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Research Article

Stability of a Functional Equation Deriving from Quadratic and Additive Functions in Non-Archimedean Normed Spaces

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We obtain the general solution of the generalized mixed additive and quadratic functional equation $f(x+my)+f(x-my)=2f(x)-2m^2f(y)+m^2f(2y)$, m is even; $f(x+y)+f(x-y)-2(m^2-1)f(y)+(m^2-1)f(2y)$, m is odd, for a positive integer m. We establish the Hyers-Ulam stability for these functional equations in non-Archimedean normed spaces when m is an even positive integer or $m=3$.

1. Introduction

The basic problem of the stability of functional equations was formulated by Ulam in 1940 in the following form. Suppose that a mapping f satisfies the additive functional equation $f(x+y)=f(x)+f(y)$ only approximately. Then does there exist an additive function which approximates f? (See also [1].) In 1941, Hyers [2] gave the following answer to this question for Banach spaces. The result of Hyers was generalized in 1950 by Aoki [3] for approximately additive mappings and in 1978 by Rassias [4] for approximately linear mappings, by considering the unbounded Cauchy differences. A further generalization was obtained by Găvruţa [5] in 1994, by replacing the Cauchy differences by a control function φ satisfying a very simple condition of convergence.

The Hyers-Ulam stability problem for the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1}$$

was first proved by Skof for a function $f:\mathcal{X}\to\mathcal{Y}$, where $\mathcal{X}$ is a normed space and $\mathcal{Y}$ is a Banach space [6]. One year later, Cholewa [7] demonstrated that Skof's theorem is still true if relevant domain is replaced by an abelian group. After that, in [8], Czerwik proved the Hyers-Ulam stability of the quadratic functional equation (1) as a special case. In [9], it was shown that a mapping $f:\mathcal{X}\to\mathcal{Y}$ is quadratic if and only if $f(kx+y)+f(kx-y)=2k^2f(x)+2f(y)$ for all $x,y\in\mathcal{X}$. Also, f is quadratic if and only if $2f((kx+ky)/2)+2f((kx-ky)/2)=k^2f(x)+k^2f(y)$ for all $x,y\in\mathcal{X}$ [10]. Cădariu and Radu investigated the stability of the Cauchy functional equation [11] and for the quadratic functional equation [12]. Stability problems of miscellaneous functional equations have been investigated by several authors during the last decades (see, e.g., [13–15]).

In [16], Eskandani et al. determined the general solution of the following mixed additive and quadratic functional equation:

$$f(x+2y)+f(x-2y)+8f(y)=2f(x)+4f(2y). \tag{2}$$

They studied the Hyers-Ulam stability of (2) in non-Archimedean Banach modules over a unital Banach algebra. In [17], Najati and Moghimi established the general solution of the mixed type additive and quadratic functional equation

$$\begin{aligned}f(2x+y)+f(2x-y)\\ =f(x+y)+f(x-y)+2f(2x)-2f(x)\end{aligned} \tag{3}$$

and investigated the stability of this equation in quasi-Banach spaces. The stability of (3) in random normed spaces is proved in [18].

In this paper, we consider the following functional equations:

$$f(x+my)+f(x-my)=2f(x)-2m^2f(y)+m^2f(2y), \tag{4}$$

where m is an even positive integer and

$$\begin{aligned} &f(x+3y)+f(x-3y) \\ &\quad =f(x+y)+f(x-y)-16f(y)+8f(2y). \end{aligned} \tag{5}$$

Indeed, (4) and (5) are different from (2) and (3). It is easily verified that the function $f(x)=\alpha x^2+\beta x$ is a solution of the functional equations (4) and (5). We show that these functional equations are mixed additive and quadratic mappings. We also prove the Hyers-Ulam stability problem for these equations. As a corollary, the hyperstability of (4) and (5) under some conditions in non-Archimedean normed spaces is shown as well.

2. General Solution of (4) and (5)

To achieve our aim in this paper, we need the following lemma which is a fundamental tool.

Lemma 1. *Let $\mathscr{X}$ and $\mathscr{Y}$ be real vector spaces.*

(i) *If an odd mapping $f:\mathscr{X}\to\mathscr{Y}$ satisfies the functional equation (4), then f is additive.*

(ii) *If an odd mapping $f:\mathscr{X}\to\mathscr{Y}$ satisfies the functional equation (5), then f is additive.*

(iii) *If an even mapping $f:\mathscr{X}\to\mathscr{Y}$ satisfies the functional equation (4), then f is quadratic.*

(iv) *If an even mapping $f:\mathscr{X}\to\mathscr{Y}$ satisfies the functional equation (5), then f is quadratic.*

Proof. (i) Letting $x=0$ in (4), we get $f(2y)=2f(y)$ for all $y\in\mathscr{X}$. This equality implies that

$$f(x+my)+f(x-my)=2f(x) \tag{6}$$

for all $x,y\in\mathscr{X}$. Replacing y by y/m in (6), we have

$$f(x+y)+f(x-y)=2f(x) \tag{7}$$

for all $x,y\in\mathscr{X}$. Substituting x,y by y,x in (7), respectively, we obtain

$$f(x+y)-f(x-y)=2f(y) \tag{8}$$

for all $x,y\in\mathscr{X}$. The equalities (7) and (8) show that

$$f(x+y)=f(x)+f(y) \quad (x,y\in\mathscr{X}). \tag{9}$$

(ii) Suppose that f satisfies (5). Similar to the part (i), by the oddness of f, we have $f(2x)=2f(x)$ for all $x\in\mathscr{X}$. Thus

$$f(x+3y)+f(x-3y)=f(x+y)+f(x-y) \tag{10}$$

for all $x,y\in\mathscr{X}$. We substitute x by $x+y$ in (10) and then x by $x-y$ in (10); we get

$$f(x+4y)+f(x-2y)=f(x+2y)+f(x), \tag{11}$$

$$f(x+2y)+f(x-4y)=f(x)+f(x-2y) \tag{12}$$

for all $x,y\in\mathscr{X}$. Then, by adding (11) to (12), we lead to

$$f(x+4y)+f(x-4y)=2f(x) \tag{13}$$

for all $x,y\in\mathscr{X}$. Similar to the part (i), we can show that f is additive.

