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Research Article

Existence Theory for *n*th Order Nonlocal Integral Boundary Value Problems and Extension to Fractional Case

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This paper is devoted to the study of the existence and uniqueness of solutions for *n*th order differential equations with nonlocal integral boundary conditions. Our results are based on a variety of fixed point theorems. Some illustrative examples are discussed. We also discuss the Caputo type fractional analogue of the higher-order problem of ordinary differential equations.

1. Introduction

Boundary value problems with nonclassical boundary conditions are often used to take into account some peculiarities of physical, chemical or other processes, which are impossible by applying classical boundary conditions. Nonlocal conditions appear when values of the function on the boundary are connected to values inside the domain. Integral nonlocal boundary conditions can be used when it is impossible to directly determine the values of the sought quantity on the boundary while the total amount or integral average on space domain is known.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint, and nonlocal boundary value problems as special cases. Integral boundary value problems occur in the mathematical modeling of a variety of physics processes and have recently received considerable attention. For some recent work on boundary value problems with integral boundary conditions we refer to [1–23] and the references cited therein.

In this paper, we discuss some existence and uniqueness results for boundary value problems of *n*th order ordinary differential equations. Precisely, in the first part of the paper we consider the following boundary value problem of nonlinear *n*th-order differential equations with multipoint integral boundary conditions

$$\begin{gathered} u^{(n)}(t) = f(t, u(t)), \quad t \in [0,1], \\ u(0) = 0, \qquad u'(0) = 0, \qquad u''(0) = 0, \dots, u^{(n-2)}(0) = 0, \\ \alpha u(1) + \beta u'(1) = \sum_{i=1}^{m} \gamma_i \int_0^{\eta_i} u(s)\, ds, \quad 0 < \eta_i < 1, \end{gathered} \tag{1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and α, β, γ_i, η_i, $(i = 1, 2, \dots, m)$ are real constants to be chosen appropriately. Existence and uniqueness results are proved by using a variety of fixed point theorems such as Schaefer's fixed point theorem, Leray-Schauder Nonlinear Alternative, Krasnoselskii's fixed point theorem, Banach's fixed point theorem, and Boyd and Wang fixed point theorem for nonlinear contractions [24]. The methods used are well known; however, their exposition in the framework of problem (1) is new.

Next, we extend our discussion to the fractional case by considering the problem consisting of the boundary conditions in (1) along with the Caputo type fractional differential equation as follows:

$$
\begin{gathered}
{}^{c}D^{q}x(t)=f(t,x(t)),\\
0<t<1,\quad n-1<q\le n,\quad n\ge 2,\quad n\in\mathbb{N}.
\end{gathered}\tag{2}
$$

Fractional calculus has emerged as an interesting mathematical modelling tool in many branches of basic sciences, engineering, and technical sciences [25–27]. Differential and integral operators of fractional order do share some of the characteristics exhibited by the processes associated with complex systems having long-memory in time. In other words, we can say that a dynamical system or process involving fractional derivatives takes into account its current as well as past states. This feature has contributed significantly to the popularity of the subject and has motivated many researchers to focus on fractional order models. For some recent development of the topic, for instance, see [13, 28–35].

The paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel. Section 3 contains the existence and uniqueness results for the boundary value problem (1). In Section 4, some illustrative examples are presented. In Section 5, we consider the Caputo type fractional analogue of problem (1).

2. An Auxiliary Lemma

Lemma 1. *Let $\alpha+(n-1)\beta\neq(1/n)\sum_{(i=1)}^{m}\gamma_i\eta_i^n$. For any $y\in C([0,1],\mathbb{R})$, the unique solution of the boundary value problem*

$$
\begin{gathered}
u^{(n)}(t)=y(t),\quad t\in[0,1],\\
u(0)=0,\qquad u'(0)=0,\qquad u''(0)=0,\dots,u^{(n-2)}(0)=0,\\
\alpha u(1)+\beta u'(1)=\sum_{i=1}^{m}\gamma_i\int_0^{\eta_i}u(s)\,ds,\quad 0<\eta_i<1,
\end{gathered}\tag{3}
$$

is given by

$$
\begin{aligned}
u(t)=\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}y(s)\,ds+\Lambda t^{n-1}
&\times\left\{\sum_{i=1}^{m}\gamma_i\int_0^{\eta_i}\frac{(\eta_i-s)^n}{n!}y(s)\,ds\right.\\
&-\alpha\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}y(s)\,ds\\
&\left.-\beta\int_0^1\frac{(1-s)^{n-2}}{(n-2)!}y(s)\,ds\right\},
\end{aligned}\tag{4}
$$

where

$$
\Lambda=\frac{1}{\alpha+(n-1)\beta-(1/n)\sum_{i=1}^{m}\gamma_i\eta_i^n}.\tag{5}
$$

Proof. It is well known that the solution of the differential equation in (3) can be written as

$$
\begin{aligned}
u(t)=\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}y(s)\,ds+c_0&\\
+c_1t+c_2t^2+\cdots+c_{n-2}t^{n-2}+c_{n-1}t^{n-1}&,
\end{aligned}\tag{6}
$$

where c_i, $i=0,1,\dots,n-1$, are arbitrary real constants. Using the boundary conditions $u(0)=u'(0)=u''(0)=\cdots=u^{(n-2)}(0)=0$ in (6), we get $c_0=c_1=c_2=\cdots=c_{n-2}=0$ and applying the boundary condition $\alpha u(1)+\beta u'(1)=\sum_{i=1}^{m}\gamma_i\int_0^{\eta_i}u(s)ds$, we find that

$$
\begin{aligned}
c_{n-1}=\Lambda\left(\sum_{i=1}^{m}\gamma_i\int_0^{\eta_i}\frac{(\eta_i-s)^n}{n!}y(s)\,ds-\alpha\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}y(s)\,ds\right.&\\
\left.-\beta\int_0^1\frac{(1-s)^{n-2}}{(n-2)!}y(s)\,ds\right)&,
\end{aligned}\tag{7}
$$

where Λ is defined by (5).

