Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 170372, 5 pages
http://dx.doi.org/10.1155/2013/170372

Research Article

The Investigation of Solutions to the Coupled Schrödinger-Boussinesq Equations

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Received 12 May 2013; Accepted 2 June 2013

Academic Editor: Shaoyong Lai

The (G'/G)-expansion method and the symbolic computation system Mathematica are employed to investigate the coupled Schrödinger-Boussinesq equations. The hyperbolic function solutions, trigonometric function solutions, and rational function solutions to the equations are obtained. The decaying properties of several solutions are analyzed.

1. Introduction

In laser and plasma physics, the important problems under interactions between a nonlinear complex Schrödinger field and a real Boussinesq field have been raised. In particular, the study of the coupled Schrödinger-Boussinesq equations has attracted much attention of mathematicians and physicists (see [1–3]). The existence of the global solution of the initial-boundary problem for the equations was investigated in [1]. The existence of a periodic solution for the equations was considered in [2]. Kılıcman and Abazari [3] used the (G'/G)-expansion method to construct periodic and soliton solutions for the Schrödinger-Boussinesq equations $iu_t+u_{xx}-auv=0$, $v_{tt}-v_{xx}+v_{xxxx}-b(|u|^2)_{xx}=0$, where a and b are real constants. The investigation of nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena (see [4–7]).

In this paper, we consider the following coupled Schrödinger-Boussinesq equations:

$$\begin{gathered} iE_t+E_{xx}+\beta_1 E=NE, \\ 3N_{tt}-N_{xxxx}+3\left(N^2\right)_{xx}+\beta_2 N_{xx}=\left(|E|^2\right)_{xx}, \end{gathered} \tag{1}$$

where $E(x,t)$ is a complex unknown function, $N(x,t)$ is a real unknown function, and β_1 and β_2 are real positive constants. System (1) is known to describe various physical processes in laser and plasma physics, such as formation, Langmuir field amplitude, intense electromagnetic waves, and modulational instabilities (see [8]). The approximate solutions and conservation law for the coupled system (1) have been studied in [9]. In [10], Chen and Xu used the F-expansion method to obtain a number of periodic wave solutions expressed by various Jacobi elliptic functions for (1). Cai et al. [11] studied same equations by the modified F-expansion method.

In the present paper, we use the (G'/G)-expansion method and the symbolic computation system Mathematica to investigate the coupled Schrödinger-Boussinesq system (1). Here, we state that the previous works do not obtain the solutions presented in this paper.

The layout of this paper is as follows. In Section 2, we give the description of the generalized (G'/G)-expansion method. In Section 3, we apply this method to solve (1). A conclusion will be obtained in Section 4.

2. Brief Description of the (G'/G)-Expansion Method

To make our presentation self-contained, we recall the (G'/G)-expansion method. The details can be found in Wang et al.'s work [12].

Step 1. For a given PDE with two independent variables t and x

$$P\left(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots\right) = 0, \tag{2}$$

we convert it into an ODE

$$P\left(u, u', u'', u''', \ldots\right) = 0. \tag{3}$$

Using travelling transformation $u(x,t) = u(\xi)$, $\xi = x - kt$. Equation (3) can be integrated as long as all terms contain derivatives where integration constants are considered to be zeros.

Step 2. Suppose that the solution of (3) can be expressed as a polynomial in (G'/G)

$$u(\xi) = \sum_{i=0}^{n} a_i \left(\frac{G'}{G}\right)^i, \tag{4}$$

where $G = G(\xi)$ satisfies the second-order ODE with respect to ξ. Namely,

$$G'' + \lambda G' + \mu G = 0, \tag{5}$$

where $a_1, \ldots, a_n \neq 0$, λ, and μ are constants to be determined later. The positive integer n can be determined by balancing the highest-order derivatives with highest-order nonlinear terms appearing in (3). It is easy to check that (5) admits three types of solutions

