma-b/2}. \tag{47}$$

Let $\Psi_1 = \Phi_1^{-1} : \rho = I + U_1(I,\theta)$, $\varphi = \theta$. The system (24) is transformed into (41). □

Lemma 8. *For I large enough, the following conclusions hold:*

(i) $|\partial^k U_1(I,\theta)/\partial I^k| \le CI^{-k+1-\gamma-b/2}$,

(ii) $U_1(I,-\theta) = U_1(I,\theta)$.

Proof. In view of

$$I = \rho + V_1(\rho,\varphi), \qquad \rho = I + U_1(I,\theta), \tag{48}$$

we obtain

$$U_1(I,\theta) = -V_1\left(I + U_1(I,\theta),\theta\right). \tag{49}$$

By $|(\partial^k/\partial\rho^k)V_1(\rho,\varphi)| \le C\rho^{-k+1-\gamma-b/2}$, we have $|(\partial/\partial\rho)V_1(\rho,\varphi)| \le C\rho^{-\gamma-b/2} \le 1/2$ for ρ large enough. Hence, U_1 is uniquely determined by the contraction mapping principle. Moreover, $U_1(\cdot,\theta) \in C^{\infty}(\mathscr{A}_{I_0})$, for some $I_0 > 0$, as a consequence of the implicit function theorem and

$$I^{-(1-\gamma-b/2)}\left|U_1(I,\theta)\right| \le C. \tag{50}$$

Above all, if $k=1$, from (47) and (49), we get

$$\begin{aligned}
\left|\frac{\partial U_1}{\partial I}\right| &= \left|\frac{\partial V_1/\partial\rho}{1+\partial V_1/\partial\rho}\right| \le \sum_{n=0}^{\infty}\left(C\rho^{-1+1-\gamma-b/2}\right)^{n+1}\\
&\le C\cdot\rho^{-1+1-\gamma-b/2}\\
&= C\cdot I^{-1+1-\gamma-b/2}\left(1+\frac{U_1}{I}\right)^{-1+1-\gamma-b/2}\\
&\le C\cdot I^{-1+1-\gamma-b/2}.
\end{aligned} \tag{51}$$

We note that

$$\frac{\partial^k U_1(I,\theta)}{\partial I^k} = \frac{\partial^k V_1\left(I+U_1(I,\theta),\theta\right)}{\partial I^k}, \tag{52}$$

and the right side hand is sum of the term

$$\frac{\partial^s V_1}{\partial\rho^s}\cdot\frac{\partial^{k_1}(I+U_1)}{\partial I^{k_1}}\cdots\frac{\partial^{k_s}(I+U_1)}{\partial I^{k_s}}, \tag{53}$$

where $1\le s\le k$, $k_1+\cdots+k_s = k$, $k_i\ge 1$ (for $1\le i\le s$). The highest order term in U_1 is the one with $s=1$, namely, $(\partial V_1/\partial\rho)\cdot(\partial^k(I+U_1)/\partial I^k)$. We move the part $(\partial V_1/\partial\rho)\cdot(\partial^k U_1/\partial I^k)$ to the left hand side of (52). Since $|(\partial/\partial\rho)V_1(\rho,\varphi)| \le 1/2$ for ρ large enough, this also provides immediately a bound on $\partial^k U_1(I,\theta)/\partial I^k$. The rest part $|(\partial V_1/\partial\rho)\cdot(\partial^k I/\partial I^k)| \le CI^{-k+1-\gamma-b/2}$.

Now, we proceed inductively by assuming that for $j\le k-1$ the estimates

$$\left|\frac{\partial^j U_1(I,\theta)}{\partial I^j}\right| \le CI^{-j+1-\gamma-b/2} \tag{54}$$

hold and we wish to conclude that the same estimate holds for $j=k$.

Indeed, if $s\ge 2$, we have

$$\begin{aligned}
&\left|\frac{\partial^s V_1}{\partial\rho^s}\cdot\frac{\partial^{k_1}(I+U_1)}{\partial I^{k_1}}\cdots\frac{\partial^{k_s}(I+U_1)}{\partial I^{k_s}}\right|\\
&\quad\le C\cdot\left(I+U_1\right)^{-s+1-\gamma-b/2}\cdot I^{-k_1+1}\cdots I^{-k_s+1}\\
&\quad\le C\cdot I^{-k+1-\gamma-b/2}
\end{aligned} \tag{55}$$

by

$$\left|\frac{\partial^j\left(I+U_1(I,\theta)\right)}{\partial I^j}\right| \le CI^{-j+1}, \quad 1\le j\le k-1. \tag{56}$$

This proves (i) of Lemma 8.

Now we check (ii). In fact, since

$$U_1(I,\theta) = -V_1(I + U_1(I,\theta),\theta),$$
$$U_1(I,-\theta) = -V_1(I + U_1(I,-\theta),\theta), \quad (57)$$

we have

$$\left|U_1(I,\theta) - U_1(I,-\theta)\right| \le \sup_{I\ge I_0}\left|\frac{\partial V_1}{\partial\rho}\right|\left|U_1(I,\theta) - U_1(I,-\theta)\right|. \quad (58)$$

From (47), we have $|(\partial/\partial\rho)V_1(\rho,\varphi)| \le 1/2$ for $I \ge I_0$ sufficiently large and therefore we obtain $U_1(I,\theta) = U_1(I,-\theta)$. □

By the estimates in Lemma 6, we can prove the following inequalities.

Lemma 9. *For* $0 \le k + m \le 4$, *the following inequalities hold:*

(1) $|(\partial^k/\partial I^k)\tilde{l}_1(I,\theta)| \le CI^{-k+2-2\gamma-3b}$,

(2) $|(\partial^{k+m}/\partial I^k\partial t^m)\tilde{l}_2(I,\theta,t)| \le CI^{-k+a}$,

(3) $|(\partial^{k+m}/\partial I^k\partial t^m)\tilde{l}_3(I,\theta,t)| \le CI^{-k+3-4b}$,

(4) $|(\partial^{k+m}/\partial I^k\partial t^m)\tilde{l}_4(I,\theta,t)| \le CI^{-k+3-4b}$,

(5) $|(\partial^k/\partial I^k)\tilde{h}_1(I,\theta)| \le CI^{-k+1-\gamma-(5/2)b}$,

(6) $|(\partial^{k+m}/\partial I^k\partial t^m)\tilde{h}_2(I,\theta,t)| \le CI^{-k+\tau}$,

(7) $|(\partial^{k+m}/\partial I^k\partial t^m)\tilde{h}_3(I,\theta,t)| \le CI^{-k+2-4b}$.

Proof. (1) From the estimates (1) and (5) of Lemmas 6 and 8, it follows that

$$\begin{aligned}
&\left|\frac{\partial^k}{\partial I^k}\tilde{l}_1(I,\theta)\right| \\
&\le \left|\frac{\partial^k}{\partial I^k}\left(\frac{\partial V_1(\rho,\varphi)}{\partial\rho}\cdot l_1(\rho,\varphi)\right)\right| + \left|\frac{\partial^k}{\partial I^k}\left(\frac{\partial V_1(\rho,\varphi)}{\partial\varphi}\cdot h_1(\rho,\varphi)\right)\right| \\
&\le C\sum_{i_1+i_2=k}\left|\frac{\partial^{i_1+1}V_1(\rho,\varphi)}{\partial I^{i_1}\partial\rho}\right|\left|\frac{\partial^{i_2}l_1}{\partial I^{i_2}}\right| + C\sum_{i_1+i_2+i_3=k}\frac{\partial^{i_1}\rho^{2b-1}}{\partial I^{i_1}}\left|\frac{\partial^{i_2}l_1}{\partial I^{i_2}}\right|\left|\frac{\partial^{i_3}h_1}{\partial I^{i_3}}\right| \\
&\le C\sum_{i_1+i_2=k}\left(\sum_{\tau_1+\cdots+\tau_s=i_1}\left|\frac{\partial^{s+1}V_1(\rho,\varphi)}{\partial\rho^{s+1}}\frac{\partial^{\tau_1}(I+U_1)}{\partial I^{\tau_1}}\cdots\frac{\partial^{\tau_s}(I+U_1)}{\partial I^{\tau_s}}\right|\right) \\
&\qquad\times\left|\frac{\partial^{i_2}l_1}{\partial I^{i_2}}\right| \\
&\quad + C\sum_{i_1+i_2+i_3=k}\left(\sum_{\tau_1+\cdots+\tau_s=i_1}\left|\frac{\partial^{s}\rho^{2b-1}}{\partial\rho^{s}}\frac{\partial^{\tau_1}(I+U_1)}{\partial I^{\tau_1}}\cdots\frac{\partial^{\tau_s}(I+U_1)}{\partial I^{\tau_s}}\right|\right) \\
&\qquad\times\left|\frac{\partial^{i_2}l_1}{\partial I^{i_2}}\frac{\partial^{i_3}h_1}{\partial I^{i_3}}\right| \\
&\le C\rho^{-k+2-2\gamma-3b} \le CI^{-k+2-2\gamma-3b}.
\end{aligned} \quad (59)$$

(2) Since

$$\begin{aligned}
\tilde{l}_2(I,\theta,t) &= l_2(\rho,\varphi,t) \\
&\quad + \frac{\partial V_1(\rho,\varphi)}{\partial\rho}\cdot\left(l_2(\rho,\varphi,t) + \varepsilon l_3(\rho,\varphi,t)\right) \\
&\quad + \frac{\partial V_1(\rho,\varphi)}{\partial\varphi}\cdot\left(h_2(\rho,\varphi,t) + \varepsilon h_3(\rho,\varphi,t)\right) \\
&\quad + \varepsilon\left(l_3(\rho,\varphi,t) - l_3(I,\theta,t)\right),
\end{aligned} \quad (60)$$

we can prove that

$$\begin{aligned}
&\left|\frac{\partial^{k+m}}{\partial I^k\partial t^m}\left(\frac{\partial V_1(\rho,\varphi)}{\partial\rho}\cdot\left(l_2(\rho,\varphi,t) + \varepsilon l_3(\rho,\varphi,t)\right)\right)\right| \\
&\qquad \le CI^{-k+a}, \\
&\left|\frac{\partial^{k+m}}{\partial I^k\partial t^m}\left(\frac{\partial V_1(\rho,\varphi)}{\partial\varphi}\cdot\left(h_2(\rho,\varphi,t) + \varepsilon h_3(\rho,\varphi,t)\right)\right)\right| \\
&\qquad \le CI^{-k+a}, \\
&\qquad\qquad \left|\frac{\partial^{k+m}}{\partial I^k\partial t^m}l_2(\rho,\varphi,t)\right| \le I^{-k+a}.
\end{aligned} \quad (61)$$

Their proofs are similar to the proofs in (1).

