

Research Article

On a Fractional Nonlinear Hyperbolic Equation Arising from Relative Theory

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We obtain the existence of a weak solution to a fractional nonlinear hyperbolic equation arising from relative theory by the Galerkin method. Its uniqueness is also discussed. Furthermore, we show the regularity of the obtained solution. In our proof, we use harmonic analysis techniques and compactness arguments.

1. Introduction

This paper is concerned with the following fractional partial differential equations in $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$:

$$\frac{\partial^2 u}{\partial t^2} - \Lambda^{2\alpha} u + |u|^\rho u = f, \quad (1)$$

$$u(x, 0) = u_0(x), \quad (2)$$

$$\frac{\partial u}{\partial t}(x, 0) = u_1(x), \quad (3)$$

where $\Lambda = (-\Delta)^{1/2}$ is the square root of the Laplacian operator, $\alpha \in (0, 1)$, $\rho \in [0, \infty)$ are two real parameters, and $f(x, t)$ is given. Equations (1)–(3) play an important role in nuclear force and relativistic theory.

The fractional diffusion operator $\Lambda^{2\alpha}$ is nonlocal except when $\alpha \in \mathbb{N}$, which means that $\Lambda^{2\alpha} u(x)$ depends not only on $u(y)$ for y near x , but on $u(y)$ for all $y \in \mathbb{R}^n$. Fractional differential equations, arising from mathematical physics such as viscoelasticity, electrochemistry, control theory, porous media, and electromagnetism, now attract the interests of many mathematicians; see [1–4] and references therein. In the past ten years, the quasi-geostrophic equation with fractional dissipation has been extensively studied; see Constantin et al. [5–9] and references therein. The relativistic equation

shares some similar difficulties with the quasi-geostrophic equation. However, the equations studied in this paper are more complicated in that the fractional diffusion operator and the nonlinear term in (1)–(3) bring new difficulties in passing to the limits of the approximate solutions, and hence, new devices must be introduced to overcome these obstacles.

When $\alpha = 1$, (1)–(3) become the standard equations, which were intensively studied in the past century. The readers are referred to [10–12] for more details.

Interestingly enough, the parabolic version of (1)–(3) with convection corresponds to the Navier-Stokes equations with damping; see [13, 14].

We now collect the notations in this paper. The square root of the negative Laplacian $-\Delta$, Λ is given by (in terms of Fourier series)

$$(\widehat{\Lambda f})_k = |k| \widehat{f}_k, \quad k \in \mathbb{Z}^n, \quad (4)$$

where \widehat{f}_k is the Fourier coefficients of f :

$$\widehat{f}_k = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(x) e^{-ik \cdot x} dx, \quad k \in \mathbb{Z}^n. \quad (5)$$

More generally, $\Lambda^{2\alpha} f$ for $\alpha \in \mathbb{R}$ can be defined as

$$(\widehat{\Lambda^{2\alpha} f})_k = |k|^{2\alpha} \widehat{f}_k, \quad k \in \mathbb{Z}^n. \quad (6)$$

We will also invoke the notion of homogeneous Sobolev space $\dot{H}^s(\mathbb{T}^n)$ ($s \in \mathbb{R}$), which comprises all tempered distributions f on \mathbb{R}^n such that

$$\|f\|_{\dot{H}^s(\mathbb{T}^n)} = \left(\sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^{2s} |\hat{u}_k|^2 \right)^{1/2}. \quad (7)$$

