Research Article

Equilibrium Asset and Option Pricing under Jump-Diffusion Model with Stochastic Volatility

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We study the equity premium and option pricing under jump-diffusion model with stochastic volatility based on the model in Zhang et al. 2012. We obtain the pricing kernel which acts like the physical and risk-neutral densities and the moments in the economy. Moreover, the exact expression of option valuation is derived by the Fourier transformation method. We also discuss the relationship of central moments between the physical measure and the risk-neutral measure. Our numerical results show that our model is more realistic than the previous model.

1. Introduction

Option pricing problem is one of the predominant concerns in the financial market. Since the advent of the Black-Scholes option pricing formula in [1], there has been an increasing amount of literature describing the theory and its practice. Due to drawbacks of the Black-Scholes model which cannot explain numerous empirical facts such as large and sudden movements in prices, heavy tails, volatility clustering, the incompleteness of markets, and the concentration of losses in a few large downward moves, many option valuation models have been proposed and tested to fit those empirical facts. Jump-diffusion models with stochastic volatility could overcome these drawbacks of the Black-Scholes model in [2–21]. Based on those advantages, in this paper, we focus on studying the jump-diffusion model with stochastic volatility.

Different from the Black-Scholes framework, we use jump diffusion to describe the price dynamics of underlying asset. The market of our model is incomplete; that is, it is not possible to replicate the payoff of every contingent claim by a portfolio, and there are several equivalent martingale measures. How to choose a consistent pricing measure from the set of equivalent martingale measures becomes an important problem. This means that we need to find some criteria to determine one from the set of equivalent martingale measures in some economically or mathematically motivated fashion. A unique martingale measure was found by various researchers via using optimal criteria, for instance, minimal martingale criterion, minimal entropy martingale criterion, and utility maximization criterion [22–31].

General equilibrium framework method is also a popular method to deal with the option pricing in an incomplete market. General equilibrium framework is initially introduced by Lucas Jr. (1978) [32], Cox et al. (1985) [33] and developed by Vasanttilak and Lee (1990) [34], Pan (2002) [35], Liu and Pan (2003) [36], Liu et al. (2005) [37], Bates (2008) [38], Santa-Clara and Yan (2004) [12], and Zhang et al. (2012) [6]. They assumed that there is a representative investor who wants to maximize an objective function in a rational expectations economy where there are one risk-free asset and one risky asset. When the market is clear, the representative investor takes all money into the risky asset. In this paper, we build a general equilibrium model which is the same as that due to Santa-Clara and Yan (2004) [12]. Under this model, we obtain an exact expression of the equity premium and the pricing kernel in a general equilibrium economy. This can be regarded as a great contribution to the literature.

The pricing kernel which acts like the physical and risk-neutral densities and moments in the economy is also a vitally important problem in mathematical finance. In some

constant volatility models with jump diffusions, Pan (2002), Liu and Pan (2003), and Liu et al. (2005) [35–37] derived the pricing kernel with some restrictions of jump sizes in a general equilibrium setting. Recently, Zhang et al. (2012) [6] presented an analytical form for the pricing kernel without any distributional assumption on the jumps. In this paper, we extend the results of Zhang et al. (2012) to the pricing kernel with stochastic volatility.

Duffie et al. (2000) [10] and Chacko and Das (2002) [39] presented a transform analysis to price the valuation of options for affine jump diffusions with stochastic volatility. Lorig and Lozano-Carbasse (2013) [21] studied option pricing in exponential Lévy-type models with stochastic volatility and stochastic jump intensity. Lewis (2008) [8] used Fourier transformation methods to obtain the transform-based solution of option price. In this paper, we employ the Fourier transformation method to get the exact expression of European options.

Finally, we get the relationship of central moments between the physical measure and the risk-neutral measure which can help us to study the negative variance risk premium, the implied volatility smirk, and the prediction of realized skewness. Some relevant work has been done by Bakshi et al. (2003) [13], Carr and Wu (2009) [14], and Neuberger (2012) [15]. However, to the best of our knowledge, except for Zhang et al. (2012) [6], there is no literature studying this relationship. In this paper, we extend it to a stochastic volatility case. This can be regarded as another contribution to the literature.

The rest of the paper is organized as follows. In Section 2, we present our jump-diffusion model with stochastic volatility. In Section 3, we study the equity premium in a general equilibrium economy. The option pricing and the relationship of central moments between the physical measure and the risk-neutral measure are studied in Section 4. Numerical results and conclusions are shown in Sections 5 and 6, respectively.

2. Our Model

In this paper, we consider the financial market with the following two basic assets.

(i) *A Bond* whose price P_t at time t is given by

$$dP_t = rP_t dt, \qquad P_0 = 1. \tag{1}$$

(ii) *A Stock* whose price S_t at time t is given by

$$\frac{dS_t}{S_t} = \mu \, dt + \sqrt{V_t} dB_t^S + (e^x - 1) \, dN_t - \lambda E (e^x - 1) \, dt,$$

$$S_0 > 0, \tag{2}$$

$$dV_t = \kappa (\theta - V_t) \, dt + \varepsilon \sqrt{V_t} dB_t^V, \quad V_0 > 0,$$

where $t \in [0, T]$ and $T > 0$; on the filtered complete space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, there are B_t^S, B_t^V which both are 1-dimensional Brownian motions with $dB_t^S dB_t^V = \rho dt$, and dN_t is

a Poisson process with the constant jump intensity λ. $E