

## Research Article

# Asymptotic Behavior of Solutions to a Linear Volterra Integrodifferential System

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We investigate the asymptotic behavior of solutions to a linear Volterra integrodifferential system  $x_i'(t) = a_i(t) + b_i(t)x_i(t) + \sum_{j=1}^n \int_0^t K_{ij}(t,s)x_j(s)ds$ ,  $t \in \mathbb{R}^+$ ,  $i = 1, 2, \dots, n$ . We show that under some suitable conditions, there exists a solution for the above integrodifferential system, which is asymptotically equivalent to some given functions. Two examples are given to illustrate our theorem.

## 1. Introduction

Throughout this paper, we denote by  $\mathbb{N}$  the set of positive integers, by  $\mathbb{R}$  the set of all real numbers, by  $\mathbb{R}^+$  the set of all nonnegative real numbers, and by  $\mathbb{R}^n$  the set of all  $n$ -dimensional real vectors. Moreover,  $BC(\mathbb{R}^+, \mathbb{R}^n)$  denotes the Banach space of all bounded and continuous functions  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^n$  with the norm

$$\|f\| = \sup_{t \in \mathbb{R}^+} \max_{1 \leq j \leq n} |f_j(t)|, \quad (1)$$

where  $f(t) = (f_1(t), \dots, f_n(t))^T$  for  $t \in \mathbb{R}^+$ .

The aim of this paper is to study some asymptotic behavior of solutions to the following linear Volterra integrodifferential system:

$$x_i'(t) = a_i(t) + b_i(t)x_i(t) + \sum_{j=1}^n \int_0^t K_{ij}(t,s)x_j(s)ds, \quad (2)$$

$$t \in \mathbb{R}^+, \quad i = 1, 2, \dots, n,$$

where  $a_i, b_i : \mathbb{R}^+ \rightarrow \mathbb{R}$  and  $K_{ij} : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ ,  $i, j = 1, 2, \dots, n$  are all continuous functions.

**Definition 1.** We call  $x = (x_1, x_2, \dots, x_n)^T : \mathbb{R}^+ \rightarrow \mathbb{R}^n$  a solution of system (2) if  $x$  is continuously differentiable and satisfies (2).

The asymptotic behavior of solutions has been an important and interesting topic in the qualitative theory of differential and difference equations. Especially, recently, many authors have made interesting and important contributions on the asymptotic behavior of solutions for Volterra type difference equations (e.g., we refer the reader to [1–10] and references therein).

Very recently, Diblík and Schmeidel [6] obtained a very interesting result concerning the asymptotic behavior of solutions for the following linear Volterra difference equation:

$$x(n+1) = a(n) + b(n)x(n) + \sum_{i=0}^n K(n,i)x(i). \quad (3)$$

More specifically, they proved that for every admissible constant  $c \in \mathbb{R}$ , there exists a solution  $x = x(n)$  of (3) such that

$$x(n) \sim \left( c + \sum_{i=0}^{n-1} \frac{a(i)}{\beta(i+1)} \right) \beta(n), \quad n \rightarrow \infty, \quad (4)$$

where  $\beta(n) = \prod_{i=0}^{n-1} b(i)$ . However, to the best of our knowledge, it seems that there is no literature concerning such asymptotic behavior of solutions for Volterra type differential equations. That is the main motivation of this paper. In this paper, we will adopt the idea in the proof of [6] to investigate

some asymptotic behaviors of solutions for Volterra differential system (2).

### 2. Main Result

Before establishing our main result, we first give an “Arzela-Ascoli” type theorem for the subsets of  $BC(\mathbb{R}^+, \mathbb{R}^n)$ .

