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Research Article

Solving a System of Linear Volterra Integral Equations Using the Modified Reproducing Kernel Method

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A numerical technique based on reproducing kernel methods for the exact solution of linear Volterra integral equations system of the second kind is given. The traditional reproducing kernel method requests that operator a satisfied linear operator equation $Au = f$, is bounded and its image space is the reproducing kernel space $W_2^1[a,b]$. It limits its application. Now, we modify the reproducing kernel method such that it can be more widely applicable. The n-term approximation solution obtained by the modified method is of high accuracy. The numerical example compared with other methods shows that the modified method is more efficient.

1. Introduction

The purpose of this paper is to solve a system of linear Volterra integral equations

$$F(s) = Gs + \int_a^b K(s,t)\,F(t)\,dt, \quad s \in [0,1], \tag{1}$$

where

$$\begin{aligned} F(s) &= [f_1(s), f_2(s), \ldots, f_n(s)]^T, \\ G(s) &= [g_1(s), g_2(s), \ldots, g_n(s)]^T, \\ K(s,t) &= [k_{i,j}], \quad i,j = 1,2,\ldots,n. \end{aligned} \tag{2}$$

In (1), the functions K and G are given, and F is the solution to be determined. We assume that (1) has a unique solution. Volterra integral equation arises in many physical applications, for example, potential theory and Dirichlet problems, electrostatics, mathematical problems of radiative equilibrium, the particle transport problems of astrophysics and reactor theory, and radiative heat transfer problems [1–5]. Several valid methods for solving Volterra integral equation have been developed in recent years, including power series method [6], Adomain's decomposition method [7], homotopy perturbation method [8, 9], block by block method [10], and expansion method [11].

Since the reproducing kernel space $W_2^1[a,b]$, which is a special Hilbert space, is constructed in 1986 [12], the reproducing kernel theory has been applied successfully to many linear and nonlinear problems, such as differential equation, population model, and many other equations appearing in physics and engineering [12–21]. The traditional reproducing kernel method is limited, because it requires that the image space of operator A in linear operator equation $Au = f$ is $W_2^1[a,b]$ and operator A must be bounded. In order to enlarge its application range, the MRKM removes the boundedness of A and weakens its image space to $L^2[a,b]$. Subsequently, we apply the MRKM to obtain the series expression of the exact solution for (1). The n-term approximation solution is provided by truncating the series. The final numerical comparisons between our method and other methods show the efficiency of the proposed method. It is worth to mention that the MRKM can be generalized to solve other system of linear equations.

2. Preliminaries

2.1. The Reproducing Kernel Space $W_2^1[0,1]$. The reproducing kernel space $W_2^1[0,1]$ consists of all absolute continuous real-valued functions, which defined on the closed interval $[0,1]$, and the first derivative functions belong to $L^2[0,1]$.

The inner product and the norm are equipped with

$$(u,v)_{w_2^1} = u(0)\,v(0) + \int_0^1 u'(x)\,v'(x)\,dx, \quad \forall u,v \in w_2^1,$$
$$\|u\|_{W_2^1} = \sqrt{(u,v)_{w_2^1}}. \tag{3}$$

Theorem 1. *$W_2^1[0,1]$ is a reproducing kernel space with reproducing kernel [22]*

$$R_x(y) = \begin{cases} 1+y, & y \le x \\ 1+x, & y > x; \end{cases} \tag{4}$$

that is, for every $x \in [0,1]$ and $u \in W_2^1$, it follows that

$$(u(y), R_x(y))_{w_2^1} = u(x). \tag{5}$$

2.2. The Reproducing Kernel Space $W_2^2[0,1]$. The reproducing kernel space $W_2^2[0,1]$ consists of all real-valued functions in which the first derivative functions are absolute continuous on the closed interval $[0,1]$ and the second derivative functions belong to $L^2[0,1]$.

The inner product and the norm are equipped with

$$(u,v)_{W_2^2} = \sum_{k=0}^{1} u^{(k)}(0)\,v^{(k)}(0) + \int_0^1 u''(x)\,v''(x)\,dx, \quad \forall u,v \in W_2^2[0,1], \tag{6}$$
$$\|u\|_{W_2^2} = \sqrt{(u,u)_{w_2^2}}.$$

Theorem 2. *$W_2^2[0,1]$ is a reproducing kernel space with reproducing kernel [22]*

$$Q(x,y) = \begin{cases} 1 + x\times y + \dfrac{x\times y^2}{2} - \dfrac{y^3}{6} & y \le x \\ 1 + x\times y + \dfrac{x^2\times y}{2} - \dfrac{x^3}{6}, & y > x; \end{cases} \tag{7}$$

that is, for every $x \in [0,1]$ and $u \in W_2^2$, it follows that

$$(u(y), Q(x,y))_{w_2^2} = u(x). \tag{8}$$

The proof of Theorems 1 and 2 can be found in [23].