$$\frac{G'}{G} = \begin{cases} \dfrac{\sqrt{\alpha}}{2}\left(\dfrac{c_1 \sinh(1/2)\sqrt{\alpha}\xi + c_2 \cosh(1/2)\sqrt{\alpha}\xi}{c_1 \cosh(1/2)\sqrt{\alpha}\xi + c_2 \sinh(1/2)\sqrt{\alpha}\xi}\right) - \dfrac{\lambda}{2}, & \alpha > 0, \\ \dfrac{\sqrt{-\alpha}}{2}\left(\dfrac{-c_1 \sin(1/2)\sqrt{-\alpha}\xi + c_2 \cos(1/2)\sqrt{-\alpha}\xi}{c_1 \cos(1/2)\sqrt{-\alpha}\xi + c_2 \sin(1/2)\sqrt{-\alpha}\xi}\right) - \dfrac{\lambda}{2}, & \alpha < 0, \\ \dfrac{c_2}{c_1 + c_2\xi} - \dfrac{\lambda}{2}, & \alpha = 0, \end{cases} \tag{6}$$

in which $\alpha = \lambda^2 - 4\mu$.

Step 3. By substituting (4) into (3) and using (5), collecting all terms with the same order of (G'/G) together, the left-hand side of (3) can be written as a polynomial in (G'/G). Letting each coefficient of this polynomial be zero yields a system of algebraic equations for $a_1, \ldots, a_n$, k, λ, and μ.

Step 4. Since the general solutions of (5) have been known, substituting $a_i, \ldots, a_n$, k, λ, and μ into (4), we can obtain travelling wave solutions of the nonlinear evolution equation (2).

3. Solutions of the Coupled Schrödinger-Boussinesq Equations

Following the procedure described in Section 2, we adopt the ansatz solution of (1) in the form

$$E(x,t) = u(x,t)\, e^{i(kx + lt + \xi_0)}, \tag{7}$$

where $u(x,t)$ is a real function, k, l are constants to be determined, and ξ_0 is an arbitrary constant. Substituting (7) into (1) yields

$$u_t + 2ku_x = 0, \tag{8}$$

$$u_{xx} - \left(l + k^2 - \beta_1\right) u = Nu, \tag{9}$$

$$3N_{tt} - N_{xxxx} + 3\left(N^2\right)_{xx} + \beta_2 N_{xx} = \left(u^2\right)_{xx}. \tag{10}$$

We take

$$u(x,t) = u(\xi) = u\left(x - 2kt + \xi_1\right), \tag{11}$$

where ξ_1 is an arbitrary constant. Substituting (11) into (9), one gets

$$N(x,t) = \frac{u''(\xi)}{u(\xi)} - \left(l + k^2 - \beta_1\right). \tag{12}$$

Suppose that

$$N(x,t) = v(\xi) = v\left(x - 2kt + \xi_1\right). \tag{13}$$

It follows from (9), (10), (11), and (13) that

$$u'' - \left(l + k^2 - \beta_1\right) u - uv = 0, \tag{14}$$

$$-v'' + \left(12k^2 + \beta_2\right) v + 3v^2 - u^2 = 0. \tag{15}$$

Balancing u'' with uv in (14) and v'' with u^2 in (15) leads to $m = 1, n = 2$. Thus we can search for the solutions of (14) and (15) in the following forms:

$$u(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2, \quad a_2 \neq 0, \tag{16}$$

$$v(\xi) = b_0 + b_1\left(\frac{G'}{G}\right) + b_2\left(\frac{G'}{G}\right)^2, \quad b_2 \neq 0. \tag{17}$$

Substituting (16) and (17) into (14) and (15), using (5), and setting the coefficients of $(G'/G)^i$ $(i = 0, \ldots, 4)$ to be zero, we obtain the algebraic system