(iii) By the assumption, the equality

$$f(x+my)+f(x-my)=2f(x)-2m^2f(y)+m^2f(2y) \tag{14}$$

holds for a fixed even positive integer m. Putting $x=y=0$ in (14), we get $f(0)=0$. Once more, by letting $x=0$ in (14), we have

$$2f(my)=-2m^2f(y)+m^2f(2y) \tag{15}$$

for all $y\in\mathscr{X}$. Interchanging x,y into mx,x in (14), respectively, we deduce that

$$f(2mx)=2f(mx)-2m^2f(x)+m^2f(2x) \tag{16}$$

for all $x\in\mathscr{X}$. Plugging (15) into (16), we have $f(2mx)=4f(mx)$ for all $x\in\mathscr{X}$ and thus $f(2x)=4f(x)$ for all $x\in\mathscr{X}$. Using the last equality and (14), we have

$$f(x+my)+f(x-my)=2f(x)+2m^2f(y) \tag{17}$$

for all $x,y\in\mathscr{X}$. Setting $x=0$ in (17), we obtain $f(my)=m^2f(y)$ for all $y\in\mathscr{X}$. Applying this equality and putting x by mx in (17), we get

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{18}$$

for all $x,y\in\mathscr{X}$. This shows that f is a quadratic mapping.

(iv) Suppose that f satisfies (5). Replacing x by $x+y$ and $x-y$ in (5), respectively, we have

$$\begin{aligned} &f(x+4y)+f(x-2y) \\ &\quad =f(x+2y)+f(x)-16f(y)+8f(2y), \end{aligned} \tag{19}$$

$$\begin{aligned} &f(x+2y)+f(x-4y) \\ &\quad =f(x)+f(x-2y)-16f(y)+8f(2y) \end{aligned} \tag{20}$$

for all $x,y\in\mathscr{X}$. The equalities (19) and (20) imply that

$$f(x+4y)+f(x-4y)=2f(x)-32f(y)+16f(2y) \tag{21}$$

for all $x,y\in\mathscr{X}$. Now, the above equality is a special case of the part (iii) when $m=4$. □

In the following theorem, we solve (4) in which m is an even positive integer and

$$\begin{aligned} f(x+my)+f(x-my) &= f(x+y)+f(x-y) \\ &\quad -2\left(m^2-1\right)f(y)+\left(m^2-1\right)f(2y), \end{aligned} \tag{22}$$

where m is an odd positive integer.

Theorem 2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be real vector spaces. Then a mapping $f:\mathcal{X}\to\mathcal{Y}$ satisfies the functional equation (2) if and only if it satisfies*

$$f(x+my)+f(x-my) = \begin{cases} 2f(x)-2m^2f(y)+m^2f(2y), & m \text{ is even}, \\ f(x+y)+f(x-y) \\ \quad -2\left(m^2-1\right)f(y)+\left(m^2-1\right)f(2y), & m \text{ is odd} \end{cases} \tag{23}$$

for all $m\ge k$, where k is a fixed positive integer with $k\ge 3$.

Proof. Suppose that $f:\mathcal{X}\to\mathcal{Y}$ satisfies the functional equation (2). Putting $x=y=0$ in (2), we get $f(0)=0$. Replacing x by $x+y$ and $x-y$ in (2), respectively, we have

$$\begin{aligned} f(x+3y)+f(x-3y) &= f(x+y)+f(x-y)-16f(y)+8f(2y). \end{aligned} \tag{24}$$

Similar to the above, we get

$$f(x+4y)+f(x-4y)=2f(x)-32f(y)+16f(2y). \tag{25}$$

Using the above method, we can deduce that

$$f(x+my)+f(x-my) = \begin{cases} 2f(x)-2p_mf(y)+p_mf(2y), & m \text{ is even}, \\ f(x+y)+f(x-y) \\ \quad -2p_mf(y)+p_mf(2y), & m \text{ is odd} \end{cases} \tag{26}$$

for all $x,y\in\mathcal{X}$ for which $p_2=4$, $p_3=8$ and

$$p_m = \begin{cases} 2p_{m-1}-p_{m-2}+4 & m \text{ is even}, \\ 2p_{m-1}-p_{m-2} & m \text{ is odd}. \end{cases} \tag{27}$$

Solving the above recurrence equations, we get

$$p_m = \begin{cases} m^2 & m \text{ is even}, \\ m^2-1 & m \text{ is odd} \end{cases} \tag{28}$$

for all positive integers $m\ge 2$.

Conversely, assume that $f:\mathcal{X}\to\mathcal{Y}$ satisfies the functional equations (4) and (22) for each $k\ge m$. Firstly, we assume that m is even. For $k=m(m-1)$ and for each $x,y\in\mathcal{X}$, we have

$$\begin{aligned} &f(x+m(m-1)y)+f(x-m(m-1)y) \\ &\quad = 2f(x)-2m^2f((m-1)y)+m^2f(2(m-1)y) \end{aligned} \tag{29}$$

for all $x,y\in\mathcal{X}$. On the other hand,

$$\begin{aligned} &f\left(x+\left(m^2-m\right)y\right)+f\left(x-\left(m^2-m\right)y\right) \\ &\quad = 2f(x)-2\left(m^2-m\right)^2f(y)+\left(m^2-m\right)^2f(2y) \end{aligned} \tag{30}$$

for all $x,y\in\mathcal{X}$. It follows from (29) and (30) that

$$\begin{aligned} &-2f((m-1)y)+f(2(m-1)y) \\ &\quad = -2(m-1)^2f(y)+(m-1)^2f(2y) \end{aligned} \tag{31}$$

for all $x,y\in\mathcal{X}$. Since $(m+1)(m-1)$ is an odd number, we have

$$\begin{aligned} &f(x+(m+1)(m-1)y)+f(x-(m+1)(m-1)y) \\ &\quad = f(x+(m-1)y)+f(x-(m-1)y) \\ &\quad\quad -2\left((m+1)^2-1\right)f((m-1)y) \\ &\quad\quad +\left((m+1)^2-1\right)f(2(m-1)y) \end{aligned} \tag{32}$$

for all $x,y\in\mathcal{X}$. Also,

$$\begin{aligned} &f\left(x+\left(m^2-1\right)y\right)+f\left(x-\left(m^2-1\right)y\right) \\ &\quad = f(x+y)+f(x-y)-2\left(\left(m^2-1\right)^2-1\right)f(y) \\ &\quad\quad +\left(\left(m^2-1\right)^2-1\right)f(2y) \end{aligned} \tag{33}$$

for all $x,y\in\mathcal{X}$. Plugging (32) into (33) and using (31), we get

$$\begin{aligned} &f(x+(m-1)y)+f(x-(m-1)y) \\ &\quad = f(x+y)+f(x-y)-2\left((m-1)^2-1\right)f(y) \\ &\quad\quad +\left((m-1)^2-1\right)f(2y). \end{aligned} \tag{34}$$