2.3. Hilbert Space E. Hilbert space E is defined by

$$E = \bigoplus_{i=1}^{n} W_2^1 = \left\{(u_1,\dots,u_n)^T \mid u_i \in w_2^1, i = 1,\dots,n\right\}. \tag{9}$$

The inner product and the norm are given by

$$(u,v)_E = \sum_{i=1}^{n} (u_i, v_i)_{w_2^1},$$
$$\|u\|_E = \sqrt{(u,u)_E}. \tag{10}$$

It is easy to prove that E is a Hilbert space.

3. The Exact Solution of (1)

3.1. Identical Transformation of (1). Consider the ith equation of (1):

$$f_i(s) - \sum_{j=1}^{n} \int_0^s K_{ij}(s,t)\,f_j(t)\,dt = g_i(s). \tag{11}$$

Define operator $A_{ij} : W_2^1 \to L^2[0,1]$, $j = 1,\dots,n$,

$$A_{ij} = \begin{cases} u(s) - \int_0^1 k_{ij}(s,t)\,u(t)\,dt, & j = i \\ -\displaystyle\int_0^s k_{ij}(s,t)\,u(t)\,dt, & j \ne i, \end{cases} \tag{12}$$

where $u \in W_2^1$. Then, (1) can be turned into

$$\begin{aligned} A_{11}f_1 + A_{12}f_2 + \cdots + A_{1n}f_{1n} &= g_1(s) \\ A_{21}f_1 + A_{22}f_2 + \cdots + A_{2n}f_{1n} &= g_2(s) \\ &\vdots \\ A_{n1}f_1 + A_{n2}f_2 + \cdots + A_{nn}f_{1n} &= g_n(s), \end{aligned} \tag{13}$$

where $F(s) = [f_1(s), f_2(s), \dots, f_n(s)]^T \in E$.

3.2. The Exact solution of (1). Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of interval $[0,1]$, and define

$$\Psi_{ij}(x) = \left(A_{j_1,y}R_x(y)\big|_{y=x_i}, A_{j_2,y}R_x(y)\big|_{y=x_i}, \dots, A_{j_n,y}R_x(y)\big|_{y=x_i} \right)^T \tag{14}$$

for every $j = 1,2,\dots,n$, $i = 1,2,\dots$; the subscript y of $A_{ij,y}$ means that the operator A_{ij} acts on the function of y. It is easy to prove that $\Psi_{ij} \in E$.

Theorem 3. *$\{\Psi_{i1}, \Psi_{i2}, \dots, \Psi_{in}\}_{i=1}^{\infty}$ is complete in E.*

Proof. Take $u = (u_1, u_2, \dots, u_n)^T \in E$ such that $(u(x), \Psi_{ij}(x)) = 0$ for every $j = 1,2,\dots,n$, $i = 1,2,\dots$.

From this fact, it holds that

$$\begin{aligned} &(u(x), \Psi_{ij}(x)) \\ &= \Big((u_1, u_2, \dots, u_n)^T, \Big(A_{j1,y}R_x(y)\big|_{y=x_i}, A_{j2,y}R_x(y)\big|_{y=x_i}, \dots, A_{jn,y}R_x(y)\big|_{y=x_i} \Big)^T \Big) \\ &= \sum_{k=1}^{n} A_{jk,y}(u_k(x), R_x(y))_{w_2^1}\big|_{y=x_i} \\ &= \sum_{k=1}^{n} A_{jk}u_k(x_i) = 0, \end{aligned} \tag{15}$$

for every $j = 1, 2, \ldots, n$. The dense $\{x_i\}_{i=1}^{\infty}$ assumes that

$$\begin{gathered} A_{11}u_1 + A_{12}u_2 + \cdots + A_{1n}u_n = 0 \\ A_{21}u_1 + A_{22}u_2 + \cdots + A_{2n}u_n = 0 \\ \vdots \\ A_{n1}u_1 + A_{n2}u_2 + \cdots + A_{nn}u_n = 0. \end{gathered} \tag{16}$$

Since (16) has a unique solution, it follows that $u = u_1, u_2, \ldots, u_n{}^T = 0$. This completes the proof. □