**Lemma 2.** *Let  $\mathcal{F} \subset BC(\mathbb{R}^+, \mathbb{R}^n)$ , satisfying (i)  $\mathcal{F}$  is uniformly bounded; (ii)  $\mathcal{F}$  is equiuniformly continuous on every compact subset of  $\mathbb{R}^+$ ; (iii) for every  $\varepsilon > 0$ , there exist  $f_\varepsilon \in BC(\mathbb{R}^+, \mathbb{R}^n)$  and  $T_\varepsilon > 0$  such that  $\|f(t) - f_\varepsilon(t)\| < \varepsilon$  for all  $f \in \mathcal{F}$  and  $t \geq T_\varepsilon$ . Then  $\mathcal{F}$  is precompact in  $BC(\mathbb{R}^+, \mathbb{R}^n)$ .*

*Proof.* By the condition (iii), for every  $k \in \mathbb{N}$ , there exist  $T_k > 0$  such that

$$\|F_1(t) - F_2(t)\| < \frac{1}{k} \tag{5}$$

for all  $F_1, F_2 \in \mathcal{F}$  and  $t \geq T_k$ .

Let  $\{f_n\}$  be a sequence in  $\mathcal{F}$ . By (i) and (ii), it follows from Arzela-Ascoli theorem that for every  $k \in \mathbb{N}$ , there exists a subsequence  $\{f_n^k\} \subset \{f_n\}$  such that  $\{f_n^k\}$  is uniformly convergent on  $[0, T_k]$ . Then, by choosing the diagonal sequence, we can get a subsequence  $\{f_m\} \subset \{f_n\}$  such that, for every  $k \in \mathbb{N}$ ,  $\{f_m\}$  is uniformly convergent on  $[0, T_k]$ .

It remains to show that  $\{f_m\}$  is uniformly convergent on  $\mathbb{R}^+$ . For every  $\varepsilon > 0$ , choose  $k_0 \in \mathbb{N}$  with  $1/k_0 < \varepsilon$ . Since  $\{f_m\}$  is uniformly convergent on  $[0, T_{k_0}]$ , for the above  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$\sup_{t \in [0, T_{k_0}]} \|f_{m_1}(t) - f_{m_2}(t)\| \leq \varepsilon \tag{6}$$

for all  $m_1, m_2 \geq N$ ; Combining this with (5), we conclude that

$$\sup_{t \in \mathbb{R}^+} \|f_{m_1}(t) - f_{m_2}(t)\| \leq \varepsilon \tag{7}$$

for all  $m_1, m_2 \geq N$ , that is,  $\{f_m\}$  is uniformly convergent on  $\mathbb{R}^+$ . This completes the proof.  $\square$

Throughout the rest of this paper, for every  $i \in \{1, 2, \dots, n\}$ , we assume that

$$A_i = \sup_{t \in \mathbb{R}^+} |A_i(t)| < +\infty, \tag{8}$$

where

$$\begin{aligned} A_i(t) &= \int_0^t \frac{a_i(s)}{\beta_i(s)} ds, \\ \beta_i(t) &= \exp\left(\int_0^t b_i(s) ds\right), \quad t \in \mathbb{R}^+, \end{aligned} \tag{9}$$

$$0 \leq M_i := \sum_{j=1}^n \int_0^{+\infty} \left(\int_0^t |K_{ij}(t,s) \frac{\beta_j(s)}{\beta_i(t)}| ds\right) dt < 1.$$

**Theorem 3.** *Assume that*

$$\begin{aligned} 0 < \liminf_{t \rightarrow +\infty} \beta_i(t) \leq \limsup_{t \rightarrow +\infty} \beta_i(t) < +\infty, \\ i = 1, 2, \dots, n. \end{aligned} \tag{10}$$

*Let  $c = (c_1, c_2, \dots, c_n)^T \in \mathbb{R}^n$  with  $c_i + A_i(t) \geq 0$  for all  $i \in \{1, 2, \dots, n\}$  and  $t \in \mathbb{R}^+$ . Then, there exists a solution  $x = (x_1, x_2, \dots, x_n)^T : \mathbb{R}^+ \rightarrow \mathbb{R}^n$  of system (2) such that*

$$x_i(t) \sim (c_i + A_i(t)) \beta_i(t), \quad t \rightarrow +\infty, \quad i = 1, 2, \dots, n, \tag{11}$$