Next, we check the last part of $\tilde{l}_2(I,\theta,t)$. We get

$$\begin{aligned}
&\left|\frac{\partial^{k+m}}{\partial I^k\partial t^m}\left(l_3(\rho,\varphi,t) - l_3(I,\theta,t)\right)\right| \\
&\quad = \left|\frac{\partial^{k+m}}{\partial I^k\partial t^m}\left(\int_0^1\frac{\partial l_3}{\partial\rho}(I + sU_1(I,\theta),\theta,t)\cdot U_1(I,\theta)\,ds\right)\right| \\
&\quad \le \int_0^1\sum_{i_1+i_2=k}\left|\frac{\partial^{i_1+m}}{\partial I^{i_1}\partial t^m}\left(\frac{\partial l_3(I+sU_1,\theta,t)}{\partial\rho}\right)\right|\left|\frac{\partial^{i_2}U_1}{\partial I^{i_2}}\right|ds \\
&\quad \le C\int_0^1\sum_{i_1+i_2=k}(I+U_1)^{-i_1+2-4b}I^{-i_2+1-\gamma-b/2}ds \\
&\quad \le CI^{-k+3-\gamma-(9/2)b} \le CI^{-k+a},
\end{aligned} \quad (62)$$

by the estimate in Lemma 6 and the definition of a.

(3) It is clearly by (3) in Lemma 6.

(4) It is clearly by (4) in Lemmas 6 and 8.

(5) We have that

$$\begin{aligned}
&\tilde{h}_1(I,\theta) = h_0'(\rho) - h_0'(I) + h_1(\rho,\varphi), \\
&\left|\frac{\partial^{k+m}}{\partial I^k\partial t^m}\left(h_0'(\rho) - h_0'(I)\right)\right| \\
&\quad \le \left|\frac{\partial^{k+m}}{\partial I^k\partial t^m}\left(\int_0^1\frac{d^2h_0(I+sU_1)}{d\rho^2}U_1(I,\theta)\,ds\right)\right| \\
&\quad \le CI^{-k-2b+1-\gamma-b/2} \le CI^{-k+1-\gamma-5b/2}.
\end{aligned} \quad (63)$$

From the last inequalities and (5) in Lemma 6, we obtain

$$\left|\frac{\partial^k}{\partial I^k}\tilde{h}_1(I,\theta)\right| \le CI^{-k+1-\gamma-(5/2)b}. \quad (64)$$

(6) Since

$$\tilde{h}_2(I,\theta,t) = h_2(\rho,\varphi,t) + \varepsilon\left(h_3(\rho,\varphi,t) - h_3(I,\theta,t)\right),$$
$$\left|\frac{\partial^{k+m}}{\partial I^k \partial t^m} h_2(\rho,\varphi,t)\right| \le CI^{-k+\tau}, \quad (65)$$

we just have to prove that

$$\left|\frac{\partial^{k+m}}{\partial I^k \partial t^m}\left(h_3(\rho,\varphi,t) - h_3(I,\theta,t)\right)\right| \le CI^{-k+\tau}. \quad (66)$$

In fact,

$$\begin{aligned}
&\left|\frac{\partial^{k+m}}{\partial I^k \partial t^m}\left(h_3(\rho,\varphi,t) - h_3(I,\theta,t)\right)\right| \\
&= \left|\frac{\partial^{k+m}}{\partial I^k \partial t^m}\left(\int_0^1 \frac{\partial h_3}{\partial \rho}\left(I + sU_1(I,\theta),\theta,t\right)\cdot U_1(I,\theta)\,ds\right)\right| \\
&\le \int_0^1 \sum_{i_1+i_2=k}\left|\frac{\partial^{i_1+m}}{\partial I^{i_1}\partial t^m}\left(\frac{\partial h_3(I+sU_1,\theta,t)}{\partial\rho}\right)\right|\left|\frac{\partial^{i_2}U_1}{\partial I^{i_2}}\right| ds \\
&\le C\int_0^1 \sum_{i_1+i_2=k}\left(I+U_1\right)^{-i_1+1-4b} I^{-i_2+1-\gamma-b/2} ds \\
&\le CI^{-k+2-\gamma-(9/2)b} \le CI^{-k+\tau},
\end{aligned} \quad (67)$$

so we have proved (6).

(7) We have

$$\left|\frac{\partial^{k+m}}{\partial I^k \partial t^m}\tilde{h}_3(I,\theta,t)\right| \le CI^{-k+2-4b}, \quad (68)$$

by (7) in Lemma 6. □

3. The Proof of Boundedness

In this section, all the solutions of (8) which are bounded will be proved via the KAM theory for reversible systems developed by Sevryuk [21] or Moser [22, 23] if $B > 0$.

We define the functions $\eta_0, \eta_1, \eta_2, \eta_3, \xi_1, \xi_2$, and ξ_3 as

$$\eta_0(I) = \frac{1}{h_0'(I)},$$

$$\eta_1(I,\theta) = -\frac{\tilde{h}_1(I,\theta)}{h_0'(I)\left(h_0'(I) + \tilde{h}_1(I,\theta)\right)},$$

$$\begin{aligned}
&\eta_2(I,\theta,t) \\
&= -\left(\tilde{h}_2(I,\theta,t)\right) \\
&\quad \times\left(\left(h_0'(I) + \tilde{h}_1(I,\theta)\right)\right. \\
&\quad\quad \left.\times\left(h_0'(I)+\tilde{h}_1(I,\theta)+\tilde{h}_2(I,\theta,t)+\varepsilon\tilde{h}_3(I,\theta,t)\right)\right)^{-1},
\end{aligned}$$

$$\begin{aligned}
&\eta_3(I,\theta,t) \\
&= -\left(\tilde{h}_3(I,\theta,t)\right) \\
&\quad \times\left(\left(h_0'(I) + \tilde{h}_1(I,\theta)\right)\right. \\
&\quad\quad \left.\times\left(h_0'(I)+\tilde{h}_1(I,\theta)+\tilde{h}_2(I,\theta,t)+\varepsilon\tilde{h}_3(I,\theta,t)\right)\right)^{-1}, \\
&\xi_1(I,\theta,t) = \left(\tilde{l}_1(I,\theta)+\tilde{l}_2(I,\theta,t)\right) \\
&\quad \cdot\left(\eta_0(I)+\eta_1(I,\theta)+\eta_2(I,\theta,t)+\varepsilon\eta_3(I,\theta,t)\right), \\
&\xi_2(I,\theta,t) = \tilde{l}_3(I,\theta,t) \\
&\quad \cdot\left(\eta_0(I)+\eta_1(I,\theta)+\eta_2(I,\theta,t)+\varepsilon\eta_3(I,\theta,t)\right), \\
&\xi_3(I,\theta,t) = \tilde{l}_4(I,\theta,t) \\
&\quad \cdot\left(\eta_0(I)+\eta_1(I,\theta)+\eta_2(I,\theta,t)+\varepsilon\eta_3(I,\theta,t)\right).
\end{aligned} \quad (69)$$

Then system (41) is equivalent to the following system:

$$\begin{aligned}
\frac{dt}{d\theta} &= \eta_0(I) + \eta_1(I,\theta) + \eta_2(I,\theta,t) + \varepsilon\eta_3(I,\theta,t), \\
\frac{dI}{d\theta} &= \xi_1(I,\theta,t) + \varepsilon\xi_2(I,\theta,t) \\
&\quad + \alpha T_0\left|S(\theta T_0)\right|^{\alpha-1}C(\theta T_0)\,\varepsilon\xi_3(I,\theta,t).
\end{aligned} \quad (70)$$

In addition, one can verify that system (70) is reversible with respect to involution $G:(t,I)\mapsto(-t,I)$.

Then some estimates on the functions η_i $(i = 0,1,2,3)$ and ξ_i $(i = 1,2,3)$ are given.

Lemma 10. *The following inequalities hold:*

(1) $cI^{2b-1} \le |\eta_0(I)| \le CI^{2b-1}$,

(2) $|(\partial^k/\partial I^k)\eta_1(I,\theta)| \le CI^{-k-1-\gamma+3b/2}$,

(3) $|(\partial^{k+m}/\partial I^k\partial t^m)\eta_2(I,\theta,t)| \le CI^{-k+\tau+4b-2}$,

(4) $|(\partial^{k+m}/\partial I^k\partial t^m)\eta_3(I,\theta,t)| \le CI^{-k}$,

(5) $|(\partial^{k+m}/\partial I^k\partial t^m)\xi_1(I,\theta,t)| \le CI^{-k+a+2b-1}$,

(6) $|(\partial^{k+m}/\partial I^k\partial t^m)\xi_2(I,\theta,t)| \le CI^{-k+2-2b}$,

(7) $|(\partial^{k+m}/\partial I^k\partial t^m)\xi_3(I,\theta,t)| \le CI^{-k+2-2b}$, *for* $0 \le k+m \le 4$.

Proof. (1) It is clear.

(2) Note that $1 - 2b > 1 - \gamma - 2b/5$, and

$$\left|\tilde{h}_1(I,\theta)\right| \le CI^{1-\gamma-(5/2)b}, \quad (71)$$

it follows that

$$\begin{aligned}\left|h_0'(I)+\widetilde{h}_1(I,\theta)\right| &\geq \left|\left|h_0'(I)\right|-\left|\widetilde{h}_1(I,\theta)\right|\right| \\ &\geq \left|h_0'(I)\right|-\left|\widetilde{h}_1(I,\theta)\right| \\ &\geq \frac{1}{T_0}I^{1-2b}-CI^{1-\gamma-(5/2)b} \\ &\geq cI^{1-2b}\end{aligned} \tag{72}$$

as $I \gg 1$.