We also recall the meaning of by the weak (weakly*) convergence in $L^p(0, T; L^q(\mathbb{T}^n))$. Since the dual of $L^r(\mathbb{T}^n)$ ($1 \leq r < \infty$) is $L^{r/(r-1)}(\mathbb{T}^n)$ and the space $C^\infty(\mathbb{T}^n)$ is dense in $L^r(\mathbb{T}^n)$ (noticing the periodic boundary conditions), we have

$$f_k \rightharpoonup f \text{ in } L^p(0, T; L^q(\mathbb{T}^3)) \quad (8)$$

with $1 \leq p, q < +\infty$ if and only if

$$\int_0^T \int_{\mathbb{T}^n} f_k \phi \, dx \, dt \longrightarrow \int_0^T \int_{\mathbb{T}^n} f \phi \, dx \, dt, \quad (9)$$

$$\forall \phi \in C_0^\infty(0, T; C^\infty(\mathbb{T}^n)),$$

$$f_k \rightharpoonup f \text{ in } L^p(0, T; L^q(\mathbb{T}^3)) \quad (10)$$

with $p = +\infty$ or $q = +\infty$ if and only if (9) holds.

We now close this introduction by outlining the rest of this paper. In Section 2, we prove the existence of a weak solution to (1)–(3); see Theorem 4. The uniqueness of such weak solutions is discussed in Section 3; see Theorem 5. Finally, a regularity result is obtained in Section 4; see Theorem 6.

2. Existence of a Weak Solution

First, let us recall the following two fundamental lemmas.

Lemma 1 (see [15]). *Let X be a Banach space, $f \in L^p(0, T; X)$, and $\partial f / \partial t \in L^p(0, T; X)$ ($1 \leq p \leq \infty$), then $f \in C([0, T]; X)$.*

Lemma 2 (see [15]). *Let \mathcal{D} be a bounded domain in $\mathbb{R}^n \times \mathbb{R}_+$, and $\{g_k\}_{k=1}^\infty$, g belong to $L^q(\mathcal{D})$ ($1 < q < \infty$) with*

$$\|g_k\|_{L^q(\mathcal{D})} \leq C, \quad g_k \longrightarrow g \text{ a.e. } \mathcal{D}. \quad (11)$$

Then $g_k \rightharpoonup g$ in $L^q(\mathcal{D})$ weakly.

Let us now give the weak formulation of (1)–(3).

Definition 3. Let $T > 0$, $u_0 \in \dot{H}^\alpha(\mathbb{T}^n) \cap L^{\rho+2}(\mathbb{T}^n)$, $u_1 \in L^2(\mathbb{T}^n)$, and $f \in L^2(0, T; L^2(\mathbb{T}^n))$. A measurable function $u(x, t)$ is said to be a weak solution on $[0, T]$ to (1)–(3) if the following conditions hold:

(1) $u \in L^\infty(0, T; \dot{H}^\alpha(\mathbb{T}^n) \cap L^{\rho+2}(\mathbb{T}^n))$ and $\partial u / \partial t \in L^\infty(0, T; L^2(\mathbb{T}^n))$;

(2) (1) holds in the sense of distributions; that is,

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^n} \frac{\partial u}{\partial t} \cdot \frac{\partial \phi}{\partial t} \, dx \, dt - \int_0^T \int_{\mathbb{T}^n} \Lambda^\alpha u \cdot \Lambda^\alpha \phi \, dx \, dt \\ & + \int_0^T \int_{\mathbb{T}^n} |u|^\rho u \phi \, dx \, dt \\ & = \int_{\mathbb{T}^n} u_1 \phi_t \, dx + \int_0^T \int_{\mathbb{T}^n} f \phi \, dx \, dt, \end{aligned} \quad (12)$$

for each $\phi \in C^\infty(\mathbb{T}^n \times (0, T))$;

(3) $u(x, 0) = u_0(x)$ a.e. in \mathbb{T}^n ;

(4) $(\partial u / \partial t)(x, 0) = u_1(x)$ a.e. in \mathbb{T}^n .