Substituting the values of $c_0,c_1,c_2,\dots,c_{n-2}$ and c_{n-1} in (6), we get (4). □

3. Some Existence and Uniqueness Results

Let $\mathscr{C}=C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]\to\mathbb{R}$ endowed with the norm defined by $\|u\|=\sup\{|u(t)|,t\in[0,1]\}$. Let $L^1([0,1],\mathbb{R})$ be the Banach space of measurable functions $x:[0,1]\to\mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1}=\int_0^1|x(t)|dt$.

In view of Lemma 1, we define an operator $\mathscr{F}:\mathscr{C}\to\mathscr{C}$ by

$$
\begin{aligned}
(\mathscr{F}u)(t)=\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}f(s,u(s))\,ds&\\
+\Lambda t^{n-1}\left\{\sum_{i=1}^{m}\gamma_i\int_0^{\eta_i}\frac{(\eta_i-s)^n}{n!}f(s,u(s))\,ds\right.&\\
-\alpha\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}f(s,u(s))\,ds&\\
\left.-\beta\int_0^1\frac{(1-s)^{n-2}}{(n-2)!}f(s,u(s))\,ds\right\}&,
\end{aligned}\tag{8}
$$

where Λ is given by (5). Observe that the problem (1) has solutions only if the operator equation $\mathscr{F}u=u$ has fixed points.

Now we are in a position to present several existence results for the problem (1). Our first result is based on Schaefer's fixed point theorem.

Lemma 2 (see [36]). *Let X be a Banach space. Assume that $T:X\to X$ is a completely continuous operator and the set $V=\{u\in X\mid u=\mu Tu,0<\mu<1\}$ is bounded. Then, T has a fixed point in X.*

Theorem 3. *Let $f : [0,1] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Assume that there exists a constant $L_1 > 0$ such that $|f(t,u(t))| \leq L_1$ for $t \in [0,1]$, $u \in \mathscr{C}$. Then, the boundary value problem* (1) *has at least one solution.*

Proof. First we show that the operator $\mathscr{F}$ defined by (8) is completely continuous. Clearly, continuity of the operator $\mathscr{F}$ follows from the continuity of f. Then, it follows by the assumption $|f(t,u(t)| \leq L_1$ that

$$\begin{aligned}
&|(\mathscr{F}u)(t)| \\
&\quad \leq \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}|f(s,u(s))|\,ds \\
&\qquad + \left|\Lambda t^{n-1}\right| \left\{ \sum_{i=1}^m \gamma_i \int_0^{\eta_i} \frac{(\eta_i - s)^n}{n!}|f(s,u(s))|\,ds \right. \\
&\qquad\qquad + |\alpha| \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}|f(s,u(s))|\,ds \\
&\qquad\qquad \left. + |\beta| \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!}|f(s,u(s))|\,ds \right\} \\
&\quad \leq L_1 \left\{ \frac{t^n}{n!} + \left|\Lambda t^{n-1}\right| \left(\frac{\sum_{i=1}^m |\gamma_i| \eta_i^{n+1}}{(n+1)!} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!} \right) \right\} \\
&\quad \leq L_1 \left\{ \frac{1}{n!} + |\Lambda| \left(\frac{\sum_{i=1}^m |\gamma_i| \eta_i^{n+1}}{(n+1)!} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!} \right) \right\} \\
&\quad := L_2,
\end{aligned} \tag{9}$$

which implies that $\|\mathscr{F}u\| \leq L_2$. Furthermore,

$$\begin{aligned}
\left|(\mathscr{F}u)'(t)\right| &\leq \int_0^t \frac{(t-s)^{n-2}}{(n-2)!}|f(s,u(s))|\,ds \\
&\quad + \left|(n-1)\Lambda t^{n-2}\right| \\
&\quad \times \left\{ \sum_{i=1}^m \gamma_i \int_0^{\eta_i} \frac{(\eta_i - s)^n}{n!}|f(s,u(s))|\,ds \right. \\
&\qquad + |\alpha| \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}|f(s,u(s))|\,ds \\
&\qquad \left. + |\beta| \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!}|f(s,u(s))|\,ds \right\} \\
&\leq L_1 \left\{ \frac{t^{n-1}}{(n-1)!} + (n-1)\left|\Lambda t^{n-2}\right| \right. \\
&\qquad \left. \times \left(\frac{\sum_{i=1}^m |\gamma_i| \eta_i^{(n+1)}}{(n+1)} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!} \right) \right\} \\
&\leq L_1 \left\{ \frac{1}{(n-1)!} + (n-1)|\Lambda| \right. \\
&\qquad \left. \times \left(\frac{\sum_{i=1}^m |\gamma_i| \eta_i^{n+1}}{(n+1)!} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!} \right) \right\} \\
&:= L_3.
\end{aligned} \tag{10}$$

Hence, for $t_1, t_2 \in [0,1]$, we have

$$\left|(\mathscr{F}u)(t_1) - (\mathscr{F}u)(t_2)\right| \leq \int_{t_2}^{t_1} \left|(\mathscr{F}u)'(s)\right| ds \leq L_3(t_1 - t_2). \tag{11}$$

Thus, by the foregoing arguments, one can infer that the operator $\mathscr{F}$ is equicontinuous on $[0,1]$. Hence, by the Arzelá-Ascoli theorem, the operator $\mathscr{F} : \mathscr{C} \rightarrow \mathscr{C}$ is completely continuous.

Next, we consider the set

$$V = \{u \in \mathscr{C} \mid u = \mu \mathscr{F} u,\ 0 < \mu < 1\}, \tag{12}$$

and show that the set V is bounded. Let $u \in V$, then, $u = \mu \mathscr{F} u$, $0 < \mu < 1$. For any $t \in [0,1]$, we have

$$\begin{aligned}
&|u(t)| = \mu |(\mathscr{F}u)(t)| \\
&\quad \leq \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}|f(s,u(s))|\,ds \\
&\qquad + |\Lambda| t^{n-1} \left\{ \sum_{i=1}^m \gamma_i \int_0^{\eta_i} \frac{(\eta_i - s)^n}{n!}|f(s,u(s))|\,ds \right. \\
&\qquad\qquad + |\alpha| \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}|f(s,u(s))|\,ds \\
&\qquad\qquad \left. + |\beta| \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!}|f(s,u(s))|\,ds \right\} \\
&\quad \leq L_1 \max_{t\in[0,1]} \left\{ \frac{t^n}{n!} + |\Lambda| t^{n-1} \left(\frac{\sum_{i=1}^m |\gamma_i| \eta_i^{n+1}}{(n+1)!} \right.\right. \\
&\qquad\qquad \left.\left. + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!} \right) \right\} \\
&\quad := M_1.
\end{aligned} \tag{13}$$

Thus, $\|u\| \leq M_1$ for any $t \in [0,1]$. So, the set V is bounded. Thus, by the conclusion of Lemma 2, the operator $\mathscr{F}$ has at least one fixed point, which implies that the boundary value problem (1) has at least one solution. □

Our next existence result is based on Leray-Schauder Nonlinear Alternative [37].

