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Research Article

The Viro Method for Construction of C^r Piecewise Algebraic Hypersurfaces

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We propose a new method to construct a real piecewise algebraic hypersurface of a given degree with a prescribed smoothness and topology. The method is based on the smooth blending theory and the Viro method for construction of Bernstein-Bézier algebraic hypersurface piece on a simplex.

1. Introduction

Let Δ be a simplicial subdivision of a region Ω in R^k. Δ is called a pure simplicial complex of dimension k and can be described as a finite collection of simplices such that the faces of each element of Δ are elements of Δ, and the intersection of any two elements of Δ is an element of Δ, and every maximal element of Δ (with respect to inclusion) is a k-dimensional simplex. We will sometimes refer to the m-dimensional elements of Δ as *m-cells* and the simplicial subdivision as a *k-complex*. If two k-dimensional simplices in Δ meet in a face of dimension $k-1$, we say they are adjacent. Δ is said to be hereditary if for every $\tau \in \Delta$ (including the empty set) any two n-dimensional simplices S, S' of Δ that contain τ can be connected by a sequence $S = S_1, S_2, \ldots, S_m = S'$ in Δ such that each S_i is k-dimensional, each S_i contains τ, and S_i and S_{i+1} are adjacent for each i (see [1, 2]).

Let Δ be a pure, hereditary k-dimensional simplicial complex in R^k, let $S_1, S_2, \ldots, S_q$ be a given, fixed, ordering of the k-cells in Δ, and let $\Omega = \bigcup_{i=1}^{q} S_i$. Now, we recall the definitions of $C^r(\Delta)$ and $C_n^r(\Delta)$ (see [1, 2]).

Definition 1. For a nonnegative integer r and a k-complex Δ, $C^r(\Delta)$ is the set of C^r functions f on Ω (i.e., functions such that all rth order partial derivatives exist and are continuous on Ω) such that, for every $\delta \in \Delta$ including those of dimension $< k$, the restriction $f|_\delta$ is a polynomial function $f|_\delta \in R[x_1, \ldots, x_k]$. $C_n^r(\Delta)$ is the subset of $f \in C^r(\Delta)$ such that the restriction of f to each cell in Δ is a polynomial function of degree k or less.

It is clear that $C^r(\Delta)$ and $C_n^r(\Delta)$ are a Noether ring and a finite dimensional linear vector space, respectively, and are called a C^r spline ring and a multivariate spline space with degree n and smoothness r, respectively. We call

$$\mathscr{Z}(f) := \{(x_1, \ldots, x_k) \in \Omega \mid f(x_1, \ldots, x_k) = 0\} \quad (1)$$

a real C^r *piecewise algebraic hypersurface* (see [1–3]), where $f \in C^r(\Delta)$.

An important direction in real algebraic geometry during the last three decades is the construction of real algebraic hypersurfaces of a given degree with prescribed topology. Central to these developments is a combinatorial construction due to the Viro method [4–6]. The Viro method is a powerful construction method of real nonsingular algebraic hypersurfaces with prescribed topology (see [4–14]). It provides a link between the topology of real algebraic varieties and toric varieties. It is based on polyhedral subdivisions of the Newton polytopes. A particular and important case of the Viro method is called combinatorial patchworking; the combinatorial patchworking is a particular case of the Viro method which is characterized by the following two properties: the subdivision used is a triangulation, and each monomial of any "block" polynomial corresponds to a vertex of the Newton simplex.

Roughly speaking, the Viro method starts with a convex (or coherent) polyhedral subdivision $\{P_i, i \in I\}$ of a polytope P and a collection $\{f_i, i \in I\}$ of real nondegenerate polynomials f_i with Newton polyhedra P_i whose truncations on common faces of Newton polyhedra coincide. Then, a Viro polynomial f with Newton polytope P is defined, and Viro's theorem asserts that the topology of the real hypersurface $Z(f)$ defined by f can be recovered by gluing together pieces of the real hypersurfaces $Z(f_i)$.

It is well known that the hypersurface generally possesses complex topological or geometric structures in CAGD and geometric modelling. Moreover, the surface can be represented in Bernstein-Bézier form since it is often defined on a simplex, and writing a polynomial in it's the Bernstein-Bézier representation has significant advantages since its coefficients reflect geometric information about the shape of the polynomial surface, and the barycentric coordinates relative to the simplex are affine invariant, and Bernstein-Bézier basis polynomials exhibit many important properties (see [3, 15–17]). Therefore, based on the Viro method and the Newton polyhedra of Bernstein-Bézier polynomial, Lai et al. in [18] established a new method for the construction of Bernstein-Bézier algebraic hypersurfaces on a simplex with a prescribed topology and presented a method to describe the topology of the Viro Bernstein-Bézier algebraic hypersurface piece.

In CAGD and geometric modelling, most of the complex curves and surfaces are expressed by piecewise polynomials with certain smoothness (see [3, 17, 18]). Thus, the aim of this paper is to establish a new method for the construction of real piecewise algebraic hypersurfaces of a given degree with certain smoothness and prescribed topology. The new Viro method is based on the work of Lai et al. in [18] and the smooth blending theory.

The paper is organized as follows. Section 2 reviews briefly the Viro method for the construction of Bernstein-Bézier algebraic hypersurface piece on a simplex. In Section 3, we define the chart of the piecewise polynomial and deal with some properties of the chart of the Bernstein-Bézier polynomial and the chart of the piecewise polynomial. Section 4 is devoted to a new method for the construction of the real piecewise algebraic hypersurfaces of a given degree with certain smoothness and prescribed topology. The method is primarily based on the work of Lai et al. in [18] and the smooth blending theory.

2. Construction of the Bernstein-Bézier Algebraic Hypersurface Piece

This section reviews briefly the Viro method for the construction of Bernstein- Bézier algebraic hypersurface piece, as stated in [18].