For the odd case m, we have

$$\begin{aligned} &f(x+(m+1)(m-1)y)+f(x-(m+1)(m-1)y) \\ &\quad = 2f(x)-2(m+1)^2f((m-1)y) \\ &\quad\quad +(m+1)^2f(2(m-1)y) \end{aligned} \tag{35}$$

for all $x,y\in\mathcal{X}$. Also

$$\begin{aligned} &f\left(x+\left(m^2-1\right)y\right)+f\left(x-\left(m^2-1\right)y\right) \\ &\quad = 2f(x)-2\left(m^2-1\right)^2f(y)+\left(m^2-1\right)^2f(2y) \end{aligned} \tag{36}$$

for all $x, y \in \mathcal{X}$. The comparison of (35) and (36) shows that

$$-2f((m-1)y) + f(2(m-1)y) = -2(m-1)^2 f(y) + (m-1)^2 f(2y) \tag{37}$$

for all $x, y \in \mathcal{X}$. For $k = m(m-1)$ and for each $x, y \in \mathcal{X}$, we have

$$\begin{aligned} f&(x + m(m-1)y) + f(x - m(m-1)y) \\ &= f(x + (m-1)y) + f(x - (m-1)y) \\ &\quad - 2(m^2-1) f((m-1)y) + (m^2-1) f(2(m-1)y) \end{aligned} \tag{38}$$

for all $x, y \in \mathcal{X}$. On the other hand,

$$\begin{aligned} f&(x + (m^2-m)y) + f(x - (m^2-m)y) \\ &= 2f(x) - 2(m^2-m)^2 f(y) + (m^2-m)^2 f(2y) \end{aligned} \tag{39}$$

for all $x, y \in \mathcal{X}$. Now, by comparing (38) with (39) and applying (37), we obtain

$$\begin{aligned} f&(x + (m-1)y) + f(x - (m-1)y) \\ &= 2f(x) - 2(m-1)^2 f(y) + (m-1)^2 f(2y) \end{aligned} \tag{40}$$

for all $x, y \in \mathcal{X}$. This completes the proof. □

Theorem 3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be real vector spaces. A mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies either* (4) *or* (5) *if and only if there exist a symmetric biadditive mapping $B : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ and an additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that $f(x) = B(x, x) + A(x)$ for all $x \in \mathcal{X}$.*

Proof. Assume that there exist a symmetric biadditive mapping $B : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ and an additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that $f(x) = B(x, x) + A(x)$ for all $x \in \mathcal{X}$. A simple computation shows that the mappings A and $B' : \mathcal{X} \to \mathcal{Y}$ given by $B'(x) = B(x, x)$ satisfy the functional equations (4) and (5). Therefore the mapping f satisfies (4) and (5).

Conversely, we decompose f into the even part and odd part by setting

$$f_e(x) = \frac{f(x) + f(-x)}{2}, \qquad f_o(x) = \frac{f(x) - f(-x)}{2} \tag{41}$$

for all $x \in \mathcal{X}$. Obviously, $f(x) = f_e(x) + f_o(x)$ for all $x \in \mathcal{X}$. One can easily check that the mappings f_e and f_o satisfy (4) and (5). It follows from Lemma 1 that the mappings f_e and f_o are quadratic and additive, respectively. Since f_e is quadratic, by [19], there exists a symmetric biadditive mapping $B : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ such that $f_e(x) = B(x, x)$ for all $x \in \mathcal{X}$. Thus $f(x) = B(x, x) + A(x)$ for all $x \in \mathcal{X}$, where $A(x) = f_o(x)$ for all $x \in \mathcal{X}$. □

3. Hyers-Ulam Stability of (4) and (5)

We recall some basic facts concerning non-Archimedean spaces and some preliminary results.

By a non-Archimedean field, we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r + s| \le \max\{|r|, |s|\}$ for all $r, s \in \mathbb{K}$. Clearly $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$.

Let $\mathcal{X}$ be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\|$, $(x \in \mathcal{X}, r \in \mathbb{K})$;

(iii) the strong triangle inequality (ultrametric), namely,

$$\|x + y\| \le \max\{\|x\|, \|y\|\} \quad (x, y \in \mathcal{X}). \tag{42}$$

Then $(\mathcal{X}, \|\cdot\|)$ is called a non-Archimedean normed space. Due to the fact that

$$\|x_n - x_m\| \le \max\{\|x_{j+1} - x_j\|;\ m \le j \le n-1\} \quad (n \ge m), \tag{43}$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space $\mathcal{X}$. By a complete non-Archimedean normed space, we mean one in which every Cauchy sequence is convergent.

In [20], Hensel discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. The most interesting example of non-Archimedean normed spaces is p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom; for all $x, y > 0$, there exists an integer n such that $x < ny$.

Let p be a prime number. For any nonzero rational number $x = p^r(m/n)$ in which m and n are coprime to the prime number p. Consider the p-adic absolute value $|x|_p = p^{-r}$ on $\mathbb{Q}$. It is easy to check that $|\cdot|$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$ which is denoted by $\mathbb{Q}_p$ is said to be the p-adic number field. One should remember that if $p > 2$, then $|2^n| = 1$ for all integers n. The stability of some functional equations in non-Archimedean spaces was investigated, for instance, in [21–24] (see also [13, 25]).