Moreover, we also have

$$\begin{aligned}\left|\frac{\partial^l}{\partial I^l}\left(h_0'(I)+\widetilde{h}_1(I,\theta)\right)\right| &\leq \left|\frac{\partial^l}{\partial I^l}h_0'(I)\right|+\left|\frac{\partial^l}{\partial I^l}\widetilde{h}_1(I,\theta)\right| \\ &\leq CI^{-l+1-2b}+CI^{-l+1-\gamma-(2/5)b} \\ &\leq CI^{-l+1-2b}.\end{aligned} \tag{73}$$

So

$$\begin{aligned}&\left|\frac{\partial^i}{\partial I^i}\left(\frac{1}{h_0'(I)+\widetilde{h}_1(I,\theta)}\right)\right| \\ &\leq C\sum_{l_1+\cdots+l_s=i}\left|\frac{(-1)^s s!}{\left(h_0'(I)+\widetilde{h}_1(I,\theta)\right)^{s+1}}\right| \\ &\qquad\times\left|\frac{\partial^{l_1}}{\partial I^{l_1}}\left(h_0'(I)+\widetilde{h}_1(I,\theta)\right)\right| \\ &\qquad\cdots\left|\frac{\partial^{l_s}}{\partial I^{l_s}}\left(h_0'(I)+\widetilde{h}_1(I,\theta)\right)\right| \\ &\leq C\sum_{l_1+\cdots+l_s=i}I^{(2b-1)(s+1)}\cdot I^{-i+(1-2b)s}\leq CI^{-i+2b-1}.\end{aligned} \tag{74}$$

From (72) and (74), it is easy to see that

$$\begin{aligned}&\left|\frac{\partial^k}{\partial I^k}\eta_1(I,\theta)\right| \\ &=\left|\frac{\partial^k}{\partial I^k}\left(\frac{\widetilde{h}_1(I,\theta)}{h_0'(I)\left(h_0'(I)+\widetilde{h}_1(I,\theta)\right)}\right)\right| \\ &\leq C\sum_{i_1+i_2+i_3=k}\left|\frac{\partial^{i_1}}{\partial I^{i_1}}\widetilde{h}_1(I,\theta)\right| \\ &\qquad\cdot\left|\frac{\partial^{i_2}}{\partial I^{i_2}}\left(\frac{1}{h_0'(I)}\right)\right|\cdot\left|\frac{\partial^{i_3}}{\partial I^{i_3}}\left(\frac{1}{h_0'(I)+\widetilde{h}_1(I,\theta)}\right)\right| \\ &\leq C\sum_{i_1+i_2+i_3=k}I^{-i_1+1-\gamma-(2/5)b}I^{-i_2-1+2b}I^{-i_3-1+2b} \\ &\leq CI^{-k-1-\gamma+(3/2)b}.\end{aligned} \tag{75}$$

(3) We have

$$\begin{aligned}\left|\frac{\partial^{k+m}}{\partial I^k\partial t^m}\widetilde{h}_2(I,\theta,t)\right| &\leq CI^{-k+\tau}, \\ \left|\frac{\partial^{k+m}}{\partial I^k\partial t^m}\widetilde{h}_3(I,\theta,t)\right| &\leq CI^{-k+2-4b}.\end{aligned} \tag{76}$$

By (72), $1-2b>\tau$ ($\tau=\max(3-6b,-b)$) and $1-2b>2-4b$, we have

$$\begin{aligned}&\left|h_0'(I)+\widetilde{h}_1(I,\theta)+\widetilde{h}_2(I,\theta,t)+\varepsilon\widetilde{h}_3(I,\theta,t)\right| \\ &\quad\geq\left|\left|h_0'(I)+\widetilde{h}_1(I,\theta)\right|-\left|\widetilde{h}_2(I,\theta,t)+\varepsilon\widetilde{h}_3(I,\theta,t)\right|\right| \\ &\quad\geq\left|h_0'(I)+\widetilde{h}_1(I,\theta)\right|-\left|\widetilde{h}_2(I,\theta,t)\right|-\varepsilon\left|\widetilde{h}_3(I,\theta,t)\right| \\ &\quad\geq cI^{1-2b}-CI^{\tau}-C\varepsilon I^{2-4b}\geq cI^{1-2b},\end{aligned} \tag{77}$$

for $I \gg 1$.

Let $h_0'(I)+\widetilde{h}_1(I,\theta)+\widetilde{h}_2(I,\theta,t)+\varepsilon\widetilde{h}_3(I,\theta,t)=H(I,\theta,t)$. We find that

$$\begin{aligned}&\left|\frac{\partial^l}{\partial t^l}\left(\frac{1}{\left(h_0'(I)+\widetilde{h}_1(I,\theta)+\widetilde{h}_2(I,\theta,t)+\varepsilon\widetilde{h}_3(I,\theta,t)\right)^{s+1}}\right)\right| \\ &\quad\leq C\sum_{i_1+\cdots+i_r=l}\left|\frac{(-1)^r r!}{(H(I,\theta,t))^{s+1+r}}\right|\left|\frac{\partial^{i_1}}{\partial t^{i_1}}(H(I,\theta,t))\right| \\ &\qquad\cdots\left|\frac{\partial^{i_r}}{\partial t^{i_r}}(H(I,\theta,t))\right| \\ &\quad\leq C\sum_{i_1+\cdots+i_r=l}I^{(2b-1)(s+1+r)}\cdot\varepsilon^r I^{(2-4b)r}\leq CI^{(2b-1)(s+1)},\end{aligned} \tag{78}$$

so

$$\begin{aligned}&\left|\frac{\partial^{k+l}}{\partial I^k\partial t^l}\left(\frac{1}{h_0'(I)+\widetilde{h}_1(I,\theta)+\widetilde{h}_2(I,\theta,t)+\varepsilon\widetilde{h}_3(I,\theta,t)}\right)\right| \\ &\quad\leq C\left|\frac{\partial^l}{\partial t^l}\left(\sum_{i_1+\cdots+i_s=k}\frac{(-1)^s s!}{(H(I,\theta,t))^{s+1}}\cdot\frac{\partial^{i_1}}{\partial I^{i_1}}(H(I,\theta,t))\right.\right. \\ &\qquad\left.\left.\cdots\frac{\partial^{i_s}}{\partial I^{i_s}}(H(I,\theta,t))\right)\right| \\ &\quad\leq C\sum_{i_1+\cdots+i_s=k}\;\sum_{j_0+j_1+\cdots+j_s=l}\left|\frac{\partial^{j_0}}{\partial t^{j_0}}\frac{(-1)^s s!}{(H(I,\theta,t))^{s+1}}\right| \\ &\qquad\times\left|\frac{\partial^{i_1+j_1}}{\partial I^{i_1}\partial t^{j_1}}(H(I,\theta,t))\right| \\ &\qquad\cdots\left|\frac{\partial^{i_s+j_s}}{\partial I^{i_s}\partial t^{j_s}}(H(I,\theta,t))\right| \\ &\quad\leq C\sum_{i_1+\cdots+i_s=k}I^{(2b-1)(s+1)}\cdot I^{-(i_1+\cdots+i_s)-(1-2b)s}\leq CI^{-k+(2b-1)}.\end{aligned} \tag{79}$$

When $m=0$, the proof of (3) is similar to the proof of (2).

When $m > 0$, then

$$\begin{aligned}
&\left|\frac{\partial^{k+m}}{\partial I^k \partial t^m}\eta_2(I,\theta,t)\right| \\
&= \left|\frac{\partial^{k+m}}{\partial I^k \partial t^m}\left(\frac{\widetilde{h}_2(I,\theta,t)}{\left(h_0'(I)+\widetilde{h}_1(I,\theta)\right)\left(h_0'(I)+\widetilde{h}_1+\widetilde{h}_2+\varepsilon\widetilde{h}_3\right)}\right)\right| \\
&\le C\left|\frac{\partial^m}{\partial t^m}\left(\sum_{i_1+i_2+i_3=k}\frac{\partial^{i_1}}{\partial I^{i_1}}\widetilde{h}_2 \cdot \frac{\partial^{i_2}}{\partial I^{i_2}}\left(\frac{1}{h_0'(I)+\widetilde{h}_1}\right)\right.\right. \\
&\qquad\left.\left. \cdot\frac{\partial^{i_3}}{\partial I^{i_3}}\left(\frac{1}{h_0'(I)+\widetilde{h}_1+\widetilde{h}_2+\varepsilon\widetilde{h}_3}\right)\right)\right| \\
&\le C\sum_{i_1+i_2+i_3=k}\sum_{l_1+l_2=m}\left|\frac{\partial^{i_1+l_1}\widetilde{h}_2(I,\theta,t)}{\partial I^{i_1}\partial t^{l_1}}\right| \\
&\qquad\cdot\left|\frac{\partial^{i_2}}{\partial I^{i_2}}\left(\frac{1}{h_0'(I)+\widetilde{h}_1(I,\theta)}\right)\right| \\
&\qquad\cdot\left|\frac{\partial^{i_3+l_2}}{\partial I^{i_3}\partial t^{l_2}}\left(\frac{1}{h_0'(I)+\widetilde{h}_1+\widetilde{h}_2+\varepsilon\widetilde{h}_3}\right)\right| \\
&\le C\sum_{i_1+i_2+i_3=k} I^{-i_1+\tau}I^{-i_2+2b-1}I^{-i_3+2b-1} \le CI^{-k+\tau+4b-2}.
\end{aligned} \tag{80}$$

(4) The proof of (4) is similar to the proof of (3).