We arrange $\Psi_{11}, \Psi_{12}, \ldots, \Psi_{1n}, \Psi_{21}, \Psi_{22}, \ldots, \Psi_{2n}, \ldots, \Psi_{i1}, \Psi_{i2}, \ldots, \Psi_{in}, \ldots$, denoted by $\{r_i\}_{i=1}^{\infty}$; that is, $r_1 = \Psi_{11}, r_2 = \Psi_{12}, \ldots, r_n = \Psi_{1n}, r_{n+1} = \Psi_{21}, r_{n+2} = \Psi_{22}, \ldots, r_{n+n} = \Psi_{2n}, \ldots$. In a general way, $r_{(i-1)n+j} = \Psi_{ij}$, $i = 1, 2, 3, \ldots$; $j = 1, 2, \ldots, n$. The orthogonal basis $\{\overline{r}_i\}_{i=1}^{\infty}$ in E from Gram-Schmidt orthogonalization of $\{r_i\}_{i=1}^{\infty}$ is as follows:

$$\overline{r}_i = \sum_{k=1}^{i} \beta_{ik} r_k, \quad i = 1, 2, \ldots. \tag{17}$$

Theorem 4. *The exact solution of* (1) *can be expressed by*

$$F(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \rho_k \overline{r}_i(x), \tag{18}$$

where $\rho_k = (F(x), r_k)_E$*; if* $r_k = \Psi_{jl}$*, then* $\rho_k = g_l(x_j)$.

Proof. Assume that $F(x)$ is the exact solution of (1). $F(x)$ can be expanded to Fourier series in terms of normal orthogonal basis $\{\overline{r}_i(x)\}_{i=1}^{\infty}$ in E:

$$F(x) = \sum_{i=1}^{\infty} (F, \overline{r}_i)_E \overline{r}_i(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} (F, r_k)_E \overline{r}_i(x); \tag{19}$$

if $\rho_k = (F, r_k)_E$, then

$$F(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ij} \rho_k \overline{r}_i(x). \tag{20}$$

When $r_k = \Psi_{jl}$, it holds that

$$\rho_k = \left(F, \Psi_{jl}\right) = \sum_{k=1}^{n} A_{lk} u_k\left(x_j\right) = g_l\left(x_j\right). \tag{21}$$

□

Corollary 5. *The approximate solution of* (1) *is*

$$F_m(x) = \sum_{i=1}^{m} \sum_{k=1}^{i} \beta_{ik} \rho_k \overline{r}_i(x) = \left(f_{1,m}, f_{2,m}, \ldots, f_{n,m}\right)^T, \tag{22}$$

and $f_{i,m}(x)$ *converges uniformly to* $f_i(x)$ *on* $[0,1]$ *as* $m \to \infty$ *for every* $i = 1, 2, \ldots, n$.

Proof. Obviously, $\|F_m - F\|_E^2 \to 0$ holds as $m \to \infty$; that is, $F_m(x)$ is the approximate solution of (1).

Note that $\sum_{i=1}^{n} \|f_{i,m} - f_i\|_{W_2^1}^2 = \|F_m - F\|_E^2 \to 0$. Combining with the expression of $R_x(y)$, we have

$$\begin{aligned} \left|f_{i,m} - f_i\right| &= \left|\left(f_{i,m}(y) - f_i(y), R_x(y)\right)_{W_2^1}\right| \\ &\leq \left\|f_{i,m} - f_i\right\|_{W_2^1} \cdot \left\|R_x(y)\right\|_{W_2^1} \\ &= \left\|f_{i,m} - f_i\right\|_{W_2^1} \sqrt{R_x(x)} \\ &\leq \sqrt{2}\left\|f_{i,m} - f_i\right\|_{W_2^1}, \quad \forall x \in [0,1]. \end{aligned} \tag{23}$$

It shows that $f_{i,m}$ converges uniformly to f_i on $[0,1]$ as $m \to \infty$ for every $i = 1, 2, \ldots, n$. So the proof is complete. □

Remark 6. If $k_{ij}(s,t) \in C([0,1] \times [0,1])$ and $g_i \in W_2^2$ in (1), then it is reasonable to regard the unknown functions as the elements of W_2^2.

4. Numerical Examples

Taking nodes $\{x_i = (i-1)/(N-1)\}_{i=1}^{N}$, $f_{i,N}$ is the approximate solutions of f_i, and $e(f_{i,N})$ denotes the absolute errors of f_i, $i = 1, 2, \ldots, n$. According to Remark 6, we solve the following two examples appearing in [11] in W_2^2.