$$\begin{gathered}
6a_2 - a_2 b_2 = 0,\\
2a_1 + 10a_2\lambda - a_2 b_1 - a_1 b_2 = 0,\\
8a_2\mu + 3a_1\lambda + 4a_2\lambda^2 - lb_2 - k^2 b_2\\
+ \beta_1 b_2 - a_2 b_0 - a_1 b_1 - a_0 b_2 = 0,\\
6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2 - lb_1 - k^2 b_1 + \beta_1 b_1 - a_1 b_0 - a_0 b_1 = 0,\\
2a_2\mu^2 + a_1\lambda\mu - lb_0 - k^2 b_0 + \beta_1 b_0 - a_0 b_0 = 0,\\
6b_2 - 3b_2^2 + a_2^2 = 0,\\
2b_1 + 10b_2\lambda - 6b_1 b_2 + 2a_1 a_2 = 0,\\
8b_2\mu + 3b_1\lambda + 4b_2\lambda^2 - 12k^2 b_2 + \beta_2 b_2\\
- 3b_1^2 + 6b_0 b_2 + a_1^2 + 2a_0 a_2 = 0,\\
6b_2\lambda\mu + 2b_1\mu + b_1\lambda^2 - 12k^2 b_1 - \beta_2 b_1 - 6b_0 b_1 + 2a_0 a_1 = 0,\\
2b_2\mu^2 + b_1\lambda\mu - 12k^2 b_0 - \beta_2 b_0 - 3b_0^2 + a_0^2 = 0.
\end{gathered} \tag{18}$$

Solving this system with the Mathematica, we find

$$\begin{gathered}
a_0 = \pm\left(\sqrt{2}\lambda^2 + 2\sqrt{2}\mu\right),\\
a_1 = \pm 6\sqrt{2}\lambda,\\
a_2 = \pm 6\sqrt{2},\\
b_0 = \lambda^2 + 2\mu,\\
b_1 = 6\lambda,\\
b_2 = 6,\\
k = \pm\frac{1}{6}\sqrt{3\left(\beta_2 + \lambda^2 - 4\mu\right)},\\
l = \frac{1}{12}\left(12\beta_1 + \beta_2 + \lambda^2 \mp 12\sqrt{2}\lambda^2 - 4\mu \pm 48\sqrt{2}\mu\right),
\end{gathered} \tag{19}$$

or

$$\begin{gathered}
a_0 = \pm 6\sqrt{2}\mu,\\
a_1 = \pm 6\sqrt{2}\lambda,\\
a_2 = \pm 6\sqrt{2}, \quad b_0 = 6\mu,\\
b_1 = 6\lambda,\\
b_2 = 6,\\
k = \pm\frac{1}{6}\sqrt{3\left(\beta_2 - \lambda^2 + 4\mu\right)},\\
l = \frac{1}{12}\left(12\beta_1 + \beta_2 - \lambda^2 \pm 12\sqrt{2}\lambda^2 + 4\mu \mp 48\sqrt{2}\mu\right),
\end{gathered} \tag{20}$$

where λ and μ are arbitrary constants.

By using (19) and (20), the solutions (16) and (17) are written as

$$\begin{gathered}
u(\xi) = \pm\left(\sqrt{2}\lambda^2 + 2\sqrt{2}\mu\right) \pm 6\sqrt{2}\lambda\left(\frac{G'}{G}\right) \pm 6\sqrt{2}\left(\frac{G'}{G}\right)^2,\\
v(\xi) = \lambda^2 + 2\mu + 6\lambda\left(\frac{G'}{G}\right) + 6\left(\frac{G'}{G}\right)^2,
\end{gathered} \tag{21}$$

where $\xi = x \pm (1/3)\sqrt{3(\beta_2 + \lambda^2 - 4\mu)}t + \xi_1$, or

$$\begin{gathered}
u(\xi) = \pm 6\sqrt{2}\mu \pm 6\sqrt{2}\lambda\left(\frac{G'}{G}\right) \pm 6\sqrt{2}\left(\frac{G'}{G}\right)^2,\\
v(\xi) = 6\mu + 6\lambda\left(\frac{G'}{G}\right) + 6\left(\frac{G'}{G}\right)^2,
\end{gathered} \tag{22}$$

where $\xi = x \pm (1/3)\sqrt{3(\beta_2 - \lambda^2 + 4\mu)}t + \xi_1$.

Substituting general solutions of (5) into (21) and (22), we obtain three types of travelling wave solutions of the coupled Schrödinger-Boussinesq equations as follows.