Throughout this paper, we denote by $\mathbb{R}_+$ (resp., $\mathbb{R}_+^*$) the set of real numbers x such that $x \geq 0$ (resp., $x > 0$) and by $\mathbb{Z}_+$ the set of nonnegative integers. Let $S = [v_1, \ldots, v_{k+1}]$ will be a fixed k-dimensional simplex with vertices $v_1, \ldots, v_{k+1}$.

It is well known (cf. [3, 15]) that for any point $p \in S$, it can be expressed uniquely as

$$p = \sum_{j=1}^{k+1} \tau_j v_j, \tag{2}$$

where $\sum_{j=1}^{k+1} \tau_j = 1$, $\tau_j \geq 0$, $j = 1, 2, \ldots, k+1$, and $\tau = (\tau_1, \ldots, \tau_{k+1})$ is the barycentric coordinates of p with respect to S (we abusively confuse a point $p = \sum_{j=1}^{k+1} \tau_j v_j \in S$ with the corresponding barycentric coordinates $\tau = (\tau_1, \ldots, \tau_{k+1})$).

Let

$$\mathbb{Z}_{+(k,n)} = \left\{ \lambda = (\lambda_1, \ldots, \lambda_{k+1}) \in (\mathbb{Z}_+)^{k+1} \mid |\lambda| = \sum_{i=1}^{k+1} \lambda_i = n \right\}. \tag{3}$$

We call a point $p \in S$ domain point if $p = (\lambda_1/n, \ldots, \lambda_{k+1}/n)$ with $(\lambda_1, \ldots, \lambda_{k+1}) \in \mathbb{Z}_{+(k,n)}$. Further on, polyhedron relative to the simplex S means a convex polyhedron in S with domain points as its vertices.

It is well known (c.f. [3, 15]) that for any polynomial f over S with k variables and degree at most n, it can be represented in the Bernstein-Bézier form as follows:

$$f(\tau) = \sum_{|\lambda|=n} b_\lambda B_{\lambda,n}(\tau), \tag{4}$$

where $b_\lambda \in \mathbb{R}$, $\lambda = (\lambda_1, \ldots, \lambda_{k+1}) \in \mathbb{Z}_{+(k,n)}$,

$$B_{\lambda,n}(\tau) = \frac{n!}{\lambda_1! \cdots \lambda_{k+1}!} \tau_1^{\lambda_1} \cdots \tau_{k+1}^{\lambda_{k+1}} \tag{5}$$

are the Bernstein basis of degree n relative to S. The $f(\tau)$ is called a Bernstein-Bézier polynomial or B-form of the polynomial f relative to the simplex S. We will refer to the polynomials in B-form as BB-polynomials and to their coefficients b_λ as BB-coefficients. The set of its zero points in S is called a Bernstein-Bézier algebraic hypersurface piece (BB-algebraic hypersurface piece for short).

For the BB-polynomial f defined in (4), set

$$\mu(f) := \left\{ \left(\frac{\lambda_1}{n}, \ldots, \frac{\lambda_{k+1}}{n} \right) \mid \lambda \in \mathbb{Z}_{+(k,n)}, b_\lambda \neq 0 \right\}. \tag{6}$$

The convex hull conv$(\mu(f))$ of $\mu(f)$ on S, denoted by $N(f)$, is called Newton polyhedron (relative to S) of the BB-polynomial f.

For a set $\Gamma \subset S$ and a BB-polynomial $f = \sum_{|\lambda|=n} b_\lambda B_{\lambda,n}(\tau)$, denote BB-polynomial $\sum_{|\lambda|=n, \lambda \in \Gamma} b_\lambda B_{\lambda,n}(\tau)$ by f^Γ. It is called the Γ-truncation of f.

A BB-polynomial f is called nondegenerate if f and any truncation f^δ on a proper face δ of $N(f)$ has a nonsingular zero set in $(\mathbb{C}^*)^{k+1}$.

Let $\mathcal{J}_S$ be a subset of the set of domain points in simplex S and $P_S = \text{conv}(\mathcal{J}_S)$. Define the moment map associated with $\mathcal{J}_S$ relative to the simplex S, $\phi_{\mathcal{J}_S} : \text{Int}(S) \to \text{Int}(P_S)$, by

$$\phi_{\mathcal{J}_S}(\tau) = \frac{\sum_{(i_1/n, \ldots, i_{k+1}/n) \in \mathcal{J}_S} \tau_1^{i_1} \cdots \tau_{k+1}^{i_{k+1}} (i_1/n, \ldots, i_{k+1}/n)}{\sum_{(i_1/n, \ldots, i_{k+1}/n) \in \mathcal{J}_S} \tau_1^{i_1} \cdots \tau_{k+1}^{i_{k+1}}}, \tag{7}$$

where $\text{Int}(P_S)$ is the complement in P_S of the union of all its proper faces.

Definition 2. Let f be a BB-polynomial with a Newton polyhedron P. Then the closure of $\phi_{\mu(f)}(\{\tau \in \text{Int}(S), f(\tau) = 0\})$ in P is called chart $Ct_S(f)$ of f, where $\phi_{\mu(f)} : \text{Int}(S) \to \text{Int}(P)$ is the moment map associated with $\mu(f)$ relative to the simplex S.

Let $P, P_1, \ldots, P_m \subset S$ be polyhedra with $P = \bigcup_{i=1}^m P_i$ and $\text{Int}(P_i) \bigcap \text{Int}(P_j) = \emptyset$ for $i \neq j$. Assume that $\nu : P \to \mathbb{R}$ is a continuous, piecewise linear, nonnegative convex function satisfying the following conditions:

(1) all the restrictions $\nu|_{P_i}$ are linear;

(2) if the restriction of ν to an open set is linear, then this set is contained in one of the P_i;

(3) $\nu(P \bigcap L_{(k,n)}) \subset \mathbb{Z}$, where $L_{(k,n)}$ is the set of the domain points.