(5) Let $\eta_0(I)+\eta_1(I,\theta)+\eta_2(I,\theta,t)+\varepsilon\eta_3(I,\theta,t)=\eta(I,\theta,t)$. By using the estimates on the functions $\widetilde{l}_i$ $(i = 1,2)$ and η_j $(j=0,1,2,3)$, it follows that

$$\begin{aligned}
&\left|\frac{\partial^{k+m}}{\partial I^k \partial t^m}\xi_1(I,\theta,t)\right| \\
&\le C\left|\frac{\partial^m}{\partial t^m}\left(\sum_{k_1+k_2=k}\frac{\partial^{k_1}\left(\widetilde{l}_1(I,\theta)+\widetilde{l}_2(I,\theta,t)\right)}{\partial I^{k_1}}\cdot\frac{\partial^{k_2}(\eta(I,\theta,t))}{\partial I^{k_2}}\right)\right| \\
&\le C\sum_{k_1+k_2=k}\left|\frac{\partial^{k_1}\widetilde{l}_1(I,\theta)}{\partial I^{k_1}}\cdot\frac{\partial^{k_2+m}(\eta_2(I,\theta,t)+\varepsilon\eta_3(I,\theta,t))}{\partial I^{k_2}\partial t^m}\right| \\
&\quad + C\sum_{l_k+k_2=k}\sum_{m_1+m_2=m}\left|\frac{\partial^{k_1+m_1}\widetilde{l}_2(I,\theta,t)}{\partial I^{k_1}\partial t^{m_1}}\cdot\frac{\partial^{k_2+m_2}(\eta(I,\theta,t))}{\partial I^{k_2}\partial t^{m_2}}\right| \\
&\le CI^{-k_1+2-\gamma-(5/2)b}\cdot\left(I^{-k_2+\tau+4b-2}+\varepsilon I^{-k_2}\right)+CI^{-k_1+a}\cdot I^{-k_2+2b-1} \\
&\le CI^{-k+a+2b-1},
\end{aligned} \tag{81}$$

when $m \neq 0$.

When $m=0$, then

$$\begin{aligned}
&\left|\frac{\partial^{k}}{\partial I^k}\xi_1(I,\theta,t)\right| \\
&\le C\sum_{k_1+k_2=k}\left|\frac{\partial^{k_1}\left(\widetilde{l}_1(I,\theta)+\widetilde{l}_2(I,\theta,t)\right)}{\partial I^{k_1}}\cdot\frac{\partial^{k_2}(\eta(I,\theta,t))}{\partial I^{k_2}}\right| \\
&\le C\left(I^{-k_1+2-\gamma-(5/2)b}+I^{-k_1+a}\right)\cdot I^{-k_2+2b-1} \le CI^{-k+a+2b-1},
\end{aligned} \tag{82}$$

by $a > 2-\gamma-5b/2$.

(6) By using the estimates on the functions $\widetilde{l}_3$ and η_i $(i=0,1,2,3)$, it follows that

$$\begin{aligned}
&\left|\frac{\partial^{k+m}}{\partial I^k \partial t^m}\xi_2(I,\theta,t)\right| \\
&\le C\left|\frac{\partial^m}{\partial t^m}\left(\sum_{k_1+k_2=k}\frac{\partial^{k_1}\left(\widetilde{l}_3(I,\theta,t)\right)}{\partial I^{k_1}}\cdot\frac{\partial^{k_2}(\eta(I,\theta,t))}{\partial I^{k_2}}\right)\right| \\
&\le C\sum_{l_k+k_2=k}\sum_{m_1+m_2=m}\left|\frac{\partial^{k_1+m_1}\widetilde{l}_3(I,\theta,t)}{\partial I^{k_1}\partial t^{m_1}}\cdot\frac{\partial^{k_2+m_2}(\eta(I,\theta,t))}{\partial I^{k_2}\partial t^{m_2}}\right| \\
&\le CI^{-k_1+3-4b}\cdot I^{-k_2+2b-1} \\
&\le CI^{-k+2-2b}.
\end{aligned} \tag{83}$$

(7) By using the estimates on the functions $\widetilde{l}_4$ and η_i $(i=0,1,2,3)$, it follows that

$$\begin{aligned}
&\left|\frac{\partial^{k+m}}{\partial I^k \partial t^m}\xi_3(I,\theta,t)\right| \\
&\le C\left|\frac{\partial^m}{\partial t^m}\left(\sum_{k_1+k_2=k}\frac{\partial^{k_1}\left(\widetilde{l}_4(I,\theta,t)\right)}{\partial I^{k_1}}\cdot\frac{\partial^{k_2}(\eta(I,\theta,t))}{\partial I^{k_2}}\right)\right| \\
&\le C\sum_{l_k+k_2=k}\sum_{m_1+m_2=m}\left|\frac{\partial^{k_1+m_1}\widetilde{l}_4(I,\theta,t)}{\partial I^{k_1}\partial t^{m_1}}\cdot\frac{\partial^{k_2+m_2}(\eta(I,\theta,t))}{\partial I^{k_2}\partial t^{m_2}}\right| \\
&\le CI^{-k_1+3-4b}\cdot I^{-k_2+2b-1} \\
&\le CI^{-k+2-2b}.
\end{aligned} \tag{84}$$

□

Let $t=t$, $\theta=\theta$, $r=\eta_0(I)$ and

$$\begin{aligned}
F_0(r,\theta) &= \eta_1(I(r),\theta), \\
F_1(r,\theta,t) &= \eta_2(I(r),\theta,t)+\varepsilon\eta_3(I(r),\theta,t), \\
F_2(r,\theta,t) &= \eta_0'(I(r))\cdot\left(\xi_1(I(r),\theta,t)+\varepsilon\xi_2(I(r),\theta,t)\right), \\
F_3(r,\theta,t) &= \varepsilon\eta_0'(I(r))\cdot\xi_3(I(r),\theta,t),
\end{aligned} \tag{85}$$

where $I(r)$ is the inverse function of $r=\eta_0(I)$.

Then system (70) is transformed into the following form:

$$\begin{aligned}
\frac{dt}{d\theta} &= r+F_0(r,\theta)+F_1(r,\theta,t), \\
\frac{dr}{d\theta} &= F_2(r,\theta,t)+\alpha T_0\left|S(\theta T_0)\right|^{\alpha-1}C(\theta T_0)\cdot F_3(r,\theta,t).
\end{aligned} \tag{86}$$

Moreover, one can verify that system (86) is reversible with respect to involution $G:(t,r)\mapsto(-t,r)$.

It is easy to see that $I \gg 1$ if and only if $r \gg 1$, and the solutions of system (86) do exist on $0 \le \theta \le 1$ when $r(0) = r \gg 1$.

By using the estimates on η_i and ξ_i $(i = 1,2,3)$ in Lemma 10, the following inequalities can be proved.

Lemma 11. *For $0 \le k+m \le 4$ and $r \gg 1$, the following inequalities hold:*

(1) $|(\partial^k/\partial r^k)F_0(r,\theta)| \le Cr^{-k-(1+\gamma-3b/2)/(2b-1)}$,

(2) $|(\partial^{k+m}/\partial r^k\partial t^m)F_1(r,\theta,t)| \le C(r^{-k+(\tau+4b-2)/(2b-1)} + \varepsilon r^{-k})$,

(3) $|(\partial^{k+m}/\partial r^k\partial t^m)F_2(r,\theta,t)| \le C(r^{-k+(a+4b-3)/(2b-1)} + \varepsilon r^{-k})$,

(4) $|(\partial^{k+m}/\partial r^k\partial t^m)F_3(r,\theta,t)| \le C\varepsilon r^{-k}$.

Proof. Above all, we know that $r = \eta_0(I) = T_0 I^{2b-1}$, so we can get $I = ((1/T_0)r)^{1/(2b-1)}$. Then we have

$$\left|\frac{d^j I}{dr^j}\right| \le Cr^{-j+1/(2b-1)},$$
$$\left|\frac{d^j\eta_0'(I(r))}{dr^j}\right| \le C\left|\frac{d^j\left(r^{1-1/(2b-1)}\right)}{dr^j}\right| \le Cr^{-j+(2b-2)/(2b-1)}. \tag{87}$$

(1) We have that

$$\begin{aligned}\left|\frac{\partial^k F_0(r,\theta)}{\partial r^k}\right| &\le \sum_{k_1+\cdots+k_s=k}\left|\frac{\partial^s\eta_1(I,\theta)}{\partial I^s}\right|\cdot\left|\frac{d^{k_1}I}{dr^{k_1}}\right|\cdots\left|\frac{d^{k_s}I}{dr^{k_s}}\right| \\ &\le CI^{-s-1-\gamma+(3/2)b}r^{-k+(1/(2b-1))s} \\ &\le Cr^{-s(1/(2b-1))-(1+\gamma-(3/2)b)/(2b-1)}r^{-k+(1/(2b-1))s} \\ &\le Cr^{-k-(1+\gamma-(3/2)b)/(2b-1)}.\end{aligned} \tag{88}$$

(2) We have that

$$\begin{aligned}\left|\frac{\partial^{k+m}F_1(r,\theta,t)}{\partial r^k\partial t^m}\right| &\le C\sum_{i_1+\cdots+i_s=k}\left|\frac{\partial^{s+m}\eta_2(I,\theta,t)}{\partial I^s\partial t^m}\frac{d^{i_1}I}{dr^{i_1}}\cdots\frac{d^{i_s}I}{dr^{i_s}}\right| \\ &\quad + C\varepsilon\sum_{j_1+\cdots+j_\nu=k}\left|\frac{\partial^{\nu+m}\eta_3(I,\theta,t)}{\partial I^\nu\partial t^m}\frac{d^{j_1}I}{dr^{j_1}}\cdots\frac{d^{j_\nu}I}{dr^{j_\nu}}\right| \\ &\le C\left(r^{-k+((\tau+4b-2)/(2b-1))} + \varepsilon r^{-k}\right).\end{aligned} \tag{89}$$