To see how item (4) in Definition 3 makes sense, we rewrite (1) as

$$\frac{\partial^2 u}{\partial t^2} = f + \Lambda^{2\alpha} u - |u|^\rho u. \quad (13)$$

Noticing that [16]

$$\Lambda^{2\alpha} \text{ is continuous from } \dot{H}^\alpha(\mathbb{T}^n) \text{ to } \dot{H}^{-\alpha}(\mathbb{T}^n), \quad (14)$$

we have

$$\Lambda^{2\alpha} u \in L^\infty(0, T; \dot{H}^{-\alpha}(\mathbb{T}^n)). \quad (15)$$

Also, due to the fact that

$$\| |u|^\rho u \|_{L^{(\rho+2)/(\rho+1)}(\mathbb{T}^n)} = \|u\|_{L^{\rho+2}(\mathbb{T}^n)}^{\rho+1}, \quad (16)$$

we deduce from (13) that

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} & \in L^2(0, T; L^2(\mathbb{T}^n)) \\ & + L^\infty(0, T; \dot{H}^{-\alpha}(\mathbb{T}^n) + L^{(\rho+2)/(\rho+1)}(\mathbb{T}^n)). \end{aligned} \quad (17)$$

Thus

$$\frac{\partial^2 u}{\partial t^2} \in L^2(0, T; \dot{H}^{-\alpha}(\mathbb{T}^n) + L^{(\rho+2)/(\rho+1)}(\mathbb{T}^n)). \quad (18)$$

Hence, by item (1) of Definition 3, (18), and Lemma 1, we gather that

$$\frac{\partial u}{\partial t} \in C([0, T]; \dot{H}^{-\alpha}(\mathbb{T}^n) + L^{(\rho+2)/(\rho+1)}(\mathbb{T}^n)). \quad (19)$$

From this, we see that item (4) of Definition 3 makes sense, as claimed.

Now, we state our existence results in the following theorem.

Theorem 4. *Let $T > 0$,*

$$u_0 \in \dot{H}^\alpha(\mathbb{T}^n) \cap L^{\rho+2}(\mathbb{T}^n), \quad (20)$$

$$u_1 \in L^2(\mathbb{T}^n), \quad (21)$$

$$f \in L^2(0, T; L^2(\mathbb{T}^n)); \quad (22)$$

then there exists at least one weak solution u on $[0, T]$ to (1)–(3), taking u_0, u_1 as initial data.

Proof. We use Galerkin method to establish the existence of such a solution.

Step 1 (construction of approximate solution). Let $\{w_i\}_{i=1}^\infty$ be a dense and total basis in $\dot{H}^\alpha(\mathbb{T}^n) \cap L^{\rho+2}(\mathbb{T}^n)$, and consider the approximate solution which has the form

$$u_m(t) = \sum_{i=1}^m c_{im}(t) w_i, \quad (23)$$

where c_{im} satisfy the following ordinary differential system:

$$\left(\frac{\partial^2 u_m}{\partial t^2}, w_j \right) - (\Lambda^{2\alpha} u_m, w_j) + (|u_m|^\rho u_m, w_j) = (f, w_j),$$

$$1 \leq j \leq m, \quad (24)$$

$$u_m(0) = u_{0m} = \sum_{i=1}^m \alpha_{im} w_i \longrightarrow u_0 \text{ in } \dot{H}^\alpha(\mathbb{T}^n) \cap L^{\rho+2}(\mathbb{T}^n),$$

$$\text{as } m \longrightarrow \infty, \quad (25)$$

$$\frac{\partial u_m}{\partial t}(0) = u_{1m} = \sum_{i=1}^m \beta_{im} w_i \longrightarrow u_1 \text{ in } L^2(\mathbb{T}^n), \quad \text{as } m \longrightarrow \infty. \quad (26)$$

Here and hereafter, we denote $(u, v) = \int_{\mathbb{T}^n} uv \, dx$ for $u, v \in L^2(\mathbb{T}^n)$.

The system (24)–(26) is nonsingular because $\{w_i\}$ are linear independent. Thus, we may apply standard theory of ordinary differential equations to obtain the existence of a local solution to (24)–(26) on $[0, t_m]$, for some $t_m > 0$. We will then, in the next step, establish some a priori estimates of the obtained solutions which will ensure that $t_m = T$.