Let m be an even positive integer. We use the abbreviation for the given mapping $f : \mathcal{X} \to \mathcal{Y}$ as follows:

$$\begin{aligned} \mathcal{D}_m f(x, y) &:= f(x + my) + f(x - my) \\ &\qquad - 2f(x) + 2m^2 f(y) - m^2 f(2y), \\ \mathcal{D}_3 f(x, y) &:= f(x + 3y) + f(x - 3y) - f(x + y) \\ &\qquad - f(x - y) + 16 f(y) - 8 f(2y). \end{aligned} \tag{44}$$

From now on, we assume that $\mathcal{X}$ is a real vector space and $\mathcal{Y}$ is a complete non-Archimedean space unless otherwise stated explicitly. In the upcoming theorem, we prove the stability of functional equations (4) and (5).

Theorem 4. *Let $\phi : \mathscr{X} \times \mathscr{X} \to [0,\infty)$ be a function such that*

$$\lim_{k\to\infty} \frac{1}{|2|^k} \phi\left(2^k x, 2^k y\right) = 0 \tag{45}$$

for all $x, y \in \mathscr{X}$. Suppose that $f : \mathscr{X} \to \mathscr{Y}$ is an odd mapping satisfying the inequality

$$\left\| \mathscr{D}_m f(x,y) \right\| \leq \phi(x,y) \tag{46}$$

for all $x, y \in \mathscr{X}$, where m is an even positive integer or $m = 3$. Then there exists a unique additive mapping $A : \mathscr{X} \to \mathscr{Y}$ such that

$$\| f(x) - A(x) \| \leq \begin{cases} \dfrac{1}{|2m^2|} \widetilde{\phi}(x), & m \text{ is even}, \\ \dfrac{1}{|8|} \widetilde{\phi}(x), & m = 3 \end{cases} \tag{47}$$

for all $x \in \mathscr{X}$, where $\widetilde{\phi}(x) = \sup\{(\phi(0, 2^j x)/|2|^j) : j \in \mathbb{N} \cup \{0\}\}$.

Proof. We prove the result when m is even; another case is similar. Putting $x = 0$ in (46), we have

$$\| 2f(y) - f(2y) \| \leq \frac{1}{|m|^2} \phi(0,y) \tag{48}$$

for all $y \in \mathscr{X}$. Replacing y by $2^n x$ in (48) and then dividing both sides by $|2|^{n+1}$, we get

$$\left\| \frac{1}{2^n} f\left(2^n x\right) - \frac{1}{2^{n+1}} f\left(2^{n+1} x\right) \right\| \leq \frac{1}{|m|^2 |2|^{n+1}} \phi\left(0, 2^n x\right) \tag{49}$$

for all $x \in \mathscr{X}$ and all nonnegative integers n. Thus the sequence $\{f(2^n x)/2^n\}$ is Cauchy by (45) and (49). Completeness of the non-Archimedean space $\mathscr{Y}$ allows us to assume that there exists a mapping A, so that

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = A(x). \tag{50}$$

For each $x \in \mathscr{X}$ and nonnegative integers n, we have

$$\begin{aligned} &\left\| \frac{f(2^n x)}{2^n} - f(x) \right\| \\ &\quad = \left\| \sum_{j=0}^{n-1} \frac{f\left(2^{j+1} x\right)}{2^{j+1}} - \frac{f\left(2^j x\right)}{2^j} \right\| \\ &\quad \leq \max \left\{ \left\| \frac{f\left(2^{j+1} x\right)}{2^{j+1}} - \frac{f\left(2^j x\right)}{2^j} \right\| : 0 \leq j < n \right\} \\ &\quad \leq \frac{1}{|2m^2|} \max \left\{ \frac{\phi\left(0, 2^j x\right)}{|2|^j} : 0 \leq j < n \right\}. \end{aligned} \tag{51}$$

Taking that n tends to approach infinity in (51) and applying (50), we can see that inequality (47) holds when m is even. It follows from (45), (46), and (50) that for all $x, y \in \mathscr{X}$,

$$\begin{aligned} \| \mathscr{D}_m A(x,y) \| &= \lim_{n\to\infty} \frac{1}{|2|^n} \| \mathscr{D}_m f(2^n x, 2^n y) \| \\ &\leq \lim_{n\to\infty} \frac{1}{|2|^n} \phi(2^n x, 2^n y) = 0. \end{aligned} \tag{52}$$

Hence, the mapping A satisfies (4). Part (i) of Lemma 1 shows that the mapping A is additive. Now, let $A' : \mathscr{X} \to \mathscr{Y}$ be another additive mapping satisfying (47). Then we have

$$\begin{aligned} &\left\| A(x) - A'(x) \right\| \\ &\quad = \lim_{k\to\infty} \frac{1}{|2|^k} \left\| A\left(2^k x\right) - A'\left(2^k x\right) \right\| \\ &\quad \leq \lim_{k\to\infty} \frac{1}{|2|^k} \max \left\{ \left\| A\left(2^k x\right) - f\left(2^k x\right) \right\|, \right. \\ &\qquad\qquad \left. \left\| f\left(2^k x\right) - A'\left(2^k x\right) \right\| \right\} \\ &\quad \leq \frac{1}{|2m^2|} \lim_{k\to\infty} \lim_{n\to\infty} \max \left\{ \frac{\phi\left(0, 2^j x\right)}{|2|^j} : k \leq j < n + k \right\} \\ &\quad = \frac{1}{|2m^2|} \lim_{k\to\infty} \sup \left\{ \frac{\phi\left(0, 2^j x\right)}{|2|^j} : k \leq j < \infty \right\} = 0 \end{aligned} \tag{53}$$

for all $x \in \mathscr{X}$. This shows the uniqueness of A. □

We have the following result which is analogous to Theorem 4 for the functional equations (4) and (5). We include the proof for (4). The proof of (5) is similar.