(3) We have that

$$\begin{aligned}&\left|\frac{\partial^{k+m}F_2(r,\theta,t)}{\partial r^k\partial t^m}\right| \\ &\le \left|\sum_{k_1+k_2=k}\frac{d^{k_1}\eta_0'(I(r))}{dr^{k_1}}\cdot\left(\frac{\partial^{k_2+m}\xi_1(I(r),\theta,t)}{\partial r^{k_2}\partial t^m} + \varepsilon\frac{\partial^{k_2+m}\xi_2(I(r),\theta,t)}{\partial r^{k_2}\partial t^m}\right)\right| \\ &\le C\sum_{k_1+k_2=k}r^{-k_1+(2b-2)/(2b-1)} \\ &\quad\times\left(\sum_{i_1+\cdots+i_s=k_2}\frac{\partial^{s+m}\xi_1(I,\theta,t)}{\partial I^s\partial t^m}\frac{d^{i_1}I}{dr^{i_1}}\cdots\frac{d^{i_s}I}{dr^{i_s}}\right. \\ &\quad\left.+\varepsilon\sum_{j_1+\cdots+j_\nu=k_2}\frac{\partial^{\nu+m}\xi_2(I,\theta,t)}{\partial I^\nu\partial t^m}\frac{d^{j_1}I}{dr^{j_1}}\cdots\frac{d^{j_\nu}I}{dr^{j_\nu}}\right) \\ &\le Cr^{-k_1+(2b-2)/(2b-1)}r^{(1/(2b-1))(-s+a+2b-1)}r^{-k_1+(1/(2b-1))s} \\ &\quad + C\varepsilon r^{-k_1+(2b-2)/(2b-1)}r^{(1/(2b-1))(-\nu+2-2b)}r^{-k_1+(1/(2b-1))\nu} \\ &\le Cr^{-k+(a+4b-3)/(2b-1)} + C\varepsilon r^{-k}.\end{aligned} \tag{90}$$

(4) We have that

$$\begin{aligned}&\left|\frac{\partial^{k+m}F_3(r,\theta,t)}{\partial r^k\partial t^m}\right| \\ &\le \varepsilon\left|\sum_{k_1+k_2=k}\frac{d^{k_1}\eta_0'(I(r))}{dr^{k_1}}\cdot\left(\frac{\partial^{k_2+m}\xi_3(I(r),\theta,t)}{\partial r^{k_2}\partial t^m}\right)\right| \\ &\le C\varepsilon\sum_{k_1+k_2=k}r^{-k_1+(2b-2)/(2b-1)} \\ &\quad\times\left(\sum_{i_1+\cdots+i_s=k_2}\frac{\partial^{s+m}\xi_3(I,\theta,t)}{\partial I^s\partial t^m}\frac{d^{i_1}I}{dr^{i_1}}\cdots\frac{d^{i_s}I}{dr^{i_s}}\right) \\ &\le C\varepsilon r^{-k}.\end{aligned} \tag{91}$$

□

Lemma 12. *The time 1 map Φ^1 of the flow Φ^θ of the system (86) is of the form*

$$\Phi^1 : r_1 = r + Q_2(r,t), \qquad t_1 = t + \widehat{\omega}(r) + Q_1(r,t), \tag{92}$$

where $\widehat{\omega}(r) = r + \int_0^1 F_0(r,\theta)d\theta$. And there exists a $\mu_0 > 0$ such that, for $0 \le k+m \le 4$, sufficiently large r and sufficiently small ε,

$$\left|\frac{\partial^{k+m}}{\partial r^k\partial t^m}Q_i(r,t)\right| \le Cr^{-\mu_0} + \varepsilon, \quad i = 1,2 \tag{93}$$

hold. Moreover, the map Φ^1 is reversible with respect to the involution $G:(t,r)\mapsto(-t,r)$.

Proof. Since

$$\begin{aligned}
&\int_0^1 \alpha T_0\left|S(\theta T_0)\right|^{\alpha-1}\left|C(\theta T_0)\right| d\theta \\
&= \lim_{\epsilon\to 0^+}\int_\epsilon^{1/4}\alpha T_0\left|S(\theta T_0)\right|^{\alpha-1}C(\theta T_0)\,d\theta \\
&\quad - \lim_{\epsilon\to 0^+}\int_{1/4}^{1/2-\epsilon}\alpha T_0\left|S(\theta T_0)\right|^{\alpha-1}C(\theta T_0)\,d\theta \\
&\quad - \lim_{\epsilon\to 0^+}\int_{1/2+\epsilon}^{3/4}\alpha T_0\left|S(\theta T_0)\right|^{\alpha-1}C(\theta T_0)\,d\theta \\
&\quad + \lim_{\epsilon\to 0^+}\int_{3/4}^{1-\epsilon}\alpha T_0\left|S(\theta T_0)\right|^{\alpha-1}C(\theta T_0)\,d\theta \\
&= \left|S\left(\frac{T_0}{4}\right)\right|^{\alpha-1}S\left(\frac{T_0}{4}\right) - \lim_{\epsilon\to 0^+}\left|S(\epsilon T_0)\right|^{\alpha-1}S(\epsilon T_0) \\
&\quad - \left[\lim_{\epsilon\to 0^+}\left|S\left(\left(\frac{1}{2}-\epsilon\right)T_0\right)\right|^{\alpha-1}S\left(\left(\frac{1}{2}-\epsilon\right)T_0\right)\right. \\
&\qquad \left. -\left|S\left(\frac{T_0}{4}\right)\right|^{\alpha-1}S\left(\frac{T_0}{4}\right)\right] \\
&\quad - \left[\left|S\left(\frac{3T_0}{4}\right)\right|^{\alpha-1}S\left(\frac{3T_0}{4}\right)\right. \\
&\qquad \left. -\lim_{\epsilon\to 0^+}\left|S\left(\left(\frac{1}{2}+\epsilon\right)T_0\right)\right|^{\alpha-1}S\left(\left(\frac{1}{2}+\epsilon\right)T_0\right)\right] \\
&\quad + \left[\lim_{\epsilon\to 0^+}\left|S((1-\epsilon)T_0)\right|^{\alpha-1}S((1-\epsilon)T_0)\right. \\
&\qquad \left. -\left|S\left(\frac{3T_0}{4}\right)\right|^{\alpha-1}S\left(\frac{3T_0}{4}\right)\right] \\
&= 4\left|S\left(\frac{T_0}{4}\right)\right|^{\alpha-1}S\left(\frac{T_0}{4}\right) = 4,
\end{aligned} \tag{94}$$

then we get $\int_0^1 \alpha T_0|S(\theta T_0)|^{\alpha-1}|C(\theta T_0)|d\theta$ is bounded.

Let $\alpha T_0|S(vT_0)|^{\alpha-1}C(vT_0) = S_1(v)$. Set $(r(\theta),t(\theta)) = \Phi^\theta(r,t)$ with $\Phi^0 = id$ for the flow:

$$t(\theta) = t + r\theta + D_1(r,t,\theta), \qquad r(\theta) = r + D_2(r,t,\theta). \tag{95}$$

Since

$$\Phi^\theta = \Phi^0 + \int_0^\theta X\cdot\Phi^v dv, \tag{96}$$

where X denotes the vector field of the system (86), we have

$$\begin{aligned}
t(\theta) &= t \\
&\quad + \int_0^\theta \left[r(v) + F_0(r(v),v) + F_1(r(v),v,t(v))\right]dv \\
&= t + r\theta \\
&\quad + \int_0^\theta \left[D_2(r,t,v) + F_0(r+D_2,v)\right. \\
&\qquad \left. +F_1(r+D_2,v,t+rv+D_1)\right]dv \\
&= t + r\theta + D_1(r,t,\theta), \\
r(\theta) &= r \\
&\quad + \int_0^\theta \left[F_2(r(v),v,t(v))\right. \\
&\qquad \left. +S_1(v)F_3(r(v),v,t(v))\right]dv \\
&= r \\
&\quad + \int_0^\theta \left[F_2(r+D_2,v,t+rv+D_1)\right. \\
&\qquad \left. +S_1(v)F_3(r+D_2,v,t+rv+D_1)\right]dv \\
&= r + D_2(r,t,\theta),
\end{aligned} \tag{97}$$

which is equivalent to the following equations for D_1 and D_2:

$$\begin{aligned}
D_1(r,t,\theta) &= \int_0^\theta \left[D_2(r,t,v) + F_0(r+D_2,v)\right. \\
&\qquad \left. +F_1(r+D_2,v,t+rv+D_1)\right]dv, \\
D_2(r,t,\theta) &= \int_0^\theta \left[F_2(r+D_2,v,t+rv+D_1)\right. \\
&\qquad \left. +S_1(v)F_3(r+D_2,v,t+rv+D_1)\right]dv.
\end{aligned} \tag{98}$$

Let $D(r,t,\theta) = (D_1(r,t,\theta), D_2(r,t,\theta))$, $|D_1(r,t,\theta)| = \sup_{(r,t,\theta)\in (R^+\times\mathbb{T}\times(0,1])}|D_1(r,t,\theta)|$. Define $\|D\| =: |D_1|/3 + 2|D_1|/3$, and $T(D) =: (T_1(D), T_2(D))$, where

$$\begin{aligned}
T_1(D) &= \int_0^\theta \left[D_2(r,t,v) + F_0(r+D_2,v)\right. \\
&\qquad \left. +F_1(r+D_2,v,t+rv+D_1)\right]dv, \\
T_2(D) &= \int_0^\theta \left[F_2(r+D_2,v,t+rv+D_1)\right. \\
&\qquad \left. +S_1(v)F_3(r+D_2,v,t+rv+D_1)\right]dv.
\end{aligned} \tag{99}$$