Lemma 4 (nonlinear alternative for single valued maps). *Let E be a Banach space, C a closed convex subset of E, V an open* *subset of* C*, and* $0 \in V$*. Suppose that* $F : \overline{V} \to C$ *is a continuous, compact (that is,* $F(\overline{V})$ *is a relatively compact subset of* C*) map. Then, either*

(i) F *has a fixed point in* $\overline{V}$*, or*

(ii) *there is a* $u \in \partial V$ *(the boundary of* v *in* C*) and* $\lambda \in (0,1)$ *with* $v = \lambda F(v)$*.*

Theorem 5. *Let* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function. Assume that*

(B_1) *there exist a function* $p \in L([0,1],\mathbb{R}^+)$*, and a nondecreasing function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $|f(t,u)| \le p(t)\psi(\|u\|)$, $\forall (t,u) \in [0,1] \times \mathbb{R}$*;*

(B_2) *there exists a constant* $M > 0$ *such that*

$$M\left(\psi(M)\left\{\frac{1}{n!}+|\Lambda| \times\left(\frac{\sum_{i=1}^{m}|\gamma_i|\eta_i^{n+1}}{(n+1)!}+\frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!}\right)\right\}\|p\|_{L^1}\right)^{-1} > 1. \tag{14}$$

Then, the boundary value problem (1) *has at least one solution on* $[0,1]$.

Proof. Consider the operator $\mathscr{F} : \mathscr{C} \to \mathscr{C}$ defined by (8). We show that $\mathscr{F}$ *maps bounded sets into bounded sets in* $C([0,1],\mathbb{R})$. For a positive number r, let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \le r\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then,

$$\begin{aligned}
&|(\mathscr{F}u)(t)| \\
&\le \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}|f(s,u(s))|\,ds \\
&\quad + |\Lambda| t^{n-1}\left\{\sum_{i=1}^{m}|\gamma_i|\int_0^{\eta_i}\frac{(\eta_i-s)^n}{n!}|f(s,u(s))|\,ds + |\alpha|\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}|f(s,u(s))|\,ds + |\beta|\int_0^1\frac{(1-s)^{n-2}}{(n-2)!}|f(s,u(s))|\,ds\right\} \\
&\le \psi(\|u\|)\left\{\frac{1}{n!}+|\Lambda|\times\left(\frac{\sum_{i=1}^{m}|\gamma_i|\eta_i^{n+1}}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\right)\right\} \times \|p\|_{L^1}.
\end{aligned} \tag{15}$$

Thus,

$$\|\mathscr{F}u\| \le \psi(r)\left\{\frac{1}{n!}+|\Lambda|\left(\frac{\sum_{i=1}^{m}|\gamma_i|\eta_i^{n+1}}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\right)\right\} \times \|p\|_{L^1}. \tag{16}$$

Next, we show that F *maps bounded sets into equicontinuous sets of* $C([0,1],\mathbb{R})$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $u \in B_r$, where B_r is a bounded set of $C([0,1],\mathbb{R})$. Then, we obtain

$$\begin{aligned}
&|(\mathscr{F}_1u)(t_2) - (\mathscr{F}_1u)(t_1)| \\
&\le \frac{1}{(n-1)!} \times \left|\int_0^{t_1}\left[(t_2-s)^{n-1}-(t_1-s)^{n-1}\right]f(s,u(s))\,ds + \int_{t_1}^{t_2}(t_2-s)^{n-1}f(s,u(s))\,ds\right| \\
&\quad + |\Lambda|\left|t_2^{n-1}-t_1^{n-1}\right| \times\left(\sum_{i=1}^{m}\gamma_i\int_0^{\eta_i}\frac{(\eta_i-s)^{n-1}}{n!}|f(s,u(s))|\,ds + |\alpha|\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}|f(s,u(s))|\,ds + |\beta|\int_0^1\frac{(1-s)^{n-2}}{(n-2)!}|f(s,u(s))|\,ds\right) \\
&\le \frac{\psi(r)}{n!}\left(\left|2(t_2-t_1)^n\right|+\left|t_1^n-t_2^n\right|\right) + \psi(r)|\Lambda|\left|t_2^{n-1}-t_1^{n-1}\right| \times\left(\frac{\sum_{i=1}^{m}|\gamma_i|\eta_i^n}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\right)\|p\|_{L^1}.
\end{aligned} \tag{17}$$

Obviously, the right-hand side of the above inequality tends to zero independently of $u \in B_r$ as $t_2 - t_1 \to 0$. As $\mathscr{F}$ satisfies the above assumptions; therefore, it follows by the Arzelá-Ascoli theorem that $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

Let u be a solution. Then, for $t \in [0,1]$, and using the computations in proving that $\mathscr{F}$ is bounded, we have

$$\begin{aligned}
|u(t)| &= |\lambda(\mathscr{F}u)(t)| \\
&\le \psi(\|u\|)\left\{\frac{1}{n!}+|\Lambda| \times\left(\frac{\sum_{i=1}^{m}|\gamma_i|\eta_i^{n+1}}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\right)\right\} \times \|p\|_{L^1}.
\end{aligned} \tag{18}$$

In consequence, we have

$$\|u\|\left(\psi(\|u\|)\times\left\{\frac{1}{n!}+|\Lambda|\left(\frac{\sum_{i=1}^{m}|\gamma_i|\eta_i^{n+1}}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\right)\right\}\|p\|_{L^1}\right)^{-1}\leq 1. \tag{19}$$

In view of (B_2), there exists M such that $\|u\|\neq M$. Let us set

$$W=\{u\in C([0,1],\mathbb{R}):\|u\|<M\}. \tag{20}$$

Note that the operator $F:\overline{W}\to C([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of W, there is no $u\in\partial W$ such that $u=\lambda F(u)$ for some $\lambda\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder-type (Lemma 4), we deduce that F has a fixed point $u\in\overline{W}$ which is a solution of the problem (1). This completes the proof. □

To prove the next existence result, we need the following fixed point theorem.