Theorem 5. *Let $\phi : \mathscr{X} \times \mathscr{X} \to [0,\infty)$ be a function such that*

$$\lim_{k\to\infty} |2|^k \phi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) = 0 \tag{54}$$

for all $x, y \in \mathscr{X}$. Suppose that $f : \mathscr{X} \to \mathscr{Y}$ is an odd mapping satisfying the inequality

$$\left\| \mathscr{D}_m f(x,y) \right\|_{\mathscr{Y}} \leq \phi(x,y) \tag{55}$$

for all $x, y \in \mathscr{X}$, where m is an even positive integer or $m = 3$. Then there exists a unique additive mapping $A : \mathscr{X} \to \mathscr{Y}$ such that

$$\| f(x) - A(x) \| \leq \begin{cases} \dfrac{1}{|m|^2} \widetilde{\phi}(x), & m \text{ is even}, \\ \dfrac{1}{|8|} \widetilde{\phi}(x), & m = 3 \end{cases} \tag{56}$$

for all $x \in \mathscr{X}$, where $\widetilde{\phi}(x) = \sup\{|2|^j \phi(0, (x/2^{j+1})) : j \in \mathbb{N} \cup \{0\}\}$.

Proof. We only obtain the result for the even integers. Similar to the proof of Theorem 4, we have

$$\| 2f(y) - f(2y) \| \leq \frac{1}{|m|^2} \phi(0,y) \tag{57}$$

for all $y \in \mathscr{X}$. If we replace y by $x/2^{n+1}$ in inequality (57) and then multiply both sides of the result to $|2|^n$, we get

$$\left\| 2^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) \right\| \leq \frac{|2|^n}{|m|^2} \phi\left(0, \frac{x}{2^{n+1}}\right) \tag{58}$$

for all $x \in \mathcal{X}$ and all nonnegative integers n. Thus, we conclude from (54) and (58) that the sequence $\{2^n f(x/2^n)\}$ is Cauchy. Since the non-Archimedean normed space $\mathcal{Y}$ is complete, this sequence converges in $\mathcal{Y}$ to the mapping A. Indeed,

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \quad (x \in \mathcal{X}). \tag{59}$$

Using induction and (57), one can show that

$$\begin{aligned}&\left\|2^n f\left(\frac{x}{2^n}\right) - f(x)\right\| \\ &\quad \le \frac{1}{|m|^2} \max\left\{|2|^j \phi\left(0, \frac{x}{2^{j+1}}\right) : 0 \le j < n\right\}\end{aligned} \tag{60}$$

for all $x \in \mathcal{X}$ and nonnegative integers n. Since the right-hand side of inequality (60) tends to be 0 as n to approach infinity, by applying (59), we deduce inequality (56). Now, similar to the proof of Theorem 4, we can complete the rest of the proof. □

Corollary 6. *Let $\alpha, r,$ and s be positive real numbers such that $r, s \ne 1$ and $|2| < 1$. Suppose that $\mathcal{X}$ is a non-Archimedean normed space and $f : \mathcal{X} \to \mathcal{Y}$ is an odd mapping fulfilling*

$$\|\mathcal{D}_m f(x, y)\| \le \alpha\left(\|x\|^r + \|y\|^s\right) \tag{61}$$

for all $x, y \in \mathcal{X}$, where m is an even positive integer or $m = 3$. Then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \le \begin{cases} \dfrac{\alpha\|x\|^s}{|2m^2|} & m \text{ is even, } r, s > 1, \\ \dfrac{\alpha\|x\|^s}{|8|} & m = 3,\ r, s > 1, \\ \dfrac{\alpha\|x\|^s}{|m|^2|2|^s} & m \text{ is even, } r, s < 1, \\ \dfrac{\alpha\|x\|^s}{|8|\,|2|^s} & m = 3,\ r, s < 1 \end{cases} \tag{62}$$

for all $x \in \mathcal{X}$.

Proof. The result follows from Theorems 4 and 5 by letting $\phi(x, y) = \alpha(\|x\|^r + \|y\|^s)$. □

In the next result, we prove the hyperstability of the functional equations (4) and (5) under some conditions. Recall that a functional equation is called *hyperstable* if every approximate solution is an exact one (see, e.g., [13, 26–29]).

Corollary 7. *Let $\alpha, r,$ and s be positive real numbers such that $r + s \ne 1$ and $|2| < 1$. Suppose that $\mathcal{X}$ is a non-Archimedean normed space and $f : \mathcal{X} \to \mathcal{Y}$ is an odd mapping fulfilling*

$$\|\mathcal{D}_m f(x, y)\| \le \alpha\|x\|^r\|y\|^s \tag{63}$$

for all $x, y \in \mathcal{X}$, where m is an even positive integer or $m = 3$. Then f is an additive mapping.

Proof. Taking $\phi(x, y) = \alpha\|x\|^r\|y\|^s$ in Theorems 4 and 5, we can obtain the desired result. □

Theorem 8. *Let $\phi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function such that*

$$\lim_{k\to\infty} \frac{1}{|4|^k} \phi\left(2^k x, 2^k y\right) = 0 \tag{64}$$

for all $x, y \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying the inequality

$$\|\mathcal{D}_m f(x, y)\| \le \phi(x, y) \tag{65}$$

for all $x, y \in \mathcal{X}$, where m is an even positive integer. Then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{|4|}\widetilde{\phi}(x) \tag{66}$$

for all $x \in \mathcal{X}$, where $\widetilde{\phi}(x) = \sup\{(\phi(0, (2^j x/m)/|4|^j), (\phi(2^j x, (2^j x/m))/|4|^j) : j \in \mathbb{N} \cup \{0\}\}$.