Next, we will prove that T is a contraction map. From the definition of $T(D)$, we have

$$
\begin{aligned}
&\left|T_1D - T_1\widetilde{D}\right| \\
&= \left|\int_0^\theta \left[D_2 - \widetilde{D}_2 + F_0\left(r + D_2, v\right) - F_0\left(r + \widetilde{D}_2\right)\right.\right. \\
&\qquad + F_1\left(r + D_2, v, t + rv + D_1\right) \\
&\qquad \left.\left. - F_1\left(r + \widetilde{D}_2, v, t + rv + \widetilde{D}_1\right)\right] dv\right| \\
&\le \left|D_2 - \widetilde{D}_2\right| \\
&\quad + \int_0^1 \left|\frac{\partial F_0\left(r + s\left(D_2 - \widetilde{D}_2\right), v\right)}{\partial r}\right| \cdot \left|D_2 - \widetilde{D}_2\right| ds \\
&\quad + \int_0^1 \left|\frac{\partial F_1\left(r + s\left(D_2 - \widetilde{D}_2\right), v, t + rv + D_1\right)}{\partial r}\right| \\
&\qquad \cdot \left|D_2 - \widetilde{D}_2\right| ds \\
&\quad + \int_0^1 \left|\frac{\partial F_1\left(r + \widetilde{D}_2, v, t + rv + s\left(D_1 - \widetilde{D}_1\right)\right)}{\partial t}\right| \\
&\qquad \cdot \left|D_1 - \widetilde{D}_1\right| ds \\
&\le \frac{6}{5}\left|D_2 - \widetilde{D}_2\right| + \frac{1}{4}\left|D_1 - \widetilde{D}_1\right|, \\
&\left|T_2D - T_2\widetilde{D}\right| \\
&\quad = \int_0^\theta \left[F_2\left(r + D_2, v, t + rv + D_1\right)\right. \\
&\qquad - F_2\left(r + \widetilde{D}_2, v, t + rv + \widetilde{D}_1\right) \\
&\qquad + S_1(v) F_3\left(r + D_2, v, t + rv + D_1\right) \\
&\qquad \left. - S_1(v) F_3\left(r + \widetilde{D}_2, v, t + rv + \widetilde{D}_1\right)\right] dv \\
&\quad \le \int_0^1 \left|\frac{\partial F_2\left(r + s\left(D_2 - \widetilde{D}_2\right), v, t + rv + D_1\right)}{\partial r}\right| \\
&\qquad \cdot \left|D_2 - \widetilde{D}_2\right| ds \\
&\qquad + \int_0^1 \left|\frac{\partial F_2\left(r + \widetilde{D}_2, v, t + rv + s\left(D_1 - \widetilde{D}_1\right)\right)}{\partial t}\right| \\
&\qquad \cdot \left|D_1 - \widetilde{D}_1\right| ds \\
&\qquad + \int_0^1 \left|S_1(v)\right| dv \\
&\qquad \cdot \int_0^1 \left|\frac{\partial F_3\left(r + s\left(D_2 - \widetilde{D}_2\right), v, t + rv + D_1\right)}{\partial r}\right| \\
&\qquad \cdot \left|D_2 - \widetilde{D}_2\right| ds \\
&\qquad + \int_0^1 \left|S_1(v)\right| dv \\
&\qquad \cdot \int_0^1 \left|\frac{\partial F_3\left(r + \widetilde{D}_2, v, t + rv + s\left(D_1 - \widetilde{D}_1\right)\right)}{\partial t}\right| \\
&\qquad \cdot \left|D_1 - \widetilde{D}_1\right| ds \\
&\quad \le \frac{3}{20}\left|D_2 - \widetilde{D}_2\right| + \frac{1}{8}\left|D_1 - \widetilde{D}_1\right|,
\end{aligned}
\tag{100}
$$

by Lemma 11 and the boundedness of $\int_0^1 |S_1(v)|dv$. Then we have

$$
\begin{aligned}
&\left\|T(D) - T\left(\widetilde{D}\right)\right\| \\
&\quad = \frac{1}{3}\left|T_1(D) - T_1\left(\widetilde{D}\right)\right| + \frac{2}{3}\left|T_2(D) - T_2\left(\widetilde{D}\right)\right| \\
&\quad \le \frac{1}{3} \times \left(\frac{6}{5}\left|D_2 - \widetilde{D}_2\right| + \frac{1}{4}\left|D_1 - \widetilde{D}_1\right|\right) \\
&\qquad + \frac{2}{3} \times \left(\frac{3}{20}\left|D_2 - \widetilde{D}_2\right| + \frac{1}{8}\left|D_1 - \widetilde{D}_1\right|\right) \\
&\quad = \frac{1}{6}\left|D_1 - \widetilde{D}_1\right| + \frac{1}{2}\left|D_2 - \widetilde{D}_2\right| \\
&\quad \le \frac{3}{4} \times \left(\frac{1}{3}\left|D_1 - \widetilde{D}_1\right| + \frac{2}{3}\left|D_2 - \widetilde{D}_2\right|\right) \\
&\quad \le \frac{3}{4}\left\|D - \widetilde{D}\right\|,
\end{aligned}
\tag{101}
$$

by the definition of the norm $\|\cdot\|$.

Using the contraction principle, one verifies easily that for $r \ge r_0$, (98) has a unique solution in the space $\{|D_1| \le 1, |D_2| \le 1\}$. Moreover, D_1 and D_2 are smooth.

Next, we will estimate $Q_1(r,t)$ and $Q_2(r,t)$ as follows:

$$
\begin{aligned}
Q_1(r,t) &= D_1(r,t,1) - \int_0^1 F_0(r,v)\, dv \\
&= \int_0^1 \left[D_2(r,t,v)\right. \\
&\qquad + \int_0^1 \frac{\partial F_0\left(r + sD_2, v\right)}{\partial r} \cdot D_2 ds \\
&\qquad \left. + F_1\left(r + D_2, v, t + rv + D_1\right)\right] dv, \\
Q_2(r,t) &= D_2(r,t,1) \\
&= \int_0^1 \left[F_2\left(r + D_2, v, t + rv + D_1\right)\right. \\
&\qquad \left. + S_1(v) F_3\left(r + D_2, v, t + rv + D_1\right)\right] dv.
\end{aligned}
\tag{102}
$$

In order to prove (93), we just need to prove that

$$\left|\frac{\partial^{k+m}}{\partial r^k \partial t^m} D_i(r,t,\theta)\right| \le C r^{-\mu_0} + C\varepsilon, \quad i = 1,2 \tag{103}$$

hold for $k + m \le 4$.

(1) When $k + m = 0$,

$$\begin{aligned}
&\left|D_2(r,t,\theta)\right| \\
&\quad \le \int_0^\theta \left(\left|F_2(r + D_2, v, t + rv + D_1)\right|\right. \\
&\qquad \left. + \left|S_1(v)\right|\left|F_3(r + D_2, v, t + rv + D_1)\right|\right) dv \\
&\quad \le C\left(r^{-\mu_0} + \varepsilon\right) + \int_0^1 \left|S_1(v)\right| dv \cdot (C\varepsilon) \le C\left(r^{-\mu_0} + \varepsilon\right), \\
&\left|D_1(r,t,\theta)\right| \\
&\quad \le \left|D_2(r,t,\theta)\right| \\
&\qquad + \int_0^\theta \left(\left|F_0(r + D_2, v)\right| + \left|F_1(r + D_2, v, t + rv + D_1)\right|\right) dv, \\
&\quad \le \left|D_2(r,t,v)\right| + \int_0^\theta \left(C r^{-\mu_0} + C r^{-\mu_0} + C\varepsilon\right) dv \\
&\quad \le \left|D_2(r,t,\theta)\right| + C\left(r^{-\mu_0} + \varepsilon\right) \le C\left(r^{-\mu_0} + \varepsilon\right),
\end{aligned} \tag{104}$$

where $\mu_0 = \min((1 + \gamma - 3b/2)/(2b - 1), (2 - 4b - \tau)/(2b - 1), (3 - 4b - a)/(2b - 1))$.

(2) When $m = 0$ and $k \ne 0$, we check the case when $k = 1$ firstly

$$\begin{aligned}
&\left|\frac{\partial D_2(r,t,\theta)}{\partial r}\right| \\
&\quad \le \int_0^\theta \left|\frac{\partial F_2(r + D_2, v, t + rv + D_1)}{\partial r}\right| \\
&\qquad \cdot \left(1 + \left|\frac{\partial D_2(r,t,v)}{\partial r}\right|\right) dv \\
&\qquad + \int_0^\theta \left|\frac{\partial F_2(r + D_2, v, t + rv + D_1)}{\partial t}\right| \\
&\qquad \cdot \left(1 + \left|\frac{\partial D_1(r,t,v)}{\partial r}\right|\right) dv \\
&\qquad + \int_0^1 \left|S_1(v)\right| dv \cdot \int_0^\theta \left|\frac{\partial F_3(r + D_2, v, t + rv + D_1)}{\partial r}\right| \\
&\qquad \cdot \left(1 + \left|\frac{\partial D_2(r,t,v)}{\partial r}\right|\right) dv \\
&\qquad + \int_0^1 \left|S_1(v)\right| dv \cdot \int_0^\theta \left|\frac{\partial F_3(r + D_2, v, t + rv + D_1)}{\partial t}\right| \\
&\qquad \cdot \left(1 + \left|\frac{\partial D_1(r,t,v)}{\partial r}\right|\right) dv \\
&\quad \le C r^{-1}\left(r^{-\mu_0} + \varepsilon\right) \cdot \left(1 + \left|\frac{\partial D_2(r,t,v)}{\partial r}\right|\right) \\
&\qquad + C\left(r^{-\mu_0} + \varepsilon\right) \cdot \left(1 + \left|\frac{\partial D_1(r,t,v)}{\partial r}\right|\right), \\
&\left|\frac{\partial D_1(r,t,\theta)}{\partial r}\right| \\
&\quad \le \left|\frac{\partial D_2(r,t,\theta)}{\partial r}\right| \\
&\qquad + \int_0^\theta \left|\frac{\partial F_0(r + D_2, v)}{\partial r}\right| \cdot \left(1 + \left|\frac{\partial D_2(r,t,v)}{\partial r}\right|\right) dv \\
&\qquad + \int_0^\theta \left|\frac{\partial F_1(r + D_2, v, t + rv + D_1)}{\partial r}\right| \cdot \left(1 + \left|\frac{\partial D_2(r,t,v)}{\partial r}\right|\right) dv \\
&\qquad + \int_0^\theta \left|\frac{\partial F_1(r + D_2, v, t + rv + D_1)}{\partial t}\right| \cdot \left(1 + \left|\frac{\partial D_1(r,t,v)}{\partial r}\right|\right) dv \\
&\quad \le \left|\frac{\partial D_2(r,t,\theta)}{\partial r}\right| + C r^{-1}\left(r^{-\mu_0} + \varepsilon\right) \cdot \left(1 + \left|\frac{\partial D_2(r,t,v)}{\partial r}\right|\right) \\
&\qquad + C\left(r^{-\mu_0} + \varepsilon\right) \cdot \left(1 + \left|\frac{\partial D_1(r,t,v)}{\partial r}\right|\right).
\end{aligned} \tag{105}$$