Then, this function ν with this property is said to *convexify* $\{P_1, \ldots, P_m\}$.

Let $f_1, \ldots, f_m$ be BB-polynomials over $\mathbb{R}$ in $k+1$ variables with $N(f_i) = P_i$. Let $f_i^{P_i \bigcap P_j} = f_j^{P_i \bigcap P_j}$ for any i, j. Then, there exists a unique BB-polynomial f with $N(f) = P = \bigcup_{i=1}^m P_i$ and $f^{P_i} = f_i$ for $i = 1, \ldots, m$. If $f(\tau) = \sum_{|\lambda|=n} b_\lambda B_{\lambda,n}(\tau)$ and ν is a function convexifying $\{P_1, \ldots, P_m\}$, we put

$$f_t(\tau) = \sum_{|\lambda|=n} b_\lambda B_{\lambda,n}(\tau)\, t^{\nu(\lambda/n)}. \tag{8}$$

The BB-polynomials f_t are said to be obtained by *patchworking* BB-polynomials $f_1, \ldots, f_m$ by ν or, briefly, f_t is a *patchwork* of BB-polynomials $f_1, \ldots, f_m$ by ν.

Let $f_1, \ldots, f_m$ be BB-polynomials in $k+1$ variables with $N(f_i) = P_i$, and let $\text{Int}(P_i) \bigcap \text{Int}(P_j) = \emptyset$ for $i \neq j$. A chart $Ct_S(f)$ of a BB-polynomial f with $N(f) = P$ is said to be obtained by *patchworking* charts of BB-polynomials $f_1, \ldots, f_m$ and it is a *patchwork* of charts of BB-polynomials $f_1, \ldots, f_m$ if $P = \bigcup_{i=1}^m P_i$ and the chart $Ct_S(f)$ of f, up to isotopy, is $\bigcup_{i=1}^m Ct_S(f_i)$.

The result about the Viro method for the construction of the Bernstein-Bézier algebraic hypersurface piece on a simplex with a prescribed topology is shown in the following proposition (see [18]).

Proposition 3 (see [18]). *Let $P, P_1, \ldots, P_m, \nu, f_1, \ldots, f_m$, and f_t be as above (f_t is a patchwork of BB-polynomials $f_1, \ldots, f_m$ by ν). If BB-polynomials $f_1, \ldots, f_m$ are nondegenerate, then there exists $t_0 > 0$ such that for any $t \in (0, t_0]$ the chart of BB-polynomial f_t is obtained by patchworking charts of BB-polynomials $f_1, \ldots, f_m$.*

3. The Chart of the Piecewise Polynomial

In this section, the chart of the piecewise polynomial is defined, and some properties of the chart of the Bernstein-Bézier polynomial and the chart of the piecewise polynomial are discussed.

Theorem 4. *Let f be a BB-polynomial with Newton polyhedron P and let Γ be a face of P. Then,*

(1) $Ct_S(f) \bigcap \Gamma = Ct_S(f^\Gamma)$;

(2) *if f^Γ is nondegenerate with respect to Γ (which is the case when, for example, f is nondegenerate with respect to P), then, the $Ct_S(f)$ intersects Γ transversally.*

Proof. Suppose that the BB-polynomial f is defined in (4) and

$$g(x) = \sum_{|\lambda|=n} \frac{n!}{\lambda_1! \cdots \lambda_{k+1}!} b_\lambda x_2^{\lambda_2} \cdots x_{k+1}^{\lambda_{k+1}}. \tag{9}$$

Define the mapping $\psi : S \to T_{(k,n)} = \{(x_2, \ldots, x_{k+1}) \in (\mathbb{R}_+)^k \mid x_2 + \cdots + x_{k+1} \le n\}$ by

$$\psi(\tau_1, \tau_2, \ldots, \tau_{k+1}) = (n\tau_2, \ldots, n\tau_{k+1}), \quad (\tau_1, \tau_2, \ldots, \tau_{k+1}) \in S. \tag{10}$$

According to [18, Lemma 3.2], then, $\psi(P)$ is the Newton polyhedron $N(g)$ of the polynomial g and $\psi(\Gamma)$ is a face of $N(g)$ if Γ is a face of P. Moreover, by [10, Remark 1.1], the *chart* $Ct_+(g)$ of g (see [6–8, 18]) and the truncation $g^{\psi(\Gamma)}$ on the face $\psi(\Gamma)$ have the following properties:

(a) $Ct_+(g) \bigcap \psi(\Gamma) = Ct_+(g^{\psi(\Gamma)})$;

(b) the *chart* $Ct_+(g)$ intersects $\psi(\Gamma)$ transversally if $g^{\psi(\Gamma)}$ is nondegenerate with respect to $\psi(\Gamma)$.

On the other hand, by [18, Lemma 3.2 and Theorem 3.5], we can get the following equalities:

$$\psi^{-1}\left(Ct_+(g) \bigcap \psi(\Gamma)\right) = Ct_S(f) \bigcap \Gamma, \quad \psi^{-1}\left(Ct_+\left(g^{\psi(\Gamma)}\right)\right) = Ct_S\left(f^\Gamma\right). \tag{11}$$

This, together with equality (10) and properties (a) and (b), shows that $Ct_S(f) \bigcap \Gamma = Ct_S(f^\Gamma)$ and the $Ct_S(f)$ intersects Γ transversally if f^Γ is nondegenerate with respect to Γ. This completes the proof. □

The following result is a generalization of Farin's theorem (see [3, 15]) on high dimensional space.