Example 7. Consider the following system of Volterra integral equations of the second kind [11]:

$$\begin{aligned} f_1(s) &= g_1(s) + \int_0^s (s-t)^3 f_1(t)\, dt + \int_0^s (s-t)^2 f_2(t)\, dt, \\ f_2(s) &= g_2(s) + \int_0^s (s-t)^4 f_1(t)\, dt \\ &\quad + \int_0^s (s-t)^3 f_2(t)\, dt, \end{aligned} \tag{24}$$

where $g_1(s)$ and $g_2(s)$ are chosen such that the exact solution is $f_1(s) = 1 + s^2$, $f_2(s) = 1 + s - s^3$. The numerical results obtained by using the present method are compared with [11] in Table 1.

Example 8. Consider the following system of linear Volterra integral equations of the second kind [11]:

$$\begin{aligned} f_1(s) &= g_1(s) + \int_0^s (\sin(s-t) - 1) f_1(t)\, dt \\ &\quad + \int_0^s (1 - t\cos s) f_2(t)\, dt, \\ f_2(s) &= g_2(s) + \int_0^s \left(f_1(t)\right) dt + \int_0^s (s-t) f_2(t)\, dt, \end{aligned} \tag{25}$$

where $g_1(s)$ and $g_2(s)$ are chosen such that the exact solution is $f_1(s) = \cos s, f_2(s) = \sin s$. The numerical results obtained by using the present method are compared with [11] in Table 2.

Table 1: Absolute errors for Example 7.

Nodes x_i	Errors $e(f_1)$ [11]	Errors $e(f_{1,100})$	Errors $e(f_2)$ [11]	Errors $e(f_{2,100})$ [11]
0.0	0	$1.58309E-10$	0	$3.98245E-10$
0.1	$2.63472E-7$	$3.92220E-12$	$2.11685E-8$	$3.94493E-10$
0.2	$1.62592E-5$	$3.21563E-10$	$2.61132E-6$	$4.72710E-10$
0.3	$1.74905E-4$	$5.95890E-10$	$4.18979E-5$	$6.19366E-10$
0.4	$8.93799E-4$	$5.11051E-10$	$2.86285E-4$	$6.95422E-10$
0.5	$3.00491E-3$	$2.46104E-10$	$1.19940E-3$	$4.19959E-10$
0.6	$7.47528E-3$	$1.98685E-9$	$3.56141E-3$	$5.90035E-10$
0.7	$1.40733E-2$	$5.01512E-9$	$7.74239E-3$	$2.83080E-9$
0.8	$1.78384E-2$	$9.62848E-9$	$1.09171E-2$	$6.94058E-9$
0.9	$4.97756E-3$	$1.61180E-8$	$2.27326E-3$	$1.36984E-8$
1.0	$3.84378E-2$	$2.49043E-8$	$3.32111E-2$	$2.42565E-8$

Table 2: Absolute errors for Example 8.

Nodes x_i	Errors $e(f_1)$ [11]	Errors $e(f_{1,100})$	Errors $e(f_2)$ [11]	Errors $e(f_{2,100})$ [11]
0.0	0	$6.93348E-11$	0	$3.60316E-11$
0.1	$1.37735E-4$	$4.53518E-09$	$1.52721E-4$	$2.75123E-08$
0.2	$9.27188E-4$	$8.84879E-09$	$1.14715E-3$	$3.10611E-08$
0.3	$2.67117E-3$	$1.28253E-08$	$3.71248E-3$	$3.53307E-08$
0.4	$5.45507E-3$	$1.65442E-08$	$8.57201E-3$	$4.03402E-08$
0.5	$9.22670E-3$	$2.00881E-08$	$1.64412E-2$	$4.61209E-08$
0.6	$1.38644E-2$	$2.35657E-09$	$2.78243E-2$	$5.27214E-08$
0.7	$1.92960E-2$	$2.71160E-08$	$4.25337E-2$	$6.02041E-08$
0.8	$2.56349E-2$	$3.09302E-08$	$5.91212E-2$	$6.86601E-08$
0.9	$3.31574E-2$	$3.52645E-08$	$7.48883E-2$	$7.82029E-08$
1.0	$4.19808E-2$	$3.67322E-08$	$8.70896E-2$	$1.02387E-07$

5. Conclusion

In this paper, we modify the traditional reproducing kernel method to enlarge its application range. The new method named MRKM is applied successfully to solve a system of linear Volterra integral equations. The numerical results show that our method is effective. It is worth to be pointed out that the MRKM is still suitable for solving other systems of linear equations.

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