3.1. The Hyperbolic Function Solutions to (1) *If* $\alpha = \lambda^2 - 4\mu > 0$. Consider

$$\begin{aligned}
E_1(x,t) &= \pm\frac{\sqrt{2}\alpha}{2}\\
&\quad\times\left[3\left(\frac{c_1\sinh(1/2)\sqrt{\alpha}\xi + c_2\cosh(1/2)\sqrt{\alpha}\xi}{c_1\cosh(1/2)\sqrt{\alpha}\xi + c_2\sinh(1/2)\sqrt{\alpha}\xi}\right)^2 - 1\right]\\
&\quad\times e^{i(kx+lt+\xi_0)},\\
N_1(x,t) &= \frac{\alpha}{2}\left[3\left(\frac{c_1\sinh(1/2)\sqrt{\alpha}\xi + c_2\cosh(1/2)\sqrt{\alpha}\xi}{c_1\cosh(1/2)\sqrt{\alpha}\xi eg + c_2\sinh(1/2)\sqrt{\alpha}\xi}\right)^2 - 1\right],
\end{aligned} \tag{23}$$

where $k = \pm(1/6)\sqrt{3(\beta_2 + \alpha)}$, $l = (1/12)(12\beta_1 + \beta_2 + \lambda^2 \mp 12\sqrt{2}\lambda^2 - 4\mu \pm 48\sqrt{2}\mu$, c_1, c_2 are arbitrary constants, and

$$\begin{aligned}
E_2(x,t) &= \pm\frac{3\sqrt{2}\alpha}{2}\\
&\quad\times\left[\left(\frac{c_1\sinh(1/2)\sqrt{\alpha}\xi + c_2\cosh(1/2)\sqrt{\alpha}\xi}{c_1\cosh(1/2)\sqrt{\alpha}\xi + c_2\sinh(1/2)\sqrt{\alpha}\xi}\right)^2 - 1\right]\\
&\quad\times e^{i(kx+lt+\xi_0)},\\
N_2(x,t) &= \frac{3\alpha}{2}\left[\left(\frac{c_1\sinh(1/2)\sqrt{\alpha}\xi + c_2\cosh(1/2)\sqrt{\alpha}\xi}{c_1\cosh(1/2)\sqrt{\alpha}\xi + c_2\sinh(1/2)\sqrt{\alpha}\xi}\right)^2 - 1\right],
\end{aligned} \tag{24}$$

where $k = \pm(1/6)\sqrt{3(\beta_2-\alpha)}$, $l = (1/12)(12\beta_1+\beta_2-\lambda^2 \pm 12\sqrt{2}\lambda^2+4\mu \mp 48\sqrt{2}\mu)$.

Remark 1. When $\xi \to \infty$, we find $|E_1(x,t)| \to \pm\sqrt{2}\alpha$, $N_1(x,t) \to \alpha$, $|E_2(x,t)| \to 0$, $N_2(x,t) \to 0$.

Remark 2. If $c_1 \neq 0$, $c_2 = 0$, $\lambda > 0$, $\mu = 0$, we find envelope solitary wave solutions for (1). Namely, $E_1(x,t)$ and $N_1(x,t)$ become

$$\begin{aligned} E_1(x,t) &= \pm\frac{\sqrt{2}}{2}\lambda^2\left(3\tanh^2\frac{\lambda}{2}\xi-1\right)e^{i(kx+lt+\xi_0)}, \\ N_1(x,t) &= \frac{1}{2}\lambda^2\left(3\tanh^2\frac{\lambda}{2}\xi-1\right). \end{aligned} \tag{25}$$

$E_2(x,t)$ and $N_2(x,t)$ are turned into

$$\begin{aligned} E_2(x,t) &= \pm\frac{3\sqrt{2}}{2}\lambda^2\left(\tanh^2\frac{\lambda}{2}\xi-1\right)e^{i(kx+lt+\xi_0)}, \\ N_2(x,t) &= \frac{3}{2}\lambda^2\left(\tanh^2\frac{\lambda}{2}\xi-1\right). \end{aligned} \tag{26}$$