Proof. It follows from (64) that $\phi(0, 0) = 0$. Thus (65) implies that $f(0) = 0$. Putting $x = 0$ in (65) and interchanging y into x, we have

$$\left\|2f(mx) + 2m^2 f(x) - m^2 f(2x)\right\| \le \phi(0, x) \tag{67}$$

for all $x \in \mathcal{X}$. Substituting x, y by mx, x in (65), respectively, we get

$$\left\|f(2mx) - 2f(mx) + 2m^2 f(x) - m^2 f(2x)\right\| \le \phi(mx, x) \tag{68}$$

for all $x \in \mathcal{X}$. It follows from (67) and (68) that

$$\|f(2mx) - 4f(mx)\| \le \max\{\phi(0, x), \phi(mx, x)\} \tag{69}$$

for all $x \in \mathcal{X}$. Thus we have

$$\|f(2x) - 4f(x)\| \le \max\left\{\phi\left(0, \frac{x}{m}\right), \phi\left(x, \frac{x}{m}\right)\right\} \tag{70}$$

for all $x \in \mathcal{X}$. Replacing x by $2^n x$ in (70) and then dividing both sides by $|4|^{n+1}$, we get

$$\begin{aligned}&\left\|\frac{1}{4^{n+1}} f\left(2^{n+1}x\right) - \frac{1}{4^n} f\left(2^n x\right)\right\| \\ &\quad \le \frac{1}{|4|} \max\left\{\frac{\phi(0, (2^n x/m))}{|4|^n}, \frac{\phi(2^n x, (2^n x/m))}{|4|^n}\right\}\end{aligned} \tag{71}$$

for all $x \in \mathcal{X}$ and all nonnegative integers n. Thus the sequence $\{f(2^n x)/4^n\}$ is Cauchy by (64) and (71). Since $\mathcal{Y}$ is complete, the sequence $\{f(2^n x)/4^n\}$ converges in $\mathcal{Y}$ for all $x \in \mathcal{X}$. So one can define the mapping $Q : \mathcal{X} \to \mathcal{Y}$ by

$$Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \quad (x \in \mathcal{X}). \tag{72}$$

For each $x \in \mathcal{X}$ and nonnegative integers n, we have

$$\begin{aligned}
&\left\| \frac{f(2^n x)}{4^n} - f(x) \right\| \\
&\quad = \left\| \sum_{j=0}^{n-1} \frac{f(2^{j+1}x)}{4^{j+1}} - \frac{f(2^j x)}{4^j} \right\| \\
&\quad \leq \max\left\{ \left\| \frac{f(2^{j+1}x)}{4^{j+1}} - \frac{f(2^j x)}{4^j} \right\| : 0 \leq j < n \right\} \\
&\quad \leq \frac{1}{|4|} \max\left\{ \frac{\phi(0,(2^j x/m))}{|4|^j}, \right. \\
&\qquad\qquad \left. \frac{\phi(2^j x,(2^j x/m))}{|4|^j} : 0 \leq j < n \right\}.
\end{aligned} \tag{73}$$

Taking n to approach infinity in (73) and using (64) and (72), we find (66). Employing (64), (65), and (72), we obtain

$$\begin{aligned}
\|\mathcal{D}_m Q(x,y)\| &= \lim_{n\to\infty} \frac{1}{|4|^n} \|\mathcal{D}_m f(2^n x, 2^n y)\| \\
&\leq \lim_{n\to\infty} \frac{1}{|4|^n} \phi(2^n x, 2^n y) = 0.
\end{aligned} \tag{74}$$

Hence, the mapping Q satisfies (4). It follows from part (iii) of Lemma 1 that the mapping Q is quadratic. If $Q' : \mathcal{X} \to \mathcal{Y}$ is another quadratic mapping satisfying (66), then

$$\begin{aligned}
&\|Q(x) - Q'(x)\| \\
&= \lim_{k\to\infty} \frac{1}{|4|^k} \|Q(2^k x) - Q'(2^k x)\| \\
&\leq \lim_{k\to\infty} \frac{1}{|4|^k} \max\{\|Q(2^k x) - f(2^k x)\|, \\
&\qquad\qquad \|f(2^k x) - Q'(2^k x)\|\} \\
&\leq \frac{1}{|4|} \lim_{k\to\infty} \lim_{n\to\infty} \max\left\{ \frac{\phi(0,(2^j x/m))}{|4|^j}, \right. \\
&\qquad\qquad \left. \frac{\phi(2^j x,(2^j x/m))}{|4|^j} : k \leq j < n+k \right\} \\
&= \frac{1}{|4|} \lim_{k\to\infty} \sup\left\{ \frac{\phi(0,(2^j x/m))}{|4|^j}, \right. \\
&\qquad\qquad \left. \frac{\phi(2^j x,(2^j x/m))}{|4|^j} : k \leq j < \infty \right\} = 0
\end{aligned} \tag{75}$$

for all $x \in \mathcal{X}$. Therefore $Q = Q'$. This completes the proof of the uniqueness of Q. □

Theorem 9. *Let $\phi : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ be a function such that*

$$\lim_{k\to\infty} |4|^k \phi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) = 0 \tag{76}$$

for all $x, y \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying $f(0) = 0$ and the inequality

$$\|\mathcal{D}_m f(x,y)\| \leq \phi(x,y) \tag{77}$$

for all $x, y \in \mathcal{X}$, where m is an even positive integer. Then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{|4|} \widetilde{\phi}(x) \tag{78}$$

for all $x \in \mathcal{X}$, where $\widetilde{\phi}(x) = \sup\{|4|^n \phi(0,(x/2^n m)), |4|^n \phi((x/2^n),(x/2^n m)) : n \in \mathbb{N}\}$.

Proof. Similar to the proof of Theorem 8, we have

$$\|f(2x) - 4f(x)\| \leq \max\left\{\phi\left(0, \frac{x}{m}\right), \phi\left(x, \frac{x}{m}\right)\right\} \tag{79}$$

for all $x \in \mathcal{X}$. Then we get

$$\left\| f(x) - 4f\left(\frac{x}{2}\right) \right\| \leq \max\left\{\phi\left(0, \frac{x}{2m}\right), \phi\left(\frac{x}{2}, \frac{x}{2m}\right)\right\} \tag{80}$$

for all $x \in \mathcal{X}$. Replacing x by $x/2^n$ in (80) and multiplying both sides to $|4|^n$, we get

$$\begin{aligned}
&\left\| 4^n f\left(\frac{x}{2^n}\right) - 4^{n+1} f\left(\frac{x}{2^{n+1}}\right) \right\| \\
&\quad \leq \frac{1}{|4|} \max\left\{ |4|^{n+1} \phi\left(0, \frac{x}{2^{n+1} m}\right), |4|^{n+1} \phi\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1} m}\right) \right\}
\end{aligned} \tag{81}$$

for all $x \in \mathcal{X}$ and all nonnegative integers n. Thus the sequence $\{4^n f(x/2^n)\}$ is Cauchy by (76). The completeness of $\mathcal{Y}$ implies that the mentioned sequence is convergent. So we consider the mapping $Q : \mathcal{X} \to \mathcal{Y}$ by

$$Q(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) \quad (x \in \mathcal{X}). \tag{82}$$