*provided that  $\liminf_{t \rightarrow +\infty} (c_i + A_i(t)) > 0$ .*

*Proof.* We define that

$$\begin{aligned} \alpha_i(0) &= \frac{(c_i + A_i) M_i}{1 - M_i}, \\ \alpha_i(t) &= (c_i + A_i + \alpha_i(0)) \end{aligned} \tag{12}$$

$$\times \sum_{j=1}^n \int_t^{+\infty} \left( \int_0^s \left| K_{ij}(s,p) \frac{\beta_j(p)}{\beta_i(s)} \right| dp \right) ds,$$

for all  $i \in \{1, 2, \dots, n\}$  and  $t \in \mathbb{R}^+$ . Moreover, we define an operator  $\rho = (\rho_1, \rho_2, \dots, \rho_n)^T$  on

$$\begin{aligned} S &= \{z \in BC(\mathbb{R}^+, \mathbb{R}^n) : c_i + A_i(t) - \alpha_i(0) \\ &\leq z_i(t) \leq c_i + A_i(t) + \alpha_i(0), \\ &i = 1, 2, \dots, n, \quad t \in \mathbb{R}^+\} \end{aligned} \tag{13}$$

by

$$\begin{aligned} (\rho_i z)(t) &= c_i + A_i(t) \\ &- \sum_{j=1}^n \int_t^{+\infty} \left( \int_0^s K_{ij}(s,p) z_j(p) \frac{\beta_j(p)}{\beta_i(s)} dp \right) ds, \\ &i = 1, 2, \dots, n, \end{aligned} \tag{14}$$

for  $t \in \mathbb{R}^+$  and  $z \in S$ . It is easy to see that  $S$  is a nonempty, closed, and convex set in  $BC(\mathbb{R}^+, \mathbb{R}^n)$ . Next, we divide the remaining proof into two steps.

*Step 1.*  $\rho(S) \subset S$ ,  $\rho$  is continuous, and  $\overline{\rho(S)}$  is compact.

Let  $z \in S$ . We have

$$\begin{aligned} |(\rho_i z)(t)| &\leq |c_i| + A_i + M_i \cdot \|z\| < +\infty, \\ t \in \mathbb{R}^+, \quad i &= 1, 2, \dots, n. \end{aligned} \tag{15}$$

In addition, since  $z \in S$ , we have

$$|z_i(t)| \leq c_i + A_i + \alpha_i(0), \quad t \in \mathbb{R}^+, \quad i = 1, 2, \dots, n. \tag{16}$$

Then, it follows that

$$\begin{aligned} & |(\rho_i z)(t) - (c_i + A_i(t))| \\ & \leq \sum_{j=1}^n \int_t^{+\infty} \left( \int_0^s \left| K_{ij}(s, p) z_j(p) \frac{\beta_j(p)}{\beta_i(s)} \right| dp \right) ds \quad (17) \\ & \leq [c_i + A_i + \alpha_i(0)] \cdot M_i = \alpha_i(0) \end{aligned}$$

for all  $i \in \{1, 2, \dots, n\}$  and  $t \in \mathbb{R}^+$ . Thus, we conclude that  $\rho(S) \subset S$ .

For every  $\varepsilon > 0$ , there exists a constant  $\delta = \varepsilon / \max_{1 \leq i \leq n} M_i$  such that for all  $z, y \in S$  with  $\|z - y\| < \delta$ , we have

$$\begin{aligned} & |(\rho_i z)(t) - (\rho_i y)(t)| \\ & = \left| \sum_{j=1}^n \int_t^{+\infty} \left( \int_0^s K_{ij}(s, p) z_j(p) \frac{\beta_j(p)}{\beta_i(s)} dp \right) ds \right. \\ & \quad \left. - \sum_{j=1}^n \int_t^{+\infty} \left( \int_0^s K_{ij}(s, p) y_j(p) \frac{\beta_j(p)}{\beta_i(s)} dp \right) ds \right| \\ & \leq \delta \cdot M_i \leq \varepsilon, \quad i \in \{1, 2, \dots, n\}, \quad t \in \mathbb{R}^+, \quad (18) \end{aligned}$$

which means that  $\rho$  is continuous.