Hence,

$$\begin{aligned}
\left|\frac{\partial D_1(r,t,\theta)}{\partial r}\right| &\le C\left(r^{-\mu_0} + \varepsilon\right), \\
\left|\frac{\partial D_2(r,t,\theta)}{\partial r}\right| &\le C\left(r^{-\mu_0} + \varepsilon\right).
\end{aligned} \tag{106}$$

Now, we proceed inductively by assuming that for $j < k - 1$ the estimates

$$\begin{aligned}
\left|\frac{\partial^j D_1(r,t,\theta)}{\partial r^j}\right| &\le C\left(r^{-\mu_0} + \varepsilon\right), \\
\left|\frac{\partial^j D_2(r,t,\theta)}{\partial r^j}\right| &\le C\left(r^{-\mu_0} + \varepsilon\right),
\end{aligned} \tag{107}$$

hold and we wish to conclude that the same estimate holds for $j = k$

$$\begin{aligned}
&\left|\frac{\partial^k D_2(r,t,\theta)}{\partial r^k}\right| \\
&\quad \le \int_0^\theta \left|\frac{\partial F_2(r + D_2, v, t + rv + D_1)}{\partial r}\right| \\
&\qquad \cdot \left|\frac{\partial^k D_2(r,t,\theta)}{\partial r^k}\right| dv \\
&\qquad + \int_0^\theta \left|\frac{\partial F_2(r + D_2, v, t + rv + D_1)}{\partial t}\right| \\
&\qquad \cdot \left|\frac{\partial^k D_1(r,t,v)}{\partial r^k}\right| dv \\
&\qquad + C\left(r^{-\mu_0} + \varepsilon\right) \\
&\qquad \cdot \sum_{k_1 + k_2 = k} \sum_{\substack{i_1 + \cdots + i_s = k_1 \\ j_1 + \cdots + j_v = k_2}} \left|\frac{\partial^{i_1}(r + D_2)}{\partial r^{i_1}}\right| \cdots \left|\frac{\partial^{i_s}(r + D_2)}{\partial r^{i_s}}\right| \\
&\qquad \times \left|\frac{\partial^{j_1}(r + D_1)}{\partial r^{j_1}}\right| \cdots \left|\frac{\partial^{j_v}(r + D_1)}{\partial r^{j_v}}\right|
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 \left|S_1(v)\right| dv \\
&\quad\cdot \int_0^\theta \left|\frac{\partial F_3(r+D_2, v, t+rv+D_1)}{\partial r}\right| \cdot \left|\frac{\partial^k D_2(r,t,v)}{\partial r^k}\right| dv \\
&+\int_0^1 \left|S_1(v)\right| dv \\
&\quad\cdot \int_0^\theta \left|\frac{\partial F_3(r+D_2, v, t+rv+D_1)}{\partial t}\right| \cdot \left|\frac{\partial^k D_1(r,t,v)}{\partial r^k}\right| dv \\
&+C\left(r^{-\mu_0}+\varepsilon\right) \\
&\quad\cdot \sum_{k_1+k_2=k} \sum_{\substack{i_1+\cdots+i_s=k_1\\ j_1+\cdots+j_\nu=k_2}} \left|\frac{\partial^{i_1}(r+D_2)}{\partial r^{i_1}}\right| \cdots \left|\frac{\partial^{i_s}(r+D_2)}{\partial r^{i_s}}\right| \\
&\qquad\qquad \times \left|\frac{\partial^{j_1}(r+D_1)}{\partial r^{j_1}}\right| \cdots \left|\frac{\partial^{j_\nu}(r+D_1)}{\partial r^{j_\nu}}\right| \\
&\le Cr^{-1}\left(r^{-\mu_0}+\varepsilon\right) \cdot \left|\frac{\partial^k D_2(r,t,v)}{\partial r^k}\right| \\
&\quad + C\left(r^{-\mu_0}+\varepsilon\right) \cdot \left|\frac{\partial^k D_1(r,t,v)}{\partial r^k}\right| + C\left(r^{-\mu_0}+\varepsilon\right), \\
&\left|\frac{\partial^k D_1(r,t,\theta)}{\partial r^k}\right| \\
&\le \left|\frac{\partial^k D_2(r,t,\theta)}{\partial r^k}\right| \\
&\quad+\int_0^\theta \left|\frac{\partial F_0(r+D_2, v)}{\partial r}\right| \cdot \left|\frac{\partial^k D_2(r,t,v)}{\partial r^k}\right| dv \\
&\quad+C\left(r^{-\mu_0}+\varepsilon\right) \cdot \sum_{i_1+\cdots+i_s=k} \left|\frac{\partial^{i_1}(r+D_2)}{\partial r^{i_1}}\right| \cdots \left|\frac{\partial^{i_s}(r+D_2)}{\partial r^{i_s}}\right| \\
&\quad+\int_0^\theta \left|\frac{\partial F_1(r+D_2, v, t+rv+D_1)}{\partial r}\right| \cdot \left|\frac{\partial^k D_2(r,t,v)}{\partial r^k}\right| dv \\
&\quad+\int_0^\theta \left|\frac{\partial F_1(r+D_2, v, t+rv+D_1)}{\partial t}\right| \cdot \left|\frac{\partial^k D_1(r,t,v)}{\partial r^k}\right| dv \\
&\quad+C\left(r^{-\mu_0}+\varepsilon\right) \\
&\quad\cdot \sum_{k_1+k_2=k} \sum_{\substack{i_1+\cdots+i_s=k_1\\ j_1+\cdots+j_\nu=k_2}} \left|\frac{\partial^{i_1}(r+D_2)}{\partial r^{i_1}}\right| \cdots \left|\frac{\partial^{i_s}(r+D_2)}{\partial r^{i_s}}\right| \\
&\qquad\qquad \times \left|\frac{\partial^{j_1}(r+D_1)}{\partial r^{j_1}}\right| \cdots \left|\frac{\partial^{j_\nu}(r+D_1)}{\partial r^{j_\nu}}\right| \\
&\le \left|\frac{\partial^k D_2(r,t,\theta)}{\partial r^k}\right| + Cr^{-1}\left(r^{-\mu_0}+\varepsilon\right) \cdot \left|\frac{\partial^k D_2(r,t,v)}{\partial r^k}\right| \\
&\quad + C\left(r^{-\mu_0}+\varepsilon\right) \cdot \left|\frac{\partial^k D_1(r,t,v)}{\partial r^k}\right| + C\left(r^{-\mu_0}+\varepsilon\right),
\end{aligned} \tag{108}
$$

where $s+\nu \le 2$. Hence,

$$
\begin{aligned}
\left|\frac{\partial^k D_1(r,t,\theta)}{\partial r^k}\right| &\le C\left(r^{-\mu_0}+\varepsilon\right), \\
\left|\frac{\partial^k D_2(r,t,\theta)}{\partial r^k}\right| &\le C\left(r^{-\mu_0}+\varepsilon\right).
\end{aligned} \tag{109}
$$

(3) We can prove that

$$
\begin{aligned}
\left|\frac{\partial^m D_1(r,t,\theta)}{\partial t^m}\right| &\le C\left(r^{-\mu_0}+\varepsilon\right), \\
\left|\frac{\partial^m D_2(r,t,\theta)}{\partial t^m}\right| &\le C\left(r^{-\mu_0}+\varepsilon\right)
\end{aligned} \tag{110}
$$

similarly to (2) when $m \ne 0$.

(4) we have that

$$
\begin{aligned}
&\left|\frac{\partial^2 D_2(r,t,\theta)}{\partial r \partial t}\right| \\
&\le \int_0^\theta \left|\frac{\partial F_2(r+D_2, v, t+rv+D_1)}{\partial r}\right| \cdot \left|\frac{\partial^2 D_2(r,t,\theta)}{\partial r \partial t}\right| dv \\
&\quad+\int_0^\theta \left|\frac{\partial F_2(r+D_2, v, t+rv+D_1)}{\partial t}\right| \cdot \left|\frac{\partial^2 D_1(r,t,\theta)}{\partial r \partial t}\right| dv \\
&\quad+C\left(r^{-\mu_0}+\varepsilon\right) \\
&\quad\cdot \left(\frac{\partial D_2(r,t,\theta)}{\partial r} + \frac{\partial D_1(r,t,\theta)}{\partial t} + \frac{\partial D_1(r,t,\theta)}{\partial r}\right. \\
&\qquad \left. + \frac{\partial D_2(r,t,\theta)}{\partial t} + 1\right) \\
&\quad+\int_0^1 \left|S_1(v)\right| dv \\
&\quad\cdot \int_0^\theta \left|\frac{\partial F_3(r+D_2, v, t+rv+D_1)}{\partial r}\right| \cdot \left|\frac{\partial^2 D_2(r,t,\theta)}{\partial r \partial t}\right| dv \\
&\quad+\int_0^1 \left|S_1(v)\right| dv \\
&\quad\cdot \int_0^\theta \left|\frac{\partial F_3(r+D_2, v, t+rv+D_1)}{\partial t}\right| \cdot \left|\frac{\partial^2 D_1(r,t,\theta)}{\partial r \partial t}\right| dv \\
&\le C\left(r^{-\mu_0}+\varepsilon\right) \cdot \left|\frac{\partial^2 D_2(r,t,\theta)}{\partial r \partial t}\right| \\
&\quad + C\left(r^{-\mu_0}+\varepsilon\right) \cdot \left|\frac{\partial^2 D_1(r,t,\theta)}{\partial r \partial t}\right| + C\left(r^{-\mu_0}+\varepsilon\right), \\
&\left|\frac{\partial^2 D_1(r,t,\theta)}{\partial r \partial t}\right| \\
&\quad \le \left|\frac{\partial^2 D_2(r,t,\theta)}{\partial r \partial t}\right| + C\left(r^{-\mu_0}+\varepsilon\right) \cdot \left|\frac{\partial^2 D_2(r,t,\theta)}{\partial r \partial t}\right| \\
&\qquad + C\left(r^{-\mu_0}+\varepsilon\right) \cdot \left|\frac{\partial^2 D_1(r,t,\theta)}{\partial r \partial t}\right| + C\left(r^{-\mu_0}+\varepsilon\right).
\end{aligned} \tag{111}
$$

Hence,

$$\left|\frac{\partial^2 D_1(r,t,\theta)}{\partial r\partial t}\right| \le C\left(r^{-\mu_0}+\varepsilon\right),$$
$$\left|\frac{\partial^2 D_2(r,t,\theta)}{\partial r\partial t}\right| \le C\left(r^{-\mu_0}+\varepsilon\right). \quad (112)$$

(5) We can prove (103) similarly to (4) for the left $k+m\le 4$. □

Proof of Boundedness. From Theorem 1.1 in [21] we can see Φ^1 possesses a sequence of invariant circles tending to infinity. So, in the original system (13), there exists a corresponding sequence of invariant tori in phase space $(x,\dot{x},t)\in\mathbb{R}^2\times\mathbb{T}$. Then any solution of system (13) is bounded because it must stay within one of those tori. □

4. The Proof of Unboundedness

In this section, we will prove that all solutions of (8) are unbounded if $B<0$. In this case, $A<0$.