For each $x \in \mathcal{X}$ and nonnegative integers n, we have

$$\begin{aligned}
&\left\| f(x) - 4^n f\left(\frac{x}{2^n}\right) \right\| \\
&\quad \leq \frac{1}{|4|} \max\left\{ |4|^j \phi\left(0, \frac{x}{2^j m}\right), |4|^j \phi\left(\frac{x}{2^j}, \frac{x}{2^j m}\right) : 1 \leq j \leq n \right\}.
\end{aligned} \tag{83}$$

Letting n approach infinity in (83) and using (76) and (82), we can see that (78) holds. The rest of the proof is similar to the proof of Theorem 8. □

Corollary 10. *Let α, r, and s be nonnegative real numbers and $|2| < 1$. Suppose that $\mathcal{X}$ is a non-Archimedean normed space and $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping fulfilling*

$$\|\mathcal{D}_m f(x,y)\| \leq \begin{cases} \alpha\|x\|^r & r \neq 2, \\ \alpha\left(\|x\|^r + \|y\|^s\right) & r, s \neq 2 \end{cases} \tag{84}$$

for all $x, y \in \mathcal{X}$, where m is an even positive integer. Then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq \begin{cases} \dfrac{\alpha\|x\|^r}{|4|} & r > 2,\ s = 0, \\ \dfrac{\alpha}{|4|}\left(\|x\|^r + \left\|\dfrac{x}{m}\right\|^s\right) & r, s > 2, \\ \dfrac{\alpha\|x\|^r}{|2|^r} & r < 2,\ s = 0, \\ \alpha\left(\dfrac{\|x\|^r}{|2|^r} + \left\|\dfrac{x}{2m}\right\|^s\right) & r, s < 2 \end{cases} \tag{85}$$

for all $x \in \mathcal{X}$.

Theorem 11. *Let $\phi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function such that*

$$\lim_{k\to\infty} \frac{1}{|4|^k}\phi\left(2^k x, 2^k y\right) = 0 \tag{86}$$

for all $x, y \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying the inequality

$$\|\mathcal{D}_3 f(x, y)\| \leq \phi(x, y) \tag{87}$$

for all $x, y \in \mathcal{X}$. Then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{|4|}\widetilde{\phi}(x) \tag{88}$$

for all $x \in \mathcal{X}$, where $\widetilde{\phi}(x) = \sup\{|2|(\phi((2^j x/4), (2^j x/4))/|4|^j), (\phi((5/4)2^j x, (2^j x/4))/|4|^j), (\phi((3/4)2^j x, (2^j x/4))/|4|^j) : j \in \mathbb{N} \cup \{0\}\}$.

Proof. Similar to the proof of Theorem 8, we can show that $f(0) = 0$. Replacing x by $x + y$ and $x - y$ in (87), respectively, we get

$$\begin{aligned} &\|f(x+4y) + f(x-2y) - f(x+2y) \\ &\quad - f(x) + 16f(y) - 8f(2y)\| \leq \phi(x+y, y), \end{aligned} \tag{89}$$

$$\begin{aligned} &\|f(x+2y) + f(x-4y) - f(x-2y) \\ &\quad - f(x) + 16f(y) - 8f(2y)\| \leq \phi(x-y, y) \end{aligned} \tag{90}$$

for all $x, y \in \mathcal{X}$. Inequalities (89) and (90) imply that

$$\begin{aligned} &\|f(x+4y) + f(x-4y) - 2f(x) + 32f(y) - 16f(2y)\| \\ &\quad \leq \max\{\phi(x+y, y), \phi(x-y, y)\} \end{aligned} \tag{91}$$

for all $x, y \in \mathcal{X}$. Interchanging x, y into $4x, x$ in (91), respectively, we obtain

$$\begin{aligned} &\|f(8x) - 2f(4x) + 32f(x) - 16f(2x)\| \\ &\quad \leq \max\{\phi(5x, x), \phi(3x, x)\} \end{aligned} \tag{92}$$

for all $x \in \mathcal{X}$. On the other hand, by putting $x = y$ in (87), we can deduce that

$$\|f(4x) - 8f(2x) + 16f(x)\| \leq \phi(x, x) \tag{93}$$

for all $x \in \mathcal{X}$. It follows from (92) and (93) that

$$\|f(8x) - 4f(4x)\| \leq \max\{|2|\phi(x, x), \phi(5x, x), \phi(3x, x)\} \tag{94}$$

for all $x \in \mathcal{X}$. Thus

$$\begin{aligned} &\left\|\frac{1}{4}f(2x) - f(x)\right\| \\ &\quad \leq \frac{1}{|4|}\max\left\{|2|\phi\left(\frac{x}{4}, \frac{x}{4}\right), \phi\left(\frac{5}{4}x, \frac{x}{4}\right), \phi\left(\frac{3}{4}x, \frac{x}{4}\right)\right\} \end{aligned} \tag{95}$$

for all $x \in \mathcal{X}$. Substituting x by $2^n x$ in (95) and then dividing both sides by $|4|^n$, we obtain

$$\begin{aligned} &\left\|\frac{1}{4^{n+1}}f\left(2^{n+1}x\right) - \frac{1}{4^n}f(2^n x)\right\| \\ &\leq \frac{1}{|4|}\max\left\{|2|\frac{\phi((2^n x/4), (2^n x/4))}{|4|^n},\right. \\ &\qquad \left.\frac{\phi((5/4)2^n x, (2^n x/4))}{|4|^n}, \frac{\phi((3/4)2^n x, (2^n x/4))}{|4|^n}\right\} \end{aligned} \tag{96}$$

for all $x \in \mathcal{X}$ and all nonnegative integers n. Thus the sequence $\{f(2^n x)/4^n\}$ is Cauchy by (86) and (96). The completeness of $\mathcal{Y}$ implies that the sequence $\{f(2^n x)/4^n\}$ is convergent. Define the mapping $Q : \mathcal{X} \to \mathcal{Y}$ via

$$Q(x) := \lim_{n\to\infty}\frac{f(2^n x)}{4^n} \quad (x \in \mathcal{X}). \tag{97}$$