Proposition 5 (see [3, 15]). *Let*

$$p^{(1)}(\tau) = \sum_{|\lambda|=n} b_\lambda^{(1)} B_{\lambda,n}(\tau), \qquad p^{(2)}(\overline{\tau}) = \sum_{|\lambda|=n} b_\lambda^{(2)} B_{\lambda,n}(\overline{\tau}) \tag{12}$$

be BB-polynomials of degree n that are defined on two adjacent k-dimensional simplices $S_1 = [v_1, v_2, \ldots, v_{k+1}]$ and $S_2 = [v_1', v_2, \ldots, v_{k+1}]$, respectively. Then $p^{(1)}(\tau)$ and $p^{(2)}(\overline{\tau})$ are C^r smoothly connected on $[v_2, \ldots, v_{k+1}]$ if and only if for all $\rho \in \{0, 1, \ldots, r\}$ and $\lambda_2, \ldots, \lambda_{k+1} \in \mathbb{Z}_+$ with $\sum_{j=2}^{k+1} \lambda_j = n - \rho$,

$$b_{\lambda^\rho}^{(2)} = \sum_{|\epsilon|=\rho} b_{\epsilon+\lambda^0}^{(1)} B_{\epsilon,\rho}\left(\tau'\right), \tag{13}$$

where $\lambda^0 = (0, \lambda_2, \dots, \lambda_{k+1})$, $\lambda^\rho = (\rho, \lambda_2, \dots, \lambda_{k+1})$, and $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_{k+1})$. Here, τ' denotes the barycentric coordinate of v_1' with respect to the simplex S_1.

Theorem 6. *Suppose that $p^{(1)}(\tau)$ and $p^{(2)}(\overline{\tau})$ defined as above are $C^r (r \geq 0)$ smoothly connected on $S_1 \bigcap S_2 = [v_2, \dots, v_{k+1}]$. Then,*

(1) *$N(p^{(1)}) \bigcap (S_1 \bigcap S_2) = N(p^{(2)}) \bigcap (S_1 \bigcap S_2)$, and $\Gamma := N(p^{(1)}) \bigcap (S_1 \bigcap S_2)$ is a face of $N(p^{(1)})$ and $N(p^{(2)})$ if $N(p^{(1)}) \bigcap (S_1 \bigcap S_2)$ is nonempty, where $N(p^{(1)})$ and $N(p^{(2)})$ are Newton polyhedra of BB-polynomials $p^{(1)}$ and $p^{(2)}$, respectively;*

(2) *$Ct_{S_1}(p^{(1)}) \bigcap \Gamma = Ct_{S_2}(p^{(2)}) \bigcap \Gamma$ for Γ above.*

Proof. By assumption and taking $\rho = 0$ in equality (13), we have that $b^{(1)}_{\lambda^0} = b^{(2)}_{\lambda^0}$ for all $\lambda_2, \dots, \lambda_{k+1} \in \mathbb{Z}_+$ with $\sum_{j=2}^{k+1} \lambda_j = n$. Thus, $\mu(p^{(1)}) \bigcap (S_1 \bigcap S_2) = \mu(p^{(2)}) \bigcap (S_1 \bigcap S_2)$, where μ is defined in (6). This implies that $N(p^{(1)}) \bigcap (S_1 \bigcap S_2) = N(p^{(2)}) \bigcap (S_1 \bigcap S_2)$, and so $\Gamma = N(p^{(1)}) \bigcap (S_1 \bigcap S_2)$ is a face of $N(p^{(1)})$ and $N(p^{(2)})$ if $N(p^{(1)}) \bigcap (S_1 \bigcap S_2)$ is nonempty.

According to the argument above, it is easy to see that Γ-truncations $(p^{(1)})^\Gamma$, $(p^{(2)})^\Gamma$ of $p^{(1)}$ and $p^{(2)}$ satisfy the equality $(p^{(1)})^\Gamma = (p^{(2)})^\Gamma$. Thus, we can get that

$$Ct_{S_1}\left(\left(p^{(1)}\right)^\Gamma\right) = Ct_{S_2}\left(\left(p^{(2)}\right)^\Gamma\right) \tag{14}$$

by the definition of the *chart*.

On the other hand, it follows from Theorem 4 that

$$\begin{gathered} Ct_{S_1}\left(p^{(1)}\right) \bigcap \Gamma = Ct_{S_1}\left(\left(p^{(1)}\right)^\Gamma\right), \\ Ct_{S_2}\left(p^{(2)}\right) \bigcap \Gamma = Ct_{S_2}\left(\left(p^{(2)}\right)^\Gamma\right). \end{gathered} \tag{15}$$

Therefore,

$$Ct_{S_1}\left(p^{(1)}\right) \bigcap \Gamma = Ct_{S_2}\left(p^{(2)}\right) \bigcap \Gamma \tag{16}$$

by equality (14). This completes the proof. □

Let Δ be a pure, hereditary simplicial complex with k-cells (i.e., k-dimensional simplex) $S_1, \dots, S_q$. Let $f \in C^r_n(\Delta)$ and each polynomial $f|_{S_\alpha}$ be expressed in the B-form

$$f|_{S_\alpha}(\tau) = \sum_{|\lambda|=n} b^{[\alpha]}_\lambda B_{\lambda,n}(\tau), \quad \alpha = 1, 2, \dots, q. \tag{17}$$

The piecewise polynomial f defined as above is called C^r piecewise BB-polynomial.

Theorem 7. *Let $f \in C^r_n(\Delta)$ be a piecewise BB-polynomial defined in (17). Then, for each adjacent pair S_α, S_β of k-cells in Δ,*

(1) *$N(f|_{S_\alpha}) \bigcap (S_\alpha \bigcap S_\beta) = N(f|_{S_\beta}) \bigcap (S_\alpha \bigcap S_\beta)$, and $\Gamma_{\alpha\beta} := N(f|_{S_\alpha}) \bigcap (S_\alpha \bigcap S_\beta)$ is a face of $N(f|_{S_\alpha})$ and $N(f|_{S_\beta})$ if $N(f|_{S_\alpha}) \bigcap (S_\alpha \bigcap S_\beta)$ is nonempty, where $N(f|_{S_\alpha})$ and $N(f|_{S_\beta})$ are Newton polyhedra of BB-polynomials $f|_{S_\alpha}$ and $f|_{S_\beta}$, respectively;*

(2) *$Ct_{S_\alpha}(f|_{S_\alpha}) \bigcap \Gamma_{\alpha\beta} = Ct_{S_\beta}(f|_{S_\beta}) \bigcap \Gamma_{\alpha\beta}$ for $\Gamma_{\alpha\beta}$ above.*

Proof. We can get the conclusion from Theorem 6 immediately. □

According to Theorems 4–7, we can define the *chart* of a piecewise BB-polynomial.