Lemma 6 (see [36]). *Let X be a Banach space. Assume that Ω is an open bounded subset of X with $0\in\Omega$ and let $T:\overline{\Omega}\to X$ be a completely continuous operator, such that*

$$\|Tu\|\leq\|u\|,\quad\forall u\in\partial\Omega. \tag{21}$$

Then, T has a fixed point in $\overline{\Omega}$.

Theorem 7. *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be continuous and there exists $\delta,r>0$ with $|f(t,u)|\leq\delta|u|$, $0<|u|<r$ and*

$$\left\{\frac{1}{n!}+|\Lambda|\left(\frac{\sum_{i=1}^{m}|\gamma_i|\eta_i^{n+1}}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\right)\right\}\delta<1. \tag{22}$$

Then, the boundary value problem (1) *has at least one solution.*

Proof. Define $\Omega=\{u\in\mathscr{C}\mid\|u\|<r\}$ and take $u\in\mathscr{C}$ such that $\|u\|=r$; that is, $u\in\partial\Omega$. As before, it can be shown that $\mathscr{F}$ is completely continuous and

$$\begin{aligned}\|\mathscr{F}u\|&\leq\max_{t\in[0,1]}\left\{\frac{t^n}{n!}+|\Lambda|t^{n-1}\times\left(\frac{\sum_{i=1}^{m}|\gamma_i|\eta_i^{n+1}}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\right)\right\}\delta\|u\| \\ &=\left\{\frac{1}{n!}+|\Lambda|\left(\frac{\sum_{i=1}^{m}|\gamma_i|\eta_i^{n+1}}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\right)\right\}\delta\|u\|,\end{aligned} \tag{23}$$

which, in view of (22), implies that $\|\mathscr{F}u\|\leq\|u\|$, $u\in\partial\Omega$. Therefore, by Lemma 6, the operator $\mathscr{F}$ has at least one fixed point, which corresponds to at least one solution of the boundary value problem (1). □

Our next existence result is based on Krasnoselskii's fixed point theorem [38].

Theorem 8 (Krasnoselskii's fixed point theorem). *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A and B be the operators such that (i) $Au+Bv\in M$ whenever $u,v\in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then, there exists $z\in M$ such that $z=Az+Bz$.*

Theorem 9. *Suppose that $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function and satisfies the following assumptions:*

(A_1) $|f(t,u)-f(t,v)|\leq L\|u-v\|$, $\forall t\in[0,1]$, $L>0$, $u,v\in\mathbb{R}$.

(A_2) $|f(t,u)|\leq\mu(t)$, $\forall(t,u)\in[0,1]\times\mathbb{R}$, *and* $\mu\in C([0,1],\mathbb{R}^+)$.

Then, the boundary value problem (1) *has at least one solution on* $[0,1]$ *if*

$$L\left\{|\Lambda|\left(\frac{\sum_{i=1}^{m}|\gamma_i|\eta_i^{n+1}}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\right)\right\}<1. \tag{24}$$

Proof. Letting $\sup_{t\in[0,1]}|\mu(t)|=\|\mu\|$, we choose a real number $\overline{r}$ satisfying the inequality

$$\overline{r}\geq\|\mu\|\left\{\frac{1}{n!}+|\Lambda|\left(\frac{\sum_{i=1}^{m}|\gamma_i|\eta_i^{n+1}}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\right)\right\}, \tag{25}$$

and consider $B_{\overline{r}}=\{u\in\mathscr{C}:\|u\|\leq\overline{r}\}$. We define the operators χ and φ on $B_{\overline{r}}$ as

$$\begin{aligned}\chi(u)(t)&=\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}f(s,u(s))\,ds,\\ \varphi(u)(t)&=\Lambda t^{n-1}\times\left\{\sum_{i=1}^{m}\gamma_i\int_0^{\eta_i}\frac{(\eta_i-s)^n}{n!}f(s,u(s))\,ds-\alpha\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}f(s,u(s))\,ds-\beta\int_0^1\frac{(1-s)^{n-2}}{(n-2)!}f(s,u(s))\right\}ds.\end{aligned} \tag{26}$$

For $u,v\in B_{\overline{r}}$, we find that

$$\begin{aligned}\|\chi u+\varphi u\|&\leq\|\mu\|\left\{\frac{1}{n!}+|\Lambda|\left(\frac{\sum_{i=1}^{m}|\gamma_i|\eta_i^{n+1}}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\right)\right\}\\ &\leq\overline{r}.\end{aligned} \tag{27}$$

Thus, $\chi u + \varphi u \in B_r$. In view of (A_1) and (24), φ is a contraction mapping. Continuity of f implies that the operator χ is continuous. Also, χ is uniformly bounded on B_r as

$$\|\chi u\| \le \frac{\|\mu\|}{n!}. \tag{28}$$

Now, we prove the compactness of the operator χ. In view of (A_1), we define

$$\sup_{(t,u)\in[0,1]\times B_r} |f(t,u)| = \overline{f}, \tag{29}$$

and consequently, for $t_1, t_2 \in [0,1]$, $t_1 < t_2$, we have

$$\begin{aligned}
&|(\chi u)(t_1) - (\chi u)(t_2)| \\
&\quad = \left| \frac{1}{(n-1)!} \int_0^{t_1} \left[(t_2 - s)^{n-1} - (t_1 - s)^{n-1} \right] f(s, u(s))\, ds \right. \\
&\qquad \left. + \frac{1}{(n-1)!} \int_{t_1}^{t_2} (t_2 - s)^{n-1} f(s, u(s))\, ds \right| \\
&\quad \le \frac{\overline{f}}{n!} \left(\left| 2(t_2 - t_1)^n \right| + \left| t_1^n - t_2^n \right| \right),
\end{aligned} \tag{30}$$

which is independent of u. Thus, χ is relatively compact on B_r. Hence, by the Arzelá-Ascoli theorem, χ is compact on B_r. Thus, all the assumptions of Theorem 8 are satisfied. So, the conclusion of Theorem 8 implies that the boundary value problem (1) has at least one solution on $[0,1]$. □

Next, we discuss the uniqueness of solutions for the problem (1). This result relies on Banach's fixed point theorem.