3.2. The Trigonometric Function Solutions to (1) *If* $\alpha < 0$. Consider

$$\begin{aligned} &E_3(x,t) \\ &= \pm\frac{\sqrt{2}\alpha}{2} \\ &\quad\times\left[3\left(\frac{-c_1\sin(1/2)\sqrt{-\alpha}\xi+c_2\cos(1/2)\sqrt{-\alpha}\xi}{c_1\cos(1/2)\sqrt{-\alpha}\xi+c_2\sin(1/2)\sqrt{-\alpha}\xi}\right)^2+1\right] \\ &\quad\times e^{i(kx+lt+\xi_0)}, \\ &N_3(x,t) \\ &= \frac{\alpha}{2}\left[3\left(\frac{-c_1\sin(1/2)\sqrt{-\alpha}\xi+c_2\cos(1/2)\sqrt{-\alpha}\xi}{c_1\cos(1/2)\sqrt{-\alpha}\xi+c_2\sin(1/2)\sqrt{-\alpha}\xi}\right)^2+1\right], \end{aligned} \tag{27}$$

where $k = \pm(1/6)\sqrt{3(\beta_2+\alpha)}$, $l = (1/12)(12\beta_1+\beta_2+\lambda^2 \mp 12\sqrt{2}\lambda^2-4\mu \pm 48\sqrt{2}\mu)$, and

$$\begin{aligned} &E_4(x,t) \\ &= \pm\frac{3\alpha}{2} \\ &\quad\times\left[\left(\frac{-c_1\sin(1/2)\sqrt{-\alpha}\xi+c_2\cos(1/2)\sqrt{-\alpha}\xi}{c_1\cos(1/2)\sqrt{-\alpha}\xi+c_2\sin(1/2)\sqrt{-\alpha}\xi}\right)^2+1\right] \\ &\quad\times e^{i(kx+lt+\xi_0)}, \\ &N_4(x,t) \\ &= -\frac{3\alpha}{2}\left[\left(\frac{-c_1\sin(1/2)\sqrt{-\alpha}\xi+c_2\cos(1/2)\sqrt{-\alpha}\xi}{c_1\cos(1/2)\sqrt{-\alpha}\xi+c_2\sin(1/2)\sqrt{-\alpha}\xi}\right)^2+1\right], \end{aligned} \tag{28}$$

where $k = \pm(1/6)\sqrt{3(\beta_2-\alpha)}$, $l = (1/12)(12\beta_1+\beta_2-\lambda^2 \pm 12\sqrt{2}\lambda^2+4\mu \mp 48\sqrt{2}\mu)$.

3.3. The Rational Function Solutions to (1) *If* $\alpha = 0$. We obtain

$$\begin{aligned} E_5(x,t) &= \pm\frac{6\sqrt{2}c_2^2}{(c_1+c_2\xi)^2}e^{i(kx+lt+\xi_0)}, \\ N_5(x,t) &= \frac{6c_2^2}{(c_1+c_2\xi)^2}, \end{aligned} \tag{29}$$

where $k = \pm(1/6)\sqrt{3(\beta_2+\alpha)}$, $l = (1/12)(12\beta_1+\beta_2+\lambda^2 \mp 12\sqrt{2}\lambda^2-4\mu \pm 48\sqrt{2}\mu)$, and

$$\begin{aligned} E_6(x,t) = E_{9,10}(x,t) &= \pm\frac{6\sqrt{2}c_2^2}{(c_1+c_2\xi)^2}e^{i(kx+lt+\xi_0)}, \\ N_6(x,t) = N_{9,10}(x,t) &= \frac{6c_2^2}{(c_1+c_2\xi)^2}, \end{aligned} \tag{30}$$

where

$$k = \pm\frac{1}{6}\sqrt{3(\beta_2-\alpha)}, \tag{31}$$

$$l = \frac{1}{12}\left(12\beta_1+\beta_2+\lambda^2 \pm 12\sqrt{2}\lambda^2+4\mu \mp 48\sqrt{2}\mu\right). \tag{32}$$

4. Conclusion

The (G'/G)-expansion method is effectively employed to deal with the coupled Schrödinger-Boussinesq equations. The hyperbolic function solutions, the trigonometric function solutions, and the rational function solutions to the equations in the case of $\alpha > 0$, $\alpha < 0$, and $\alpha = 0$ are obtained. In particular, the well-known soliton solutions are only the special case of the hyperbolic-type solutions. We find several properties of solutions when $\xi \to \infty$.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (11171241, 11071177, 11226162), the Key Project of Chinese Ministry of Education (Grant no. 211162), and the Sichuan Province Science Foundation for Youths (no. 2012JQ0011).

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