Step 2 (a priori estimates). Taking the inner product of (24) with $c'_{im}(t)$ in $L^2(\mathbb{R}^n)$, we obtain

$$\frac{1}{2} \frac{d}{dt} \left[\left\| \frac{\partial u_m}{\partial t}(t) \right\|_{L^2(\mathbb{T}^n)}^2 + (\Lambda^\alpha u_m(t), \Lambda^\alpha u_m(t)) \right]$$

$$+ \frac{1}{\rho+2} \frac{d}{dt} \int_{\mathbb{T}^n} |u_m(x, t)|^{\rho+2} dx = \left(f(t), \frac{\partial u_m}{\partial t}(t) \right).$$

$$(27)$$

Integrating over $[0, t]$ and invoking Hölder inequality then yield

$$\frac{1}{2} \left[\left\| \frac{\partial u_m}{\partial t}(t) \right\|_{L^2(\mathbb{T}^n)}^2 + \|\Lambda^\alpha u_m(t)\|_{L^2(\mathbb{T}^n)}^2 \right]$$

$$+ \frac{1}{\rho+2} \|u_m(t)\|_{L^{\rho+2}(\mathbb{T}^n)}^{\rho+2}$$

$$\leq \frac{1}{2} \left[\left\| \frac{\partial u_m}{\partial t}(0) \right\|_{L^2(\mathbb{T}^n)}^2 + \|\Lambda^\alpha u_m(0)\|_{L^2(\mathbb{T}^n)}^2 \right]$$

$$+ \frac{1}{\rho+2} \|u_m(0)\|_{L^{\rho+2}(\mathbb{T}^n)}^{\rho+2}$$

$$+ \int_0^t \|f(\tau)\|_{L^2(\mathbb{T}^n)} \left\| \frac{\partial u_m}{\partial \tau}(\tau) \right\|_{L^2(\mathbb{T}^n)} d\tau,$$

$$\forall t \in [0, T]. \quad (28)$$

Due to (20), (25), and (26), we have that $\|(\partial u_m / \partial t)(0)\|_{L^2(\mathbb{T}^n)}^2$, $\|u_m(0)\|_{L^{\rho+2}(\mathbb{T}^n)}^{\rho+2}$, and $\|u_m(0)\|_{L^{\rho+2}(\mathbb{T}^n)}^{\rho+2}$ are uniformly bounded. Thus, (28) becomes

$$\frac{1}{2} \left[\left\| \frac{\partial u_m}{\partial t}(t) \right\|_{L^2(\mathbb{T}^n)}^2 + \|\Lambda^\alpha u_m(t)\|_{L^2(\mathbb{T}^n)}^2 \right]$$

$$+ \frac{1}{\rho+2} \|u_m(t)\|_{L^{\rho+2}(\mathbb{T}^n)}^{\rho+2}$$

$$\leq C + \frac{1}{2} \int_0^t \|f(\tau)\|_{L^2(\mathbb{T}^n)}^2 d\tau$$

$$+ \frac{1}{2} \int_0^t \left\| \frac{\partial u_m}{\partial \tau}(\tau) \right\|_{L^2(\mathbb{T}^n)}^2 d\tau, \quad \forall t \in [0, T]. \quad (29)$$

Noticing that $f \in L^2(0, T; L^2(\mathbb{T}^n))$, we have

$$\frac{1}{2} \left[\left\| \frac{\partial u_m}{\partial t}(t) \right\|_{L^2(\mathbb{T}^n)}^2 + \|\Lambda^\alpha u_m(t)\|_{L^2(\mathbb{T}^n)}^2 \right]$$

$$+ \frac{1}{\rho+2} \|u_m(t)\|_{L^{\rho+2}(\mathbb{T}^n)}^{\rho+2}$$

$$\leq \frac{1}{2} \int_0^t \left\| \frac{\partial u_m}{\partial \tau}(\tau) \right\|_{L^2(\mathbb{T}^n)}^2 d\tau + C, \quad \forall t \in [0, T]. \quad (30)$$