Consider (8) which is equivalent to the following system:

$$\dot{x}=y,$$
$$\dot{y}=-A|x|^{\alpha-1}x-f(x)y-\varepsilon\hat{e}(t)|x|^{\alpha-1}x+p(t). \quad (113)$$

Replacing (18) by an "auxiliary" system

$$\dot{x}=y,$$
$$\dot{y}=A|x|^{\alpha-1}x. \quad (114)$$

Under the transformation (21), the system (113) is transformed into the form

$$\frac{d\rho}{dt}=-\frac{1}{2-2b}\rho^{2(1-b)}\hat{g}'(\varphi)+\hat{h}_1(\rho,\varphi,t),$$
$$\frac{d\varphi}{dt}=\rho^{1-2b}\hat{g}(\varphi)+\hat{g}_1(\rho,\varphi,t), \quad (115)$$

where

$$\hat{g}(\varphi)=(1-b)d+2bdA\left|S(\varphi T_0)\right|^{\alpha+1},$$
$$\hat{h}_1(\rho,\varphi,t)=-T_0d\rho f\left(\rho^bS(\varphi T_0)\right)S'(\varphi T_0)C(\varphi T_0)$$
$$-T_0d\varepsilon\rho^{2-2b}\left|S(\varphi T_0)\right|^{\alpha-1}S(\varphi T_0)S'(\varphi T_0)\hat{e}(t)$$
$$+T_0d\rho^bS'(\varphi T_0)p(t),$$
$$\hat{g}_1(\rho,\varphi,t)=bdf\left(\rho^bS(\varphi T_0)\right)S(\varphi T_0)C(\varphi T_0)$$
$$+bd\varepsilon\rho^{1-2b}\left|S(\varphi T_0)\right|^{\alpha+1}\hat{e}(t)$$
$$-bd\rho^{b-1}S(\varphi T_0)p(t). \quad (116)$$

Thus, the system (115) can be written in the form

$$\frac{d\rho}{dt}=-\frac{1}{2(1-b)}\hat{g}'(\varphi)\rho^{2(1-b)}+O\left(\varepsilon\rho^{2(1-b)}\right),$$
$$\frac{d\varphi}{dt}=\rho^{1-2b}\hat{g}(\varphi)+O\left(\varepsilon\rho^{1-2b}\right). \quad (117)$$

From the equality

$$\frac{1}{2}C^2(t)+\frac{-A}{\alpha+1}|S(t)|^{\alpha+1}=\frac{1}{2},\quad \forall t\in\mathbb{R}, \quad (118)$$

it follows that

$$0\le\left|S(\varphi T_0)\right|^{\alpha+1}\le-\frac{\alpha+1}{2A}. \quad (119)$$

Hence, the function $\hat{g}(\varphi)$ is C^1, 1-periodic and change the sign. Since $|S(T_0-\varphi T_0)|=|S(\varphi T_0)|$ for any $\varphi\in[0,1]$, there exists $\varphi_1\in(0,1/2)$ such that

$$\left|S(T_0-\varphi_1T_0)\right|^{\alpha+1}=\left|S(\varphi_1T_0)\right|^{\alpha+1}=-\frac{\alpha+1}{4A}. \quad (120)$$

That is, $\hat{g}(\varphi_1)=\hat{g}(1-\varphi_1)=0$. In view of

$$S(T_0-\varphi T_0)=-S(\varphi T_0),\qquad C(T_0-\varphi T_0)=C(\varphi T_0), \quad (121)$$

we find

$$\hat{g}'(\varphi_1)\cdot\hat{g}'(1-\varphi_1)$$
$$=-(\alpha+1)^2(2bdAT_0)^2\left|S(\varphi_1T_0)\right|^{2(\alpha-1)}S^2(\varphi_1T_0)C^2(\varphi_1T_0)$$
$$<0. \quad (122)$$

Hence, we obtain that $\hat{g}'(\varphi_1)$ or $\hat{g}'(1-\varphi_1)$ is negative. This proves that there exists a φ^* such that $\hat{g}(\varphi^*)=0$ and $\hat{g}'(\varphi^*)<0$. Therefore, there are $\upsilon>0$ and $\delta_0>0$ such that $\hat{g}'(\varphi)<-\delta_0$ for $\varphi\in[\varphi^*-\upsilon,\varphi^*+\upsilon]$ and $\hat{g}(\varphi)>0$ for $\varphi\in(\varphi^*-\upsilon,\varphi^*)$, $\hat{g}(\varphi)<0$ for $\varphi\in(\varphi^*,\varphi^*+\upsilon)$. Let

$$\mathcal{K}_{J,\upsilon}=\left\{(\rho,\varphi)\in\mathbb{R}^+\times\mathbb{T}:\rho>J,\varphi\in\left[\varphi^*-\upsilon,\varphi^*+\upsilon\right]\right\}. \quad (123)$$

Then, if J is sufficiently large, on the set $\mathcal{K}_{J,\upsilon}$, we have

$$-\frac{1}{2(1-b)}\hat{g}'(\varphi)\rho^{2(1-b)}+O\left(\varepsilon\rho^{2(1-b)}\right)>\frac{\delta_0}{2}\cdot\rho^{2(1-b)}, \quad (124)$$

$$\rho^{1-2b}\hat{g}(\varphi)+O\left(\varepsilon\rho^{1-2b}\right)>0,$$
$$\text{for } \rho\ge J,\ \varphi\in\left[\varphi^*-\upsilon,\varphi^*-\frac{\upsilon}{2}\right],$$
$$\rho^{1-2b}\hat{g}(\varphi)+O\left(\varepsilon\rho^{1-2b}\right)<0,$$
$$\text{for } \rho\ge J,\ \varphi\in\left[\varphi^*+\frac{\upsilon}{2},\varphi^*+\upsilon\right]. \quad (125)$$

From (117) and (124) we obtain, for $t \geq 0$,

$$\begin{aligned}\rho(t,\rho_0,\varphi_0) &= \rho_0 + \int_0^t \left(-\frac{1}{2(1-b)}\widehat{g}'(\varphi)\rho^{2(1-b)} + O\left(\varepsilon\rho^{2(1-b)}\right)\right)dt \\ &> \rho_0 + \int_0^t \frac{\delta_0}{2}\cdot\rho^{2(1-b)}dt \geq \rho_0 > J.\end{aligned} \tag{126}$$

Moreover, for $\rho(t,\rho_0,\varphi_0) > J$ and $\varphi(t,\rho_0,\varphi_0) \in [\varphi^* - \upsilon, \varphi^* - \upsilon/2] \cup [\varphi^* + \upsilon/2, \varphi^* + \upsilon]$, we have

$$\begin{aligned}\rho^{1-2b}\widehat{g}(\varphi) + O\left(\varepsilon\rho^{1-2b}\right) &= \rho^{1-2b}\widehat{g}'(\overline{\varphi})(\varphi - \varphi^*) + O\left(\varepsilon\rho^{1-2b}\right) \\ &< -\frac{\delta_0}{2}(\varphi - \varphi^*)\rho^{1-2b}.\end{aligned} \tag{127}$$

From (126) and (127), it follows that any solution $(\rho(t,\rho_0,\varphi_0),\varphi(t,\rho_0,\varphi_0))$ of (115) with the initial condition $(\rho(0,\rho_0,\varphi_0),\varphi(0,\rho_0,\varphi_0)) = (\rho_0,\varphi_0) \in \mathscr{K}_{J,\upsilon}$ always stays in $\mathscr{K}_{J,\upsilon}$ and satisfies $\rho(t,\rho_0,\varphi_0) > \delta t + \rho(0)$ with $\delta = \delta_0^{3-2b}/2$, for all $t \geq 0$. The proof of Theorem 3 is completed.

5. The Proof of Theorem 4

In this section, we will prove Theorem 4 by using the abstract result on the existence of quasi-periodic solutions proved in [24] in the context Aubry-Mather theory for reversible systems. We only need to show that the Poincaré map (92) has the monotone property; that is,

$$\frac{\partial t_1}{\partial r}(r,t) > 0. \tag{128}$$

We can get that

$$\left|\frac{\partial F_0(r,\theta)}{\partial r}\right| \leq Cr^{-1-(1+\gamma-3b/2)/(2b-1)} \tag{129}$$

by Lemma 11, and

$$\left|\frac{\partial Q_2(r,t)}{\partial r}\right| \leq r^{-\mu_0} + \varepsilon \tag{130}$$

by Lemma 12. Then we have

$$\frac{\partial t_1}{\partial r}(r,t) = 1 + \int_0^1 \frac{\partial F_0}{\partial r}d\theta + \frac{\partial Q_2}{\partial r} \longrightarrow c_0, \quad \text{as } r \longrightarrow +\infty, \tag{131}$$

where $c_0 \geq 1 - \varepsilon$. Therefore, we have

$$\frac{\partial t_1}{\partial r}(r,t) > 0 \tag{132}$$

as $r \gg 1$ and $\varepsilon \ll 1$. This proves the validity of (128).

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