Theorem 10. *Assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the condition (A_1).*

If

$$L \left\{ \frac{1}{n!} + |\Lambda| \left(\frac{\sum_{i=1}^m |\gamma_i| \eta_i^{n+1}}{(n+1)!} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!} \right) \right\} < 1, \tag{31}$$

then, the boundary value problem (1) *has a unique solution.*

Proof. Fixing

$$\sup_{t\in[0,1]} |f(t,0)| = M < \infty, \tag{32}$$

and selecting

$$\begin{aligned}
R &\ge \frac{MQ}{1 - LQ}, \\
Q &= \frac{1}{n!} + |\Lambda| \left(\frac{\sum_{i=1}^m |\gamma_i| \eta_i^{n+1}}{(n+1)!} \right. \\
&\qquad \left. + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!} \right),
\end{aligned} \tag{33}$$

we show that $\mathcal{F} B_R \subset B_r$, where $B_R = \{u \in \mathcal{C} : \|u\| \le R\}$. For $u \in B_R$, we have for $t \in [0,1]$,

$$\begin{aligned}
&|(\mathcal{F}u)(t)| \\
&\le \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} |f(s,u(s))|\, ds + |\Lambda| t^{n-1} \right. \\
&\qquad \times \left(\sum_{i=1}^m |\gamma_i| \int_0^{\eta_i} \frac{(\eta_i - s)^n}{n!} |f(s,u(s))|\, ds \right. \\
&\qquad + |\alpha| \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} |f(s,u(s))|\, ds \\
&\qquad \left.\left. + |\beta| \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} |f(s,u(s))|\, ds \right) \right\} \\
&\le \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} \left(|f(s,u(s)) \right. \right. \\
&\qquad \left. - f(s,0)| + |f(s,0)| \right) ds \\
&\qquad + |\Lambda| t^{n-1} \\
&\qquad \times \left(\sum_{i=1}^m |\gamma_i| \int_0^{\eta_i} \frac{(\eta_i - s)^n}{n!} \right. \\
&\qquad \times \left(|f(s,u(s)) - f(s,0)| \right. \\
&\qquad \left. + |f(s,0)| \right) ds \\
&\qquad + |\alpha| \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} \left(|f(s,u(s)) \right. \\
&\qquad - f(s,0)| \\
&\qquad \left. + |f(s,0)| \right) ds \\
&\qquad + |\beta| \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} \\
&\qquad \times \left(|f(s,u(s)) - f(s,0)| \right. \\
&\qquad \left.\left.\left. + |f(s,0)| \right) ds \right) \right\} \\
&\le (Lr + M) \\
&\qquad \times \sup_{t\in[0,1]} \left\{ \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} ds + |\Lambda| t^{n-1} \right. \\
&\qquad \times \left(\sum_{i=1}^m |\gamma_i| \int_0^{\eta_i} \frac{(\eta_i - s)^n}{n!} ds \right. \\
&\qquad + |\alpha| \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} ds \\
&\qquad \left.\left. + |\beta| \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} ds \right) \right\}
\end{aligned}$$

$$\leq (Lr+M)\left\{\frac{1}{n!}+|\Lambda|\left(\frac{\sum_{i=1}^{m}|\gamma_i|\eta_i^{n+1}}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\right)\right\}$$
$$\leq R. \tag{34}$$

Thus, we get $\mathscr{F}u \in B_R$. Now, for $u, v \in \mathscr{C}$ and for each $t \in [0,1]$, we obtain

$$\begin{aligned}
&|(\mathscr{F}u)(t)-(\mathscr{F}v)(t)| \\
&\leq \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\left|f(s,u(s))-f(s,v(s))\right|ds\right. \\
&\quad + |\Lambda| t^{n-1} \\
&\quad \times\left(\sum_{i=1}^{m}|\gamma_i|\int_0^{\eta_i}\frac{(\eta_i-s)^n}{n!}\left|f(s,u(s))-f(s,v(s))\right|ds\right. \\
&\quad + |\alpha|\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}\left|f(s,u(s))-f(s,v(s))\right|ds \\
&\quad \left.\left.+ |\beta|\int_0^1\frac{(1-s)^{n-2}}{(n-2)!}\left|f(s,u(s))-f(s,v(s))\right|ds\right)\right\} \\
&\leq L\left\{\frac{1}{n!}+|\Lambda|\left(\frac{\sum_{i=1}^{m}\gamma_i\eta_i^{n+1}}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\right)\right\} \\
&\quad \times \|u-v\|.
\end{aligned} \tag{35}$$

Since

$$L\left\{\frac{1}{n!}+|\Lambda|\left(\frac{\sum_{i=1}^{m}|\gamma_i|\eta_i^{n+1}}{(n+1)!}+\frac{|\alpha|}{n!}+\frac{|\beta|}{(n-1)!}\right)\right\}<1, \tag{36}$$

$\mathscr{F}$ is a contraction; therefore, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

We give another uniqueness result for the problem (1) by using Banach's fixed point theorem and Hölder's inequality. In the following, we denote by $L_{1/p}([0,1],\mathbb{R}^+)$ the space of $1/p$-Lebesgue measurable functions from $[0,1]$ to $\mathbb{R}^+$ with norm $\|\mu\|_p = (\int_0^1 |\mu(s)|^{1/p}ds)^p$.

Theorem 11. *Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following Lipschitz condition:*

(A_3) $|f(t,x)-f(t,y)| \leq m(t)|x-y|$, *for all* $(t,x),(t,y) \in [0,1]\times\mathbb{R}$, *where* $m \in L_{1/\gamma}([0,1],\mathbb{R}^+)$, $\gamma \in (0,1)$.