Next, we show that  $\rho(S)$  is precompact. Firstly, for every  $x \in S$ , we have

$$\begin{aligned} \|\rho x\| &= \sup_{t \in \mathbb{R}^+} \max_{1 \leq i \leq n} |(\rho_i x)(t)| \\ & \leq \max_{1 \leq i \leq n} [c_i + A_i + \alpha_i(0)], \quad (19) \end{aligned}$$

which means that  $\rho(S)$  is uniformly bounded. Secondly, for every  $z \in S, t_1, t_2 \in \mathbb{R}^+$  and  $i = 1, 2, \dots, n$ , we have

$$\begin{aligned} & |(\rho_i z)(t_1) - (\rho_i z)(t_2)| \\ & = \left| \sum_{j=1}^n \int_{t_1}^{+\infty} \left( \int_0^s K_{ij}(s, p) z_j(p) \frac{\beta_j(p)}{\beta_i(s)} dp \right) ds \right. \\ & \quad \left. - \sum_{j=1}^n \int_{t_2}^{+\infty} \left( \int_0^s K_{ij}(s, p) z_j(p) \frac{\beta_j(p)}{\beta_i(s)} dp \right) ds \right| \\ & \leq \max_{1 \leq i \leq n} [c_i + A_i + \alpha_i(0)] \\ & \quad \cdot \left| \sum_{j=1}^n \int_{t_1}^{t_2} \left( \int_0^s \left| K_{ij}(s, p) \frac{\beta_j(p)}{\beta_i(s)} \right| dp \right) ds \right|, \quad (20) \end{aligned}$$

which yields that  $\rho(S)$  is equiuniformly continuous on every compact subsets of  $\mathbb{R}^+$ . Thirdly, by the definition of  $M_i$ , for

every  $\varepsilon > 0$ , there exists  $T > 0$  such that for all  $t \geq T$  and  $z \in S$ , we have

$$\begin{aligned} & \sum_{j=1}^n \int_t^{+\infty} \left( \int_0^s \left| K_{ij}(s, p) \frac{\beta_j(p)}{\beta_i(s)} \right| dp \right) ds \\ & < \frac{\varepsilon}{\max_{1 \leq i \leq n} [c_i + A_i + \alpha_i(0)]}, \quad i = 1, 2, \dots, n, \quad (21) \end{aligned}$$

which yields that

$$\|\rho_i z - \rho_i 0\| < \varepsilon, \quad i = 1, 2, \dots, n, \quad (22)$$

and thus  $\|\rho z - \rho 0\| < \varepsilon$ . Then, by Lemma 2, we know that  $\rho(S)$  is precompact.

*Step 2.* By Step 1 and Schauder's fixed-point theorem,  $\rho$  has a fixed point in  $S$ ; that is, there exists  $z^0 = (z_1^0, z_2^0, \dots, z_n^0)^T \in S$  such that

$$\begin{aligned} & z_i^0(t) = c_i + A_i(t) \\ & \quad - \sum_{j=1}^n \int_t^{+\infty} \left( \int_0^s K_{ij}(s, p) z_j^0(p) \frac{\beta_j(p)}{\beta_i(s)} dp \right) ds, \quad (23) \end{aligned}$$

for all  $i \in \{1, 2, \dots, n\}$  and  $t \in \mathbb{R}^+$ . Noting that

$$\sup_{z \in S} \|z\| \leq \max_{1 \leq i \leq n} [c_i + A_i + \alpha_i(0)], \quad (24)$$

we have

$$\begin{aligned} & |z_i^0(t) - (c_i + A_i(t))| \\ & \leq \max_{1 \leq i \leq n} [c_i + A_i + \alpha_i(0)] \\ & \quad \cdot \sum_{j=1}^n \int_t^{+\infty} \left( \int_0^s \left| K_{ij}(s, p) \frac{\beta_j(p)}{\beta_i(s)} \right| dp \right) ds, \quad (25) \end{aligned}$$

for all  $i \in \{1, 2, \dots, n\}$  and  $t \in \mathbb{R}^+$ . Then, it is easy to see that

$$\lim_{t \rightarrow +\infty} |z_i^0(t) - (c_i + A_i(t))| = 0, \quad i = 1, 2, \dots, n, \quad (26)$$