By a simple computation, one can show that

$$\begin{aligned} &\left\|\frac{f(2^n x)}{4^n} - f(x)\right\| \\ &\leq \frac{1}{|4|}\max\left\{|2|\frac{\phi\left(\left(2^j x/4\right), \left(2^j x/4\right)\right)}{|4|^j},\right. \\ &\qquad \frac{\phi\left((5/4)2^j x, \left(2^j x/4\right)\right)}{|4|^j}, \\ &\qquad \left.\frac{\phi\left((3/4)2^j x, \left(2^j x/4\right)\right)}{|4|^j} : 0 \leq j < n\right\} \end{aligned} \tag{98}$$

for all $x \in \mathcal{X}$ and for all $n \geq 0$. Taking n to approach infinity in (98) and applying (86) and (97), we find (88). By (86), (87), and (97), we have

$$\begin{aligned} \|\mathcal{D}_3 Q(x, y)\| &= \lim_{n\to\infty}\frac{1}{|4|^n}\|\mathcal{D}_3 f(2^n x, 2^n y)\| \\ &\leq \lim_{n\to\infty}\frac{1}{|4|^n}\phi(2^n x, 2^n y) = 0. \end{aligned} \tag{99}$$

Hence, the mapping Q satisfies (5). It follows from part (iv) of Lemma 1 that the mapping Q is quadratic. Similar to the proof of Theorem 8, one can show that Q is unique. □

Theorem 12. *Let $\phi : \mathcal{X}\times\mathcal{X} \to [0,\infty)$ be a function such that*

$$\lim_{k\to\infty} |4|^k \phi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) = 0 \qquad (100)$$

for all $x, y \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying the inequality

$$\left\| \mathcal{D}_3 f(x, y) \right\| \le \phi(x, y) \qquad (101)$$

for all $x, y \in \mathcal{X}$. Then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\left\| f(x) - Q(x) \right\| \le \frac{1}{|4|}\widetilde{\phi}(x) \quad (x \in \mathcal{X}), \qquad (102)$$

where $\widetilde{\phi}(x) = \sup\{|2||4|^j\phi((x/2^{j+3}), (x/2^{j+3})), |4|^j\phi(5(x/2^{j+3}), (x/2^{j+3})), |4|^j\phi(3(x/2^{j+3}), (x/2^{j+3})) : j \in \mathbb{N}\cup\{0\}\}$.

Proof. Similar to the proof of Theorem 8, one can obtain

$$\left\| f(8x) - 4f(4x) \right\| \le \max\{|2|\phi(x, x), \phi(5x, x), \phi(3x, x)\} \qquad (103)$$

for all $x \in \mathcal{X}$. Thus

$$\left\| f(x) - 4f\left(\frac{x}{2}\right) \right\| \le \max\left\{ |2|\phi\left(\frac{x}{8}, \frac{x}{8}\right), \phi\left(\frac{5}{8}x, \frac{x}{8}\right), \phi\left(\frac{3}{8}x, \frac{x}{8}\right) \right\} \qquad (104)$$

for all $x \in \mathcal{X}$. Substituting x by $x/2^n$ in (104) and then multiplying both sides to $|4|^n$, we have

$$\begin{aligned} &\left\| 4^n f\left(\frac{x}{2^n}\right) - 4^{n+1} f\left(\frac{x}{2^{n+1}}\right) \right\| \\ &\quad \le \max\left\{ |2|\,|4|^n\phi\left(\frac{x}{2^{n+3}}, \frac{x}{2^{n+3}}\right), \right. \\ &\qquad \left. |4|^n\phi\left(5\frac{x}{2^{n+3}}, \frac{x}{2^{n+3}}\right), |4|^n\phi\left(3\frac{x}{2^{n+3}}, \frac{x}{2^{n+3}}\right) \right\} \end{aligned} \qquad (105)$$

for all $x \in \mathcal{X}$ and all nonnegative integers n. The last inequality and (100) imply that the sequence $\{f(2^n x)/4^n\}$ is Cauchy. So, this sequence converges to the mapping $Q : \mathcal{X} \to \mathcal{Y}$. In other words,

$$Q(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n} \quad (x \in \mathcal{X}). \qquad (106)$$

We can also show that

$$\begin{aligned} &\left\| f(x) - 4^n f\left(\frac{x}{2^n}\right) \right\| \\ &\quad \le \frac{1}{|4|} \max\left\{ |2|\,|4|^j\phi\left(\frac{x}{2^{j+3}}, \frac{x}{2^{j+3}}\right), |4|^j\phi\left(5\frac{x}{2^{j+3}}, \frac{x}{2^{j+3}}\right), \right. \\ &\qquad \left. |4|^j\phi\left(3\frac{x}{2^{j+3}}, \frac{x}{2^{j+3}}\right) : 0 \le j < n \right\} \end{aligned} \qquad (107)$$

for all $x \in \mathcal{X}$ and for all $n \ge 0$. Taking n to approach infinity in (107) and applying (100) and (101), we see that inequality (102) holds. Now, similar to the proof of Theorem 8, one can show that Q is a unique quadratic mapping. □

Corollary 13. *Let α and r be positive real numbers such that $r \ne 2$ and $|2| < 1$. Suppose that $\mathcal{X}$ is a non-Archimedean normed space and $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping fulfilling*

$$\left\| \mathcal{D}_3 f(x, y) \right\| \le \alpha \|y\|^r \qquad (108)$$

for all $x, y \in \mathcal{X}$. Then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\left\| f(x) - Q(x) \right\| \le \begin{cases} \dfrac{|2|\,\|x\|^r}{|4|^{r+1}}\alpha & r > 2, \\[2ex] \dfrac{\|x\|^r}{|2|\,|8|^r}\alpha & r < 2 \end{cases} \qquad (109)$$

for all $x \in \mathcal{X}$.

Proof. Letting $\phi(x, y) = \alpha\|y\|^r$ in Theorems 11 and 12, one can obtain the required result. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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