Then, the boundary value problem (1) *has a unique solution, provided that*

$$\begin{aligned}
\|m\|_p &\left\{\frac{1}{(n-1)!}\left(\frac{1-\gamma}{n-\gamma}\right)^{1-\gamma}\right. \\
&+|\Lambda|\left(\frac{1}{n!}\sum_{i=1}^{m}|\gamma_i|\eta_i^{n+1-\gamma}\left(\frac{1-\gamma}{n+1-\gamma}\right)^{1-\gamma}\right. \\
&+\frac{|\alpha|}{(n-1)!}\left(\frac{1-\gamma}{n-\gamma}\right)^{1-\gamma} \\
&\left.\left.+\frac{|\beta|}{(n-2)!}\left(\frac{1-\gamma}{n-1-\gamma}\right)^{1-\gamma}\right)\right\}<1.
\end{aligned} \tag{37}$$

Proof. For $x, y \in \mathscr{C}$ and for each $t \in [0,1]$ together with Hölder's inequality, we have

$$\begin{aligned}
&|(\mathscr{F}u)(t)-(\mathscr{F}v)(t)| \\
&\leq \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\left|f(s,u(s))-f(s,v(s))\right|ds \\
&\quad + |\Lambda| t^{n-1} \\
&\quad \times\left(\sum_{i=1}^{m}|\gamma_i|\int_0^{\eta_i}\frac{(\eta_i-s)^n}{n!}\left|f(s,u(s))-f(s,v(s))\right|ds\right. \\
&\quad + |\alpha|\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}\left|f(s,u(s))-f(s,v(s))\right|ds \\
&\quad \left.+ |\beta|\int_0^1\frac{(1-s)^{n-2}}{(n-2)!}\left|f(s,u(s))-f(s,v(s))\right|ds\right) \\
&\leq \left[\frac{1}{(n-1)!}\left(\int_0^1 (1-s)^{(n-1)/(1-\gamma)}ds\right)^{1-\gamma}\right. \\
&\quad \times\left(\int_0^1 (m(s))^{1/\gamma}ds\right)^{\gamma} \\
&\quad + |\Lambda| t^{n-1}\left(\sum_{i=1}^{m}|\gamma_i|\frac{1}{n!}\right. \\
&\quad \times\left(\int_0^{\eta_i}(\eta_i-s)^{n/(1-\gamma)}ds\right)^{1-\gamma} \\
&\quad \times\left(\int_0^1 (m(s))^{1/\gamma}ds\right)^{\gamma}
\end{aligned}$$

$$
\begin{aligned}
&+|\alpha| \frac{1}{(n-1)!} \\
&\times\left(\int_0^1(1-s)^{(n-1)/(1-\gamma)} ds\right)^{1-\gamma} \\
&\times\left(\int_0^1(m(s))^{1/\gamma} ds\right)^{\gamma} ds \\
&+|\beta| \frac{1}{(n-2)!} \\
&\times\left(\int_0^1(1-s)^{(n-2)/(1-\gamma)} ds\right)^{1-\gamma} \\
&\left.\left.\times\left(\left(\int_0^1(m(s))^{1/\gamma} ds\right)^{\gamma}\right)\right]\|u-v\| \\
\leq\|m\|_p &\left\{\frac{1}{(n-1)!}\left(\frac{1-\gamma}{n-\gamma}\right)^{1-\gamma}\right. \\
&+|\Lambda|\left(\frac{1}{n!} \sum_{i=1}^m\left|\gamma_i\right| \eta_i^{n+1-\gamma}\left(\frac{1-\gamma}{n+1-\gamma}\right)^{1-\gamma}\right. \\
&+\frac{|\alpha|}{(n-1)!}\left(\frac{1-\gamma}{n-\gamma}\right)^{1-\gamma} \\
&\left.\left.+\frac{|\beta|}{(n-2)!}\left(\frac{1-\gamma}{n-1-\gamma}\right)^{1-\gamma}\right)\right\} \\
&\times\|u-v\| .
\end{aligned} \tag{38}
$$

By the given condition (37), it follows that $\mathcal{F}$ is a contraction mapping. Hence, the Banach fixed point theorem applies and $\mathcal{F}$ has a fixed point which is the unique solution of the problem (1). This completes the proof. □

Finally, we discuss the uniqueness of solutions for the problem (1) by using a fixed point theorem for nonlinear contractions due to Boyd and Wong.

Definition 12. Let E be a Banach space and let $G : E \to E$ be a mapping. G is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\xi) < \xi$ for all $\xi > 0$ with the following property:

$$\|Gx - Gy\| \leq \Psi(\|x-y\|), \quad \forall x, y \in E. \tag{39}$$

Lemma 13 (see Boyd and Wong [24]). *Let E be a Banach space and let $G : E \to E$ be a nonlinear contraction. Then, G has a unique fixed point in E.*

Theorem 14. *Assume that*

(A_4) $|f(t,x) - f(t,y)| \leq h(t)(|x-y|/(H^* + |x-y|))$, $t \in [0,1]$, $x, y \geq 0$, *where* $h : [0,1] \to \mathbb{R}^+$ *is continuous, where*

$$H^* = \|h\|_{L^1}\left\{\frac{1}{n!} + |\Lambda|\left(\frac{\sum_{i=1}^m |\gamma_i| \eta_i^{n+1}}{(n+1)!} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!}\right)\right\}. \tag{40}$$

Then, the boundary value problem (1) *has a unique solution.*

Proof. We consider the operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ defined by (8).

Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be the continuous nondecreasing function satisfying $\Psi(0) = 0$ and $\Psi(\xi) < \xi$ for all $\xi > 0$ which is defined by

$$\Psi(\xi) = \frac{H^*\xi}{H^* + \xi}, \quad \forall \xi \geq 0. \tag{41}$$

For $x, y \in \mathcal{C}$, $s \in [0,1]$, we have

$$|f(s, x(s)) - f(s, y(s))| \leq \frac{\|h\|_{L^1}}{H^*}\Psi(\|x-y\|), \tag{42}$$

and so

$$
\begin{aligned}
|\mathcal{F}x(t) - \mathcal{F}y(t)| \leq & \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} h(s) \frac{|x(s)-y(s)|}{H^* + |x(s)-y(s)|} ds \\
&+|\Lambda|\left\{\sum_{i=1}^m |\gamma_i|\right. \\
&\times \int_0^{\eta_i} \frac{(\eta_i - s)^n}{n!} h(s) \\
&\times \frac{|x(s)-y(s)|}{H^* + |x(s)-y(s)|} ds \\
&+|\alpha| \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} h(s) \\
&\times \frac{|x(s)-y(s)|}{H^* + |x(s)-y(s)|} ds \\
&+|\beta| \int_0^1 \frac{(1-s)^{n-2}}{(n-2)!} h(s) \\
&\left.\times \frac{|x(s)-y(s)|}{H^* + |x(s)-y(s)|} ds\right\} \\
\leq & \frac{H^*\|x-y\|}{H^* + \|x-y\|},
\end{aligned} \tag{43}
$$

where we have used (40). By the definition of Ψ, it follows that $\|\mathcal{F}x - \mathcal{F}y\| \leq \Psi(\|x-y\|)$. This shows that $\mathcal{F}$ is a nonlinear contraction. Thus, by Lemma 13, the operator $\mathcal{F}$ has a unique fixed point in $\mathcal{C}$, which in turn is a unique solution of the problem (1). □

4. Example

Example 1. Consider the boundary value problem

$$u'''(t) = f(t, u(t)), \quad t \in [0,1],$$
$$u(0) = 0, \qquad u'(0) = 0, \tag{44}$$
$$u(1) + u'(1) = \sum_{i=1}^{3} \gamma_i \int_0^{\eta_i} u(s)\, ds, \quad 0 < \eta_i < 1,$$

where $n = 3$, $\alpha = 1$, $\beta = 1$, $\eta_1 = 1/4$, $\eta_2 = 1/2$, $\eta_3 = 3/4$, $\gamma_1 = 1$, $\gamma_2 = 1/3$, and $\gamma_3 = 2/3$.

We find that

$$\Lambda = \frac{1}{\alpha + (n-1)\beta - (1/n)\sum_{i=1}^{m} \gamma_i \eta_i^n} \approx 0.346362,$$
$$\rho = \frac{1}{n!} + |\Lambda| \left(\frac{\sum_{i=1}^{m} |\gamma_i| \eta_i^{n+1}}{(n+1)!} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!} \right) \approx 0.400976. \tag{45}$$

(**a**) Let

$$f(t,u) = \frac{e^{-t}}{2(1+e^t)} \cdot \frac{|u|}{1+|u|}, \quad t \in [0,1], u \in \mathbb{R}. \tag{46}$$

Since $|f(t,u) - f(t,v)| \le (1/4)|u - v|$, then, (A_1) is satisfied with $L = 1/4$. Since

$$L \left\{ \frac{1}{n!} + |\Lambda| \left(\frac{\sum_{i=1}^{m} |\gamma_i| \eta_i^{n+1}}{(n+1)!} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!} \right) \right\} \approx 0.100244 < 1, \tag{47}$$

therefore, by Theorem 10, the problem (44) with f given by (46) has a unique solution.

(**b**) Let

$$f(t, u(t)) = t^2 \tan^{-1} u(t) + u(t), \quad t \in [0,1]. \tag{48}$$

Choose $\gamma = 1/2 \in (0,1)$. Since

$$\left| f(t,x) - f(t,y) \right| = t^2 \left| \tan^{-1} x - \tan^{-1} y \right| + |x - y| \le \left(t^2 + 1\right) |x - y|, \tag{49}$$

then, (A_3) is satisfied with $m(t) = (t^2 + 1) \in L_2([0,1], \mathbb{R}^+)$. We can show that

$$\|m\|_p = \left(\int_0^1 \left(s^2+1\right)^2 ds \right)^{1/2} \approx 1.366260,$$
$$\|m\|_p \left\{ \frac{1}{(n-1)!} \left(\frac{1-\gamma}{n-\gamma} \right)^{1-\gamma} + |\Lambda| \left(\frac{1}{n!} \sum_{i=1}^{m} |\gamma_i| \eta_i^{n+1-\gamma} \left(\frac{1-\gamma}{n+1-\gamma} \right)^{1-\gamma} + \frac{|\alpha|}{(n-1)!} \left(\frac{1-\gamma}{n-\gamma} \right)^{1-\gamma} + \frac{|\beta|}{(n-2)!} \left(\frac{1-\gamma}{n-1-\gamma} \right)^{1-\gamma} \right) \right\} \approx 0.692906 < 1. \tag{50}$$

Thus, by Theorem 11, the problem (44) with f defined by (48) has a unique solution.

(**c**) Let

$$f(t,u) = \frac{t}{10\pi} \sin(\pi u) + \frac{(t+1)u^2}{1+u^2}, \quad t \in [0,1]. \tag{51}$$

Clearly,

$$\left| f(t,u) \right| = \left| \frac{t}{10\pi} \sin(\pi u) + \frac{(t+1)u^2}{1+u^2} \right| \le (t+1) \left(\frac{\|u\|}{10} + 1 \right). \tag{52}$$

Choosing $p(t) = t + 1$, $\psi(\|x\|) = (\|x\|/10) + 1$, we obtain

$$\frac{M}{\left\{ \frac{1}{n!} + |\Lambda| \left(\frac{\sum_{i=1}^{m} |\gamma_i| \eta_i^{n+1}}{(n+1)!} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!} \right) \right\} \|p\|_{L^1}} > \frac{M}{10} + 1, \tag{53}$$

which implies that $M > 0.639955$. Hence, by Theorem 5, the boundary value problem (44) with f defined by (51) has at least one solution on $[0,1]$.

(**d**) Let

$$f(t,u) = \frac{t|u|}{1+|u|}, \quad 0 < t < 1. \tag{54}$$

We choose $h(t) = (1 + t)$ and find that

$$H^* = \|h\|_{L^1} \left\{ \frac{1}{n!} + |\Lambda| \left(\frac{\sum_{i=1}^{m} |\gamma_i| \eta_i^{n+1}}{(n+1)!} + \frac{|\alpha|}{n!} + \frac{|\beta|}{(n-1)!} \right) \right\} \approx 0.601464. \tag{55}$$

Clearly,

$$\begin{aligned}\left|f(t,x)-f(t,y)\right| &= \left|\frac{t\left(|x|-|y|\right)}{1+|x|+|y|+|x||y|}\right| \\ &\le \frac{0.601464\left|x-y\right|}{0.601464+\left|x-y\right|}.\end{aligned} \tag{56}$$

Hence, by Theorem 14, the boundary value problem (44) with f defined by (54) has a unique solution on $[0,1]$.