Combining this with

$$\liminf_{t \rightarrow +\infty} (c_i + A_i(t)) > 0, \quad (27)$$

we have

$$\lim_{t \rightarrow +\infty} \frac{z_i^0(t)}{c_i + A_i(t)} = 1, \quad i = 1, 2, \dots, n, \quad (28)$$

that is,

$$z_i^0(t) \sim c_i + A_i(t), \quad t \rightarrow +\infty, \quad i = 1, 2, \dots, n. \quad (29)$$

Now, define a function  $x = (x_1, x_2, \dots, x_n)^T : \mathbb{R}^+ \rightarrow \mathbb{R}^n$  by

$$x_i(t) = z_i^0(t) \beta_i(t), \quad i = 1, 2, \dots, n, \quad t \in \mathbb{R}^+. \quad (30)$$

It follows from (23) that

$$\begin{aligned} \frac{d}{dt} z_i^0(t) &= \frac{a_i(t)}{\beta_i(t)} + \sum_{j=1}^n \int_0^t K_{ij}(t, p) z_j^0(p) \frac{\beta_j(p)}{\beta_i(t)} dp, \\ i &= 1, 2, \dots, n, \quad t \in \mathbb{R}^+, \end{aligned} \tag{31}$$

which yields that

$$\begin{aligned} \frac{x_i'(t) \beta_i(t) - x_i(t) \beta_i'(t)}{\beta_i^2(t)} &= \frac{a_i(t)}{\beta_i(t)} + \sum_{j=1}^n \int_0^t K_{ij}(t, s) \frac{x_j(s)}{\beta_i(t)} ds, \\ i &= 1, 2, \dots, n, \quad t \in \mathbb{R}^+. \end{aligned} \tag{32}$$

Then, we get

$$\begin{aligned} x_i'(t) &= a_i(t) + b_i(t) x_i(t) \\ &+ \sum_{j=1}^n \int_0^t K_{ij}(t, s) x_j(s) ds, \\ i &= 1, 2, \dots, n, \quad t \in \mathbb{R}^+, \end{aligned} \tag{33}$$

which means that  $x$  is a solution to system (2). In addition, combining (28) with the assumption

$$0 < \liminf_{t \rightarrow +\infty} \beta_i(t) \leq \limsup_{t \rightarrow +\infty} \beta_i(t) < +\infty, \tag{34}$$

$$i = 1, 2, \dots, n,$$

we get

$$\lim_{t \rightarrow +\infty} \frac{x_i(t)}{(c_i + A_i(t)) \beta_i(t)} = 1, \quad i = 1, 2, \dots, n, \tag{35}$$

which yields (11). □

*Example 4.* Let  $n = 1$ , and for all  $t, s \in \mathbb{R}^+$ ,

$$\begin{aligned} a_1(t) &= \exp(\sin \pi t) \cos t, \\ b_1(t) &= \pi \cos \pi t, \\ K_{11}(t, s) &= \frac{\exp(\sin \pi t)}{(1+t+s)^3 \exp(\sin \pi s)}. \end{aligned} \tag{36}$$

Then, for all  $t \in \mathbb{R}^+$ , we have  $\beta_1(t) = \exp(\sin \pi t)$ ,

$$\begin{aligned} A_1(t) &= \int_0^t \frac{a_1(s)}{\beta_1(s)} ds = \sin t, \\ A_1 &= \sup_{t \in \mathbb{R}^+} \{|A_1(t)|\} = 1 \in (0, +\infty), \\ M_1 &= \int_0^{+\infty} \left( \int_0^t \left| K_{11}(t, s) \frac{\beta(s)}{\beta(t)} \right| ds \right) dt \end{aligned}$$

$$\begin{aligned} &= \int_0^{+\infty} \left( \int_0^t \frac{1}{(1+t+s)^3} ds \right) dt \\ &= \int_0^{+\infty} \frac{1}{2} \left[ \frac{1}{(1+t)^2} - \frac{1}{(1+2t)^2} \right] dt \\ &= \frac{1}{2} \int_0^{+\infty} \frac{1}{(1+t)^2} dt - \frac{1}{2} \int_0^{+\infty} \frac{1}{(1+2t)^2} dt \\ &= \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \in (0, 1). \end{aligned} \tag{37}$$