Applying Gronwall's inequality then yields

$$\left\| \frac{\partial u_m}{\partial t}(t) \right\|_{L^2(\mathbb{T}^n)} \leq C, \quad (31)$$

and hence,

$$\|u_m(t)\|_{L^{\rho+2}(\mathbb{T}^n)} \leq C, \quad \|\Lambda^\alpha u_m(t)\|_{L^2(\mathbb{T}^n)} \leq C. \quad (32)$$

Step 3 (passage to limit $m \rightarrow \infty$).

By (31) and (32), we have, up to a subsequence, still denoted by u_m , that

$$u_m \rightharpoonup u \text{ weakly } * \text{ in } L^\infty(0, T; \dot{H}^\alpha(\mathbb{T}^n) \cap L^{\rho+2}(\mathbb{T}^n)),$$

$$\text{as } m \longrightarrow \infty,$$

$$\frac{\partial u_m}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ weakly } * \text{ in } L^\infty(0, T; L^2(\mathbb{T}^n)), \quad \text{as } m \longrightarrow \infty, \quad (33)$$

and also that u_m are uniformly bounded in $L^2(0, T; \dot{H}^\alpha(\mathbb{T}^n))$. Thus by Lemma 1, we find that

$$u_m \longrightarrow u \text{ strongly in } L^2(0, T; L^2(\mathbb{T}^n)), \quad \text{a.e. } \mathbb{T}^n,$$

$$\text{as } m \longrightarrow \infty. \quad (34)$$

Hence, there exists a function $w(x, t)$ such that

$$|u_m|^\rho u_m \longrightarrow w(x, t) \text{ weakly } * \text{ in } L^\infty$$

$$\times (0, T; L^{(\rho+2)/(\rho+1)}(\mathbb{T}^n)), \quad \text{as } m \longrightarrow \infty. \quad (35)$$

By Lemma 2 and the fact that

$$|u_m|^\rho u_m \longrightarrow |u|^\rho u, \quad \text{a.e. in } \mathbb{T}^n, \quad (36)$$

we know then that

$$w = |u|^\rho u. \quad (37)$$

Fixing j , we now pass to the limit $m \rightarrow \infty$ in (24) to deduce that

$$\begin{aligned} & \left(\frac{\partial^2 u}{\partial t^2}(t), w_j \right) + \left(\Lambda^{2\alpha} u(t), w_j \right) \\ & + \left(|u(t)|^\rho u(t), w_j \right) = \left(f(t), w_j \right). \end{aligned} \quad (38)$$

A simple density argument then shows that

$$\left(\frac{\partial^2 u}{\partial t^2}(t), v \right) + \left(\Lambda^{2\alpha} u(t), v \right) + \left(|u(t)|^\rho u(t), v \right) = \left(f(t), v \right), \quad (39)$$

for all $v \in \dot{H}^\alpha(\mathbb{T}^n) \cap L^{\rho+2}(\mathbb{T}^n)$.

Up to now, we have proved items (1) and (2) in Definition 3. Let us turn our attention to items (3) and (4) in Definition 3.

By (33) and Lemma 1, we know $u_m(0) \rightharpoonup u(0)$ weakly in $L^2(\mathbb{T}^n)$, and from (25), $u_m(0) \rightarrow u_0$ in $\dot{H}^\alpha(\mathbb{T}^n) \cap L^{\rho+2}(\mathbb{T}^n)$; thus item (3) of Definition 3 is verified.