5. Fractional Case

In this section, we consider a Caputo type fractional analogue of problem (1) given by

$$\begin{gathered}{}^{c}D^{q}x(t)=f(t,x(t)), \\ 0<t<1,\quad n-1<q\le n,\quad n\ge 2,\quad n\in\mathbb{N}, \\ x(0)=0,\quad x'(0)=0,\quad x''(0)=0,\ldots,x^{(n-2)}(0)=0, \\ \alpha u(1)+\beta u'(1)=\sum_{i=1}^{m}\gamma_i\int_0^{n_i}u(s)\,ds,\quad 0<\eta_i<1,\end{gathered} \tag{57}$$

where ${}^{c}D^{q}$ denotes the Caputo fractional derivative of order q. Before proceeding further, we recall some basic definitions of fractional calculus [25–27].

Definition 15. For an at least n-times continuously differentiable function $g:[0,\infty)\to\mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$\begin{gathered}{}^{c}D^{q}g(t)=\frac{1}{\Gamma(n-q)}\times\int_0^t(t-s)^{n-q-1}g^{(n)}(s)\,ds, \\ n-1<q<n,\ n=[q]+1,\end{gathered} \tag{58}$$

where $[q]$ denotes the integer part of the real number q.

Definition 16. The Riemann-Liouville fractional integral of order q is defined as

$$I^{q}g(t)=\frac{1}{\Gamma(q)}\int_0^t\frac{g(s)}{(t-s)^{1-q}}ds,\quad q>0, \tag{59}$$

provided that the integral exists.

It is well known [26] that the general solution of the fractional differential equation

$$\begin{gathered}{}^{c}D^{q}u(t)=y(t), \\ 0<t<1,\quad n-1<q\le n,\quad n\ge 2,\quad n\in\mathbb{N},\end{gathered} \tag{60}$$

with $y\in C([0,1],\mathbb{R})$ can be written as

$$u(t)=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}y(s)\,ds+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}, \tag{61}$$

where $c_0,c_1,c_2,\ldots,c_{n-1}$ are arbitrary constants. Using the boundary conditions for the problem (57), we find that $c_0=c_1=c_2=\cdots=c_{n-2}=0$ and

$$\begin{aligned}c_{n-1}=\Lambda\Bigg(&\sum_{i=1}^{m}\gamma_i\int_0^{\eta_i}\left(\int_0^s\frac{(s-r)^{q-1}}{\Gamma(q)}y(r)\,dr\right)ds \\ &-\alpha\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}y(s)\,ds \\ &-\beta\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}y(s)\,ds\Bigg),\end{aligned} \tag{62}$$

where $\Lambda=[\alpha+(n-1)\beta-(1/n)\sum_{i=1}^{m}\gamma_i\eta_i^{n}]^{-1}$. Substituting these values in (61) yields

$$\begin{aligned}u(t)=&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}y(s)\,ds \\ &+\Lambda t^{n-1} \\ &\times\Bigg(\sum_{i=1}^{m}\gamma_i\int_0^{\eta_i}\left(\int_0^s\frac{(s-r)^{q-1}}{\Gamma(q)}y(r)\,dr\right)ds \\ &-\alpha\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}y(s)\,ds \\ &-\beta\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}y(s)\,ds\Bigg).\end{aligned} \tag{63}$$

Integrating the second term in (63) with respect to s after interchanging the order of integration, we obtain

$$\begin{aligned}u(t)=&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}y(s)\,ds \\ &+\Lambda t^{n-1} \\ &\times\Bigg(\sum_{i=1}^{m}\gamma_i\int_0^{\eta_i}\frac{(\eta_i-r)^{q}}{\Gamma(q+1)}y(r)\,dr \\ &-\alpha\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}y(s)\,ds \\ &-\beta\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}y(s)\,ds\Bigg).\end{aligned} \tag{64}$$

Replacing $y(s)$ with $f(s,u(s))$ in (64), the solution of the problem (57) is given by

$$\begin{aligned}u(t)=&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,u(s))\,ds \\ &+\Lambda t^{n-1}\end{aligned}$$

$$\times \left(\sum_{i=1}^{m} \gamma_i \int_0^{\eta_i} \frac{(\eta_i - s)^q}{\Gamma(q+1)} f(s, u(s))\, ds \right.$$
$$- \alpha \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, u(s))\, ds$$
$$\left. -\beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s, u(s))\, ds \right). \tag{65}$$

In relation to the problem (57), we define an operator $\mathcal{F}_q : C([0,1], \mathbb{R}) \rightarrow C([0,1], \mathbb{R})$ by

$$\begin{aligned}\left(\mathcal{F}_q u\right)(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, u(s))\, ds \\ &+ \Lambda t^{n-1} \\ &\times \left(\sum_{i=1}^{m} \gamma_i \int_0^{\eta_i} \frac{(\eta_i - s)^q}{\Gamma(q+1)} f(s, u(s))\, ds \right. \\ &- \alpha \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, u(s))\, ds \\ &\left. -\beta \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s, u(s))\, ds \right).\end{aligned} \tag{66}$$

By taking $q = n$ in (66), the resulting operator reduces to the one given by (8) for a nth order classical boundary value problem. Thus, all the results for the fractional problem (57), analogous to the classical problem (1), can be obtained with the aid of the operator $\mathcal{F}_q$ given by (66). For example, Theorem 10 has the following fractional analogue.

Theorem 17. *Assume that $f : [0,1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying the condition (A_1).*

If

$$L \left\{ \frac{1}{\Gamma(q+1)} + |\Lambda| \left(\frac{\sum_{i=1}^{m} |\gamma_i| \eta_i^{q+1}}{\Gamma(q+2)} + \frac{|\alpha|}{\Gamma(q+1)} + \frac{|\beta|}{\Gamma(q)} \right) \right\} < 1, \tag{67}$$

then, the boundary value problem (57) *has a unique solution.*

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