In addition, it is easy to see that

$$\begin{aligned} 0 < e^{-1} &= \liminf_{t \rightarrow +\infty} \beta_1(t) \\ &\leq \limsup_{t \rightarrow +\infty} \beta_1(t) = e < +\infty. \end{aligned} \tag{38}$$

Thus, by Theorem 3, we conclude that for every  $c > 1$ , there exists a solution  $x : \mathbb{R}^+ \rightarrow \mathbb{R}$  for (2) such that

$$x(t) \sim (c + \sin t) \exp(\sin \pi t), \quad t \rightarrow +\infty. \tag{39}$$

*Remark 5.* It is needed to note that in the above example,  $(c + \sin t) \exp(\sin \pi t)$  is not a solution to (2).

*Example 6.* Consider the following system:

$$\begin{aligned} x_i'(t) &= a_i(t) + b_i(t) x_i(t) \\ &+ \sum_{j=1}^2 \int_0^t K_{ij}(t, s) x_j(s) ds, \quad i = 1, 2, \end{aligned} \tag{40}$$

where

$$\begin{aligned} a_1(t) &= \exp(\sin \pi t) \cos t, \\ a_2(t) &= -\exp(\cos \pi t) \sin t, \\ b_1(t) &= \pi \cos \pi t, \\ b_2(t) &= -\pi \sin \pi t, \end{aligned} \tag{41}$$

$$K_{ij}(t, s) = \frac{(-1)^{i+j} \exp(\sin \pi t)}{16 (1+t+s)^3 \exp(\sin \pi s)}$$

for all  $i, j = 1, 2$ , and  $t, s \in \mathbb{R}^+$ . By a direct calculation, we get

$$\begin{aligned} \beta_1(t) &= \exp(\sin \pi t), \\ \beta_2(t) &= \exp(-1 + \cos \pi t), \\ A_1(t) &= \int_0^t \frac{a_1(s)}{\beta_1(s)} ds = \sin t, \quad A_1 = 1, \\ A_2(t) &= \int_0^t \frac{a_2(s)}{\beta_2(s)} ds = (-1 + \cos t) e, \quad t \in \mathbb{R}^+, \quad A_2 = 2e, \end{aligned}$$

$$\begin{aligned}
 M_1 &= \sum_{j=1}^2 \int_0^{+\infty} \left( \int_0^t \left| K_{1j}(t,s) \frac{\beta_j(s)}{\beta_1(t)} \right| ds \right) dt \\
 &\leq \frac{1+e}{64} < 1, \\
 M_2 &= \sum_{j=1}^2 \int_0^{+\infty} \left( \int_0^t \left| K_{2j}(t,s) \frac{\beta_j(s)}{\beta_2(t)} \right| ds \right) dt \\
 &\leq \frac{e+2e^3}{64} < 1.
 \end{aligned}
 \tag{42}$$

Moreover, we have

$$\begin{aligned}
 0 < e^{-1} &= \liminf_{t \rightarrow +\infty} \beta_1(t) \\
 &\leq \limsup_{t \rightarrow +\infty} \beta_1(t) = e < +\infty, \\
 0 < e^{-2} &= \liminf_{t \rightarrow +\infty} \beta_2(t) \\
 &\leq \limsup_{t \rightarrow +\infty} \beta_2(t) = 1 < +\infty.
 \end{aligned}
 \tag{43}$$

Then, by Theorem 3, for every  $c = (c_1, c_2)^T \in \mathbb{R}^2$  with  $c_1 > 1$  and  $c_2 > 2e$ , there exists a solution  $x = (x_1, x_2) : \mathbb{R}^+ \rightarrow \mathbb{R}^2$  of system (40) such that

$$\begin{aligned}
 x_1(t) &\sim (c_1 + \sin t) \exp(\sin \pi t), \quad t \rightarrow +\infty, \\
 x_2(t) &\sim [c_2 + (-1 + \cos t)e] \exp(-1 + \cos \pi t), \quad t \rightarrow +\infty.
 \end{aligned}
 \tag{44}$$

### Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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