By (24) and (39), we see that

$$\left(\frac{\partial^2 u_m}{\partial t^2}(0), w_j \right) \longrightarrow \left(\frac{\partial^2 u}{\partial t^2}(0), w_j \right) \text{ weakly*}, \quad (40)$$

and by Lemma 1, we have

$$\left(\frac{\partial u_m}{\partial t}(0), w_j \right) \longrightarrow \left(\frac{\partial u}{\partial t}(0), w_j \right). \quad (41)$$

On the other hand, (26) implies that

$$\left(\frac{\partial u_m}{\partial t}(0), w_j \right) \longrightarrow (u_1, w_j). \quad (42)$$

Hence

$$\left(\frac{\partial u}{\partial t}(0), w_j \right) = (u_1, w_j), \quad \forall j \in \mathbb{N}. \quad (43)$$

This verifies item (4) in Definition 3. \square

3. Uniqueness of Weak Solutions

In this section, we will discuss the uniqueness of weak solutions of (1)–(3). We only obtain partial results in case $0 \leq \rho \leq 2\alpha/(n - 2\alpha)$. More precisely, we have the following theorem.

Theorem 5. *Assuming as in Theorem 4, then there exists an unique weak solution of (1)–(3), in case $0 \leq \rho \leq 2\alpha/(n - 2\alpha)$.*

Proof. Let u, v be two weak solutions for (1)–(3) given in Theorem 4 with the same datum. Then $w = u - v$ satisfies

$$\frac{\partial^2 w}{\partial t^2} + \Lambda^{2\alpha} w = |v|^\rho v - |u|^\rho u, \quad (44)$$

$$w(0) = 0, \quad \frac{\partial w}{\partial t}(0) = 0. \quad (45)$$

Also, we have

$$\begin{aligned} w & \in L^\infty(0, T; \dot{H}^\alpha \cap L^{\rho+2}(\mathbb{T}^n)), \\ \frac{\partial w}{\partial t} & \in L^\infty(0, T; L^2(\mathbb{T}^n)). \end{aligned} \quad (46)$$

Taking the inner product of (44) with $\partial w / \partial t$ in $L^2(\mathbb{T}^n)$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left[\left\| \frac{\partial w}{\partial t} \right\|_{L^2(\mathbb{T}^n)}^2 + \|w\|_{\dot{H}^\alpha(\mathbb{T}^n)}^2 \right] \\ & = \int_{\mathbb{T}^n} (|v|^\rho v - |u|^\rho u) \frac{\partial w}{\partial t} dx. \end{aligned} \quad (47)$$

Invoking Hölder and Sobolev inequalities, we obtain

$$\begin{aligned} & \int_{\mathbb{T}^n} (|v|^\rho v - |u|^\rho u) \frac{\partial w}{\partial t} dx \\ & \leq (\rho + 1) \int_{\mathbb{T}^n} \sup(|u|^\rho, |v|^\rho) |w| \left| \frac{\partial w}{\partial t} \right| dx \\ & \leq C \left(\| |u|^\rho \|_{L^{n/\alpha}(\mathbb{T}^n)} + \| |v|^\rho \|_{L^{n/\alpha}(\mathbb{T}^n)} \right) \|w\|_{L^q(\mathbb{T}^n)} \left\| \frac{\partial w}{\partial t} \right\|_{L^2(\mathbb{T}^n)} \\ & \leq C \left(\|u\|_{\dot{H}^\alpha(\mathbb{T}^n)}^\rho + \|v\|_{\dot{H}^\alpha(\mathbb{T}^n)}^\rho \right) \|w\|_{L^q(\mathbb{T}^n)} \left\| \frac{\partial w}{\partial t} \right\|_{L^2(\mathbb{T}^n)} \\ & \leq C \left(\|u\|_{\dot{H}^\alpha(\mathbb{T}^n)}^\rho + \|v\|_{\dot{H}^\alpha(\mathbb{T}^n)}^\rho \right) \|w\|_{\dot{H}^\alpha(\mathbb{T}^n)} \left\| \frac{\partial w}{\partial t} \right\|_{L^2(\mathbb{T}^n)}, \end{aligned} \quad (48)$$

where $(1/q) + (\alpha/n) + (1/2) = 1$, and we use $0 \leq \rho \leq 2\alpha/(n - 2\alpha)$.

Thus, (47) becomes

$$\begin{aligned} & \left\| \frac{\partial w}{\partial t} \right\|_{L^2(\mathbb{T}^n)}^2 + \|w\|_{\dot{H}^\alpha(\mathbb{T}^n)}^2 \\ & \leq C \int_0^t \left(\left\| \frac{\partial w}{\partial t} \right\|_{L^2(\mathbb{T}^n)}^2 + \|w\|_{\dot{H}^\alpha(\mathbb{T}^n)}^2 \right) d\tau, \end{aligned} \quad (49)$$

and we get $w = 0, u = v$ as desired. \square

4. Regularity of the Weak Solution

Now we discuss the regularity of solutions for (1)–(3). If the initial value and force are more regular, then so is the solution.

Theorem 6. *Let*

$$\begin{aligned} u_0 &\in \dot{H}^{2\alpha}(\mathbb{T}^n) \cap \dot{H}^\alpha(\mathbb{T}^n) \cap L^{\rho+2}(\mathbb{T}^n), \\ u_1 &\in \dot{H}^\alpha(\mathbb{T}^n) \cap L^2(\mathbb{T}^n), \end{aligned} \quad (50)$$

$$f \in L^\infty(0, T; L^2(\mathbb{T}^n)), \quad \frac{\partial f}{\partial t} \in L^2(0, T; L^2(\mathbb{T}^n)),$$

and $0 \leq \rho \leq 2\alpha/(n - 2\alpha)$. Then there exists a unique weak solution for (1)–(3). Furthermore,

$$\begin{aligned} u &\in L^\infty(0, T; \dot{H}^{2\alpha}(\mathbb{T}^n)), \\ \frac{\partial u}{\partial t} &\in L^\infty(0, T; \dot{H}^\alpha(\mathbb{T}^n)), \\ \frac{\partial^2 u}{\partial t^2} &\in L^\infty(0, T; L^2(\mathbb{T}^n)). \end{aligned} \quad (51)$$

Proof. We just establish the a priori bounds, since the verification follows directly from passing to the limit for Galerkin approximate solutions.

Step 1 (bounds for initial data). Observing that

$$0 \leq \rho \leq \frac{n}{n - 2\alpha} \implies \frac{n}{2(\rho + 1)} \geq -\alpha + \frac{n}{2}, \quad (52)$$

we see by Sobolev inequality that

$$\| |u_0|^\rho u_0 \|_{L^2(\mathbb{T}^n)} \leq \| u_0 \|_{L^{2(\rho+1)}(\mathbb{T}^n)}^{\rho+1} \leq \| u_0 \|_{\dot{H}^\alpha(\mathbb{T}^n)}^{\rho+1}. \quad (53)$$

By formula (13), we have

$$\frac{\partial^2 u}{\partial t^2}(0) \in L^2(\mathbb{T}^n). \quad (54)$$

Step 2 (bounds for $\partial u/\partial t$, $\partial^2 u/\partial t^2$). Differentiating (1) with respect to t , we find that

$$\frac{\partial^3 u}{\partial t^3} - \Lambda^{2\alpha} \frac{\partial u}{\partial t} + (\rho + 1) |u|^\rho \frac{\partial u}{\partial t} = \frac{\partial f}{\partial t}. \quad (55)$$

Taking the inner product of (55) with $\partial^2 u/\partial t^2$, we obtain by integration by parts that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left[\left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(\mathbb{T}^n)}^2 + \left\| \Lambda^\alpha \frac{\partial u}{\partial t} \right\|_{L^2(\mathbb{T}^n)}^2 \right] \\ &\leq \left(\frac{\partial f}{\partial t}, \frac{\partial^2 u}{\partial t^2} \right) - (\rho + 1) \left(|u|^\rho \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2} \right) \\ &\equiv I_1 + I_2. \end{aligned} \quad (56)$$

The first term I_1 can be easily dominated by using Hölder inequality as

$$I_1 \leq \left\| \frac{\partial f}{\partial t} \right\|_{L^2(\mathbb{T}^n)} \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(\mathbb{T}^n)} \leq \frac{1}{2} \left\| \frac{\partial f}{\partial t} \right\|_{L^2(\mathbb{T}^n)}^2 + \frac{1}{2} \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(\mathbb{T}^n)}^2. \quad (57)$$

To tackle I_2 , we invoke Hölder and Sobolev inequalities to deduce that

$$\begin{aligned} I_2 &\leq C \| |u|^\rho \|_{L^p(\mathbb{T}^n)} \left\| \frac{\partial u}{\partial t} \right\|_{L^q(\mathbb{T}^n)} \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(\mathbb{T}^n)} \\ &\leq C \| u \|_{L^{\rho p}(\mathbb{T}^n)}^\rho \left\| \frac{\partial u}{\partial t} \right\|_{L^q(\mathbb{T}^n)} \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(\mathbb{T}^n)} \\ &\leq C \| \Lambda^\alpha u \|_{L^2(\mathbb{T}^n)}^\rho \left\| \Lambda^\alpha \frac{\partial u}{\partial t} \right\|_{L^2(\mathbb{T}^n)} \left\| \frac{\partial u}{\partial t} \right\|_{L^q(\mathbb{T}^n)} \\ &\leq C \left(\left\| \Lambda^\alpha \frac{\partial u}{\partial t} \right\|_{L^2(\mathbb{T}^n)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^q(\mathbb{T}^n)}^2 \right), \end{aligned} \quad (58)$$

where p, q are chosen so that

$$\frac{n}{\rho p} = -\alpha + \frac{n}{2} = \frac{n}{q}, \quad (59)$$

and thus

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{2} \leq 1. \quad (60)$$

Gathering (57) and (58) into (56), it follows that

$$\begin{aligned} &\frac{d}{dt} \left[\left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(\mathbb{T}^n)}^2 + \left\| \Lambda^\alpha \frac{\partial u}{\partial t} \right\|_{L^2(\mathbb{T}^n)}^2 \right] \\ &\leq \left\| \frac{\partial f}{\partial t} \right\|_{L^2(\mathbb{T}^n)}^2 + C \left(\left\| \Lambda^\alpha \frac{\partial u}{\partial t} \right\|_{L^2(\mathbb{T}^n)}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^q(\mathbb{T}^n)}^2 \right). \end{aligned} \quad (61)$$

Gronwall inequality then implies that

$$\begin{aligned} &\left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^2(\mathbb{T}^n)}^2 + \left\| \Lambda^\alpha \frac{\partial u}{\partial t} \right\|_{L^2(\mathbb{T}^n)}^2 \\ &\leq e^{Ct} \left[\left\| \frac{\partial^2 u}{\partial t^2}(0) \right\|_{L^2(\mathbb{T}^n)}^2 + \left\| \Lambda^\alpha u_1 \right\|_{L^2(\mathbb{T}^n)}^2 \right] \\ &\quad + e^{Ct} \int_0^t \left\| \frac{\partial f}{\partial s}(s) \right\|_{L^2(\mathbb{T}^n)}^2 ds < \infty. \end{aligned} \quad (62)$$

Step 3 (bounds for u). Rewriting (1) as

$$\Lambda^{2\alpha} u = \frac{\partial^2 u}{\partial t^2} + |u|^\rho u - f, \quad (63)$$

we have $u \in L^\infty(0, T; \dot{H}^{2\alpha}(\mathbb{T}^n))$, in view of similar inequalities satisfied by $|u|^\rho u$ as (53).

This completes the proof of Theorem 6. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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