Definition 8. Let $f \in C^r_n(\Delta)$ be a piecewise BB-polynomial defined in (17), and let $N(f|_{S_\alpha})$ be the Newton polyhedron of the BB-polynomial $f|_{S_\alpha}$. The chart $Ct(f)$ of f is the closure of $\bigcup_{\alpha=1}^q \phi_{\mu(f|_{S_\alpha})}(\{\tau \in \mathrm{Int}(S_\alpha), f|_{S_\alpha}(\tau) = 0\})$, where $\mu(f|_{S_\alpha}) = \{(\lambda_1/n, \dots, \lambda_{k+1}/n) \mid \lambda \in \mathbb{Z}_{+(k,n)}, b^{[\alpha]}_\lambda \neq 0\}$ and $\phi_{\mu(f|_{S_\alpha})} : \mathrm{Int}(S_\alpha) \to \mathrm{Int}(N(f|_{S_\alpha}))$ is the moment map associated with $\mu(f|_{S_\alpha})$ relative to the simplex S_α.

Obviously, we can get the following conclusion from Theorems 4–7 and the definition above immediately.

Theorem 9. *Let $f \in C^r_n(\Delta)$ be a piecewise BB-polynomial defined in (17). Then, the chart $Ct(f)$ of f is $\bigcup_{\alpha=1}^q Ct_{S_\alpha}(f|_{S_\alpha})$.*

4. The Construction of Real C^r Piecewise Algebraic Hypersurfaces

In this section, we propose a new method for the construction of real piecewise algebraic hypersurfaces of a given degree with certain smoothness and prescribed topology. The method is primarily based on the work of Lai et al. in [18] and the smooth blending theory.

Let Δ be a pure, hereditary simplicial complex with k-cells (i.e., k-dimensional simplex) $S_1, \dots, S_q$.

For each adjacent pair $S_\alpha = [v_{(\alpha,1)}, v_{(\alpha,2)}, \dots, v_{(\alpha,k+1)}]$, $S_\beta = [v_{(\beta,1)}, v_{(\beta,2)}, \dots, v_{(\beta,k+1)}]$ of k-cells in Δ, assume that $S_\alpha \cap S_\beta = [v_{(\alpha,1)}, \dots, v_{(\alpha,i-1)}, \widehat{v_{(\alpha,i)}}, v_{(\alpha,i+1)}, \dots, v_{(\alpha,k+1)}] = [v_{(\beta,1)}, \dots, v_{(\beta,j-1)}, \widehat{v_{(\beta,j)}}, v_{(\beta,j+1)}, \dots, v_{(\beta,k+1)}]$, where the hat means that the corresponding vertex is omitted. Obviously, there exist two arrangements $l_1, l_2, \dots, l_{k+1}$ and $d_1, d_2, \dots, d_{k+1}$ of numbers $1, 2, \dots, k+1$ such that $v_{(\alpha,l_1)} = v_{(\alpha,i)}$, $v_{(\beta,d_1)} = v_{(\beta,j)}$ and $v_{(\alpha,l_w)} = v_{(\beta,d_w)}$ for $w = 2, \dots, k+1$.

Now, we define two one-to-one transformations $\eta_{\alpha\beta}$, $\eta_{\beta\alpha} : (\mathbb{R}_+)^{k+1} \to (\mathbb{R}_+)^{k+1}$, by $\eta_{\alpha\beta}(\lambda) = (\lambda_{l_1}, \lambda_{l_2}, \dots, \lambda_{l_{k+1}})$, and $\eta_{\beta\alpha}(\lambda) = (\lambda_{d_1}, \lambda_{d_2}, \dots, \lambda_{d_{k+1}})$, respectively, where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{k+1}) \in (\mathbb{R}_+)^{k+1}$.

For any given $\alpha \in \{1, \dots, q\}$, let $f^{(\alpha,1)}, \dots, f^{(\alpha,m_\alpha)}$ be BB-polynomials over the simplex $S_\alpha = [v_{(\alpha,1)}, v_{(\alpha,2)}, \dots, v_{(\alpha,k+1)}]$ with $N(f^{(\alpha,\zeta)}) = P_{(\alpha,\zeta)}$, $\zeta = 1, \dots, m_\alpha$, and let

$$f^{[\alpha]}_t(\tau) = \sum_{|\lambda|=n} b^{[\alpha]}_\lambda B_{\lambda,n}(\tau)\, t^{\nu^{[\alpha]}(\lambda/n)} \tag{18}$$

be a *patchworking* of BB-polynomials $f^{(\alpha,1)}, \dots, f^{(\alpha,m_\alpha)}$ by $\nu^{[\alpha]}$.

Define the piecewise BB-polynomial f_t on the domain Ω by

$$f_t\big|_{S_\alpha}(\tau) = f_t^{[\alpha]}(\tau), \qquad \alpha = 1, 2, \dots, q. \tag{19}$$

Theorem 10. *Suppose that $f^{(1,1)}, \dots, f^{(1,m_1)}, f^{(2,1)}, \dots, f^{(q,m_q)}$ are non-degenerate, that $f_t^{[\alpha]}$ is a patchworking of BB-polynomials $f^{(\alpha,1)}, \dots, f^{(\alpha,m_\alpha)}$ by $\nu^{[\alpha]}$, $\alpha = 1, \dots, q$, and that S_α, S_β, $v_{(\alpha,i)}$, $v_{(\beta,j)}$, $\eta_{\alpha\beta}$, $\eta_{\beta\alpha}$, $P_{(\alpha,\zeta)}$, $f_t^{[\alpha]}$, and f_t are defined as above, and $Int(P_{(\alpha,\zeta)}) \bigcap Int(P_{(\alpha,\varsigma)}) = \emptyset$ for $\zeta \neq \varsigma$. If for each adjacent pair S_α, S_β of k-cells in Δ and all $\rho \in \{0, 1, \dots, r\}$ and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{k+1}) \in \mathbb{Z}_{+(k,n)}$ with $\sum_{i=2}^{k+1} \lambda_i = n - \rho$,*

$$\begin{aligned} & b^{[\beta]}_{(\eta_{\beta\alpha}^{-1}(\lambda^\rho))} t^{\nu^{[\beta]}(\eta_{\beta\alpha}^{-1}(\lambda^\rho)/n)} \\ &\quad = \sum_{|\epsilon|=\rho} b^{[\alpha]}_{\eta_{\alpha\beta}^{-1}(\epsilon+\lambda^0)} B_{\epsilon,\rho}\left(\tau'_{\beta\alpha}\right) t^{\nu^{[\alpha]}(\eta_{\alpha\beta}^{-1}(\epsilon+\lambda^0)/n)}, \quad t \in \mathbb{R}_+^*, \end{aligned} \tag{20}$$

where $\lambda^\rho = (\rho, \lambda_2, \dots, \lambda_{k+1})$ and $\tau'_{\beta\alpha}$ denotes the barycentric coordinate of $v_{(\beta,j)}$ with respect to the simplex $S_\alpha = [v_{(\alpha,l_1)}, v_{(\alpha,l_2)}, \dots, v_{(\alpha,l_{k+1})}]$, then,

(1) $f_t \in C_n^r(\Delta)$;

(2) *if for any BB-polynomial $f^{(\alpha,i)}$ with $N(f^{(\alpha,i)}) \bigcap (S_\alpha \bigcap S_\beta) \neq \emptyset$, $i \in \{1, \dots, m_\alpha\}$, there exists a BB-polynomial $f^{(\beta,j)}$ in $\{f^{(\beta,1)}, \dots, f^{(\beta,m_\beta)}\}$ such that $N(f^{(\beta,j)}) \bigcap (S_\alpha \bigcap S_\beta) = N(f^{(\alpha,i)}) \bigcap (S_\alpha \bigcap S_\beta)$, then there is $t_0 > 0$ such that for any $t \in (0, t_0]$, the chart $Ct(f_t)$ of piecewise BB-polynomial f_t, up to isotopy, is $\bigcup_{\alpha=1}^q \bigcup_{i=1}^{m_\alpha} Ct_{S_\alpha}(f^{(\alpha,i)})$.*

Proof. Let $\tau = (\tau_1, \dots, \tau_{k+1})$ and $\chi = (\chi_1, \chi_2, \dots, \chi_{k+1})$ be the barycentric coordinates of point $p \in S_\beta$ with respect to $S_\beta = [v_{(\beta,1)}, v_{(\beta,2)}, \dots, v_{(\beta,k+1)}]$ and $S_\beta = [v_{(\beta,d_1)}, v_{(\beta,d_2)}, \dots, v_{(\beta,d_{k+1})}]$, respectively. Then,

$$\chi = (\chi_1, \chi_2, \dots, \chi_{k+1}) = \eta_{\beta\alpha}(\tau) = \left(\tau_{d_1}, \tau_{d_2}, \dots, \tau_{d_{k+1}}\right) \tag{21}$$

by the definition of the transformation $\eta_{\beta\alpha}$, and so

$$\begin{aligned} f_t^{[\beta]}(p) &= f_t^{[\beta]}(\tau) \\ &= \sum_{|\lambda|=n} b_\lambda^{[\beta]} \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_{k+1}!} \\ &\quad \times \tau_1^{\lambda_1}\tau_2^{\lambda_2}\cdots\tau_{k+1}^{\lambda_{k+1}} t^{\nu^{[\beta]}(\lambda/n)} \\ &= \sum_{|\lambda|=n} b_\lambda^{[\beta]} \frac{n!}{\lambda_{d_1}!\lambda_{d_1}!\cdots\lambda_{d_{k+1}}!} \\ &\quad \times (\chi_1)^{\lambda_{d_1}}(\chi_2)^{\lambda_{d_2}}\cdots(\chi_{k+1})^{\lambda_{d_{k+1}}} t^{\nu^{[\beta]}(\lambda/n)}. \end{aligned} \tag{22}$$

Set $\mu = (\mu_1, \mu_2, \dots, \mu_{k+1}) := \eta_{\beta\alpha}(\lambda)$. Then, $\lambda = \eta_{\beta\alpha}^{-1}(\mu)$ and $(\mu_1, \mu_2, \dots, \mu_{k+1}) = (\lambda_{d_1}, \lambda_{d_2}, \dots, \lambda_{d_{k+1}})$. Therefore, it follows from equality (22) that

$$\begin{aligned} f_t^{[\beta]}(p) &= \sum_{|\mu|=n} b^{[\beta]}_{\eta_{\beta\alpha}^{-1}(\mu)} t^{\nu^{[\beta]}(\eta_{\beta\alpha}^{-1}(\mu)/n)} \\ &\quad \times \frac{n!}{\mu_1!\mu_1!\cdots\mu_{k+1}!}(\chi_1)^{\mu_1}(\chi_2)^{\mu_2}\cdots(\chi_{k+1})^{\mu_{k+1}}. \end{aligned} \tag{23}$$

This shows that the polynomial $f_t^{[\beta]}$ over $S_\beta = [v_{(\beta,d_1)}, v_{(\beta,d_2)}, \dots, v_{(\beta,d_{k+1})}]$ can be represented as follows:

$$f_t^{[\beta]}(\chi) = \sum_{|\mu|=n} b^{[\beta]}_{\eta_{\beta\alpha}^{-1}(\mu)} t^{\nu^{[\beta]}(\eta_{\beta\alpha}^{-1}(\mu)/n)} B_{\mu,n}(\chi). \tag{24}$$

By a similar argument above, we have that the polynomial $f_t^{[\alpha]}$ over $S_\alpha = [v_{(\alpha,l_1)}, \dots, v_{(\alpha,l_{k+1})}]$ can be represented as follows

$$f_t^{[\alpha]}(\overline{\chi}) = \sum_{|\mu|=n} b^{[\alpha]}_{\eta_{\alpha\beta}^{-1}(\mu)} t^{\nu^{[\alpha]}(\eta_{\alpha\beta}^{-1}(\mu)/n)} B_{\mu,n}(\overline{\chi}), \tag{25}$$

where $\overline{\chi} = (\overline{\chi}_1, \overline{\chi}_2, \dots, \overline{\chi}_{k+1})$ is the barycentric coordinate of point $p \in S_\alpha$ with respect to $S_\alpha = [v_{(\alpha,l_1)}, v_{(\alpha,l_2)}, \dots, v_{(\alpha,l_{k+1})}]$.

Since $v_{(\alpha,l_w)} = v_{(\beta,d_w)}$ for $w = 2, \dots, k+1$, $S_\alpha \bigcap S_\beta = [v_{(\alpha,l_2)}, \dots, v_{(\alpha,l_{k+1})}]$; thus, according to Proposition 5, $f_t \in C_n^r(\Delta)$ if and only if the BB-coefficients $b^{[\beta]}_{\eta_{\beta\alpha}^{-1}(\mu)} t^{\nu^{[\beta]}(\eta_{\beta\alpha}^{-1}(\mu)/n)}$ of $f_t^{[\beta]}$ and the BB-coefficients $b^{[\alpha]}_{\eta_{\alpha\beta}^{-1}(\mu)} t^{\nu^{[\alpha]}(\eta_{\alpha\beta}^{-1}(\mu)/n)}$ of $f_t^{[\alpha]}$ satisfy equality (20). This proves property (1).

Below we will show that property (2) holds.

For each adjacent pair S_α, S_β of k-cells in Δ, set $\Gamma_{\alpha\beta} = N(f^{(\alpha,i)}) \bigcap (S_\alpha \bigcap S_\beta)$ when $N(f^{(\alpha,i)}) \bigcap (S_\alpha \bigcap S_\beta) \neq \emptyset$; then, $\Gamma_{\alpha\beta}$ is a face of $N(f^{(\alpha,i)})$ and $N(f^{(\beta,j)})$ by the assumption that $N(f^{(\beta,j)}) \bigcap (S_\alpha \bigcap S_\beta) = N(f^{(\alpha,i)}) \bigcap (S_\alpha \bigcap S_\beta)$. Since $f_t^{[\alpha]}$ (resp., $f_t^{[\beta]}$) is a patchworking of BB-polynomials $f^{(\alpha,1)}, \dots, f^{(\alpha,m_\alpha)}$ (resp., $f^{(\beta,1)}, \dots, f^{(\beta,m_\beta)}$) by $\nu^{[\alpha]}$ (resp., $\nu^{[\beta]}$) and $\mathrm{Int}(P_{(\alpha,\zeta)}) \bigcap \mathrm{Int}(P_{(\alpha,\varsigma)}) = \mathrm{Int}(P_{(\beta,\zeta)}) \bigcap \mathrm{Int}(P_{(\beta,\varsigma)}) = \emptyset$ for $\zeta \neq \varsigma$, then, for $\Gamma_{\alpha\beta}$-truncations $(f^{(\alpha,i)})^{\Gamma_{\alpha\beta}}$, $(f^{(\beta,j)})^{\Gamma_{\alpha\beta}}$, $(f_t^{[\alpha]})^{\Gamma_{\alpha\beta}}$ and $(f_t^{[\beta]})^{\Gamma_{\alpha\beta}}$ of $f^{(\alpha,i)}$, $f^{(\beta,j)}$, $f_t^{[\alpha]}$, and $f_t^{[\beta]}$, respectively, we have that

$$\begin{aligned} \left(f^{(\alpha,i)}\right)^{\Gamma_{\alpha\beta}} &= \left(f_t^{[\alpha]}\right)^{\Gamma_{\alpha\beta}}\Big|_{t=1}, \\ \left(f^{(\beta,j)}\right)^{\Gamma_{\alpha\beta}} &= \left(f_t^{[\beta]}\right)^{\Gamma_{\alpha\beta}}\Big|_{t=1}. \end{aligned} \tag{26}$$

Since $f_t^{[\alpha]}$ and $f_t^{[\beta]}$ are C^r smoothly connected on $S_\alpha \bigcap S_\beta$, $N(f^{(\beta,j)}) \bigcap (S_\alpha \bigcap S_\beta) = N(f^{(\alpha,i)}) \bigcap (S_\alpha \bigcap S_\beta)$ and (20) implies that $(f_t^{[\alpha]})^{\Gamma_{\alpha\beta}} = (f_t^{[\beta]})^{\Gamma_{\alpha\beta}}$. Thus, $(f^{(\alpha,i)})^{\Gamma_{\alpha\beta}} = (f^{(\beta,j)})^{\Gamma_{\alpha\beta}}$ by (26).

Using a similar argument in the proof of Theorem 6, we have $Ct_{S_\alpha}(f^{(\alpha,i)}) \bigcap \Gamma_{\alpha\beta} = Ct_{S_\beta}(f^{(\beta,j)}) \bigcap \Gamma_{\alpha\beta}$ by $(f^{(\alpha,i)})^{\Gamma_{\alpha\beta}} = (f^{(\beta,j)})^{\Gamma_{\alpha\beta}}$. This shows that

$$\bigcup_{i=1}^{m_\alpha} Ct_{S_\alpha}\left(f^{(\alpha,i)}\right) \bigcap \left(S_\alpha \bigcap S_\beta\right) = \bigcup_{j=1}^{m_\beta} Ct_{S_\beta}\left(f^{(\beta,j)}\right) \bigcap \left(S_\alpha \bigcap S_\beta\right). \tag{27}$$

On the other hand, since $f_t^{[\alpha]}$ is a patchworking of BB-polynomials $f^{(\alpha,1)}, \ldots, f^{(\alpha,m_\alpha)}$ by $\nu^{[\alpha]}$, we get from the assumption and Proposition 3 that there is $t_0^{[\alpha]} > 0$ such that for any $t \in (0, t_0^{[\alpha]}]$ the chart $Ct_{S_\alpha}(f_t^{[\alpha]})$ of BB-polynomial $f_t^{[\alpha]}$ (i.e., $f_t|_{S_\alpha}$), up to isotopy, is $\bigcup_{i=1}^{m_\alpha} Ct_{S_\alpha}(f^{(\alpha,i)})$. This conclusion together with Theorems 7–9 and equality (27) implies that there is $t_0 > 0$ such that for any $t \in (0, t_0]$ the chart $Ct(f_t)$ of piecewise BB-polynomial f_t, up to isotopy, is $\bigcup_{\alpha=1}^{q} \bigcup_{i=1}^{m_\alpha} Ct_{S_\alpha}(f^{(\alpha,i)})$, where $t_0 = \min\{t_0^{[1]}, \ldots, t_0^{[q]}\}$. This completes the proof. □

Assume that P is a k-dimensional simplex with domain points in S as its vertices and that the BB-polynomial f is a real $k+1$-nomial (this means that the nonzero coefficients in f correspond to the only vertices of P). Denote by $M(P)$ the set of middle points of edges of P. For any point $v \in M(P)$ on an edge with endpoints $(i_1/n, \ldots, i_{k+1}/n)$, $(j_1/n, \ldots, j_{k+1}/n)$, we assign that

$$\delta(v) = \operatorname{sign}\left(b_{(i_1,\ldots,i_{k+1})} \cdot b_{(j_1,\ldots,j_{k+1})}\right), \tag{28}$$

where $b_{(i_1,\ldots,i_{k+1})}, b_{(j_1,\ldots,j_{k+1})}$ are the corresponding BB-coefficients in f. Put

$$M_-(P) = \{v \in M(P) \mid \delta(v) = -1\}. \tag{29}$$

Theorem 11. *Suppose that* $f^{(1,1)}, \ldots, f^{(1,m_1)}, \ldots, f^{(q,1)}, \ldots, f^{(q,m_q)}, P_{(1,1)}, \ldots, P_{(1,m_1)}, \ldots, P_{(q,1)}, \ldots, P_{(q,m_q)}, f_t^{[\alpha]}$, *and* f_t *satisfy the conditions in Theorem 10 and that for each* $\alpha \in \{1, 2, \ldots, q\}$, $f^{(\alpha,1)}, \ldots, f^{(\alpha,m_\alpha)}$ *are real* $(k+1)$*-nomial and* $P_{(\alpha,1)}, \ldots, P_{(\alpha,m_\alpha)}$ *with dimension* k *form a triangulation of* $N(f_t^{[\alpha]})$*. Then,* $f_t \in C_n^r(\Delta)$ *and there exists a constant* $t_0 > 0$ *such that for any* $t \in (0, t_0]$ *the chart* $Ct(f)$ *of* f*, up to isotopy, is* $\bigcup_{\alpha=1}^{q} \bigcup_{i=1}^{m_\alpha} conv(M_-(P_{(\alpha,i)}))$*, where* $M_-(P_{(\alpha,i)})$ *is defined as* (29).

Proof. By assumption and [18, Theorem 4.4], we know that there is $t_0^{[\alpha]} > 0$ such that for any $t \in (0, t_0^{[\alpha]}]$ the chart $Ct_{S_\alpha}(f_t^{[\alpha]})$ of BB-polynomial $f_t^{[\alpha]}$, up to isotopy, is $\bigcup_{i=1}^{m_\alpha} \operatorname{conv}(M_-(P_{(\alpha,i)}))$. □

The remainder of the proof can be completed by a similar approach in the proof of Theorem 10.

According to Theorems 10 and 11, if we want to construct a C^r piecewise algebraic hypersurface $f(\tau) = 0$ with a prescribed complex topology and a degree on a partition Δ, we can just construct some Bernstein-Bézier algebraic hypersurfaces pieces $f^{(\alpha,1)}(\tau) = 0, \ldots, f^{(\alpha,m_\alpha)}(\tau) = 0$ with simple topology and a convex function $\nu^{[\alpha]}$ on each k-cell S_α, where $f^{(\alpha,1)}, \ldots, f^{(\alpha,m_\alpha)}$, and $\nu^{[\alpha]}$ satisfy the conditions of Theorem 10 or Theorem 11, $\alpha = 1, 2, \ldots, q$, and then, the BB-polynomial $f^{[\alpha]}(\tau)$ is obtained by a patchworking of BB-polynomials $f^{(\alpha,1)}, \ldots, f^{(\alpha,m_\alpha)}$ by $\nu^{[\alpha]}$. Thus, we can get $f(\tau)$ by defining $f|_{S_\alpha}(\tau) := f^{[\alpha]}(\tau)$.

Conflict of Interests

The authors do not have any possible conflict of interests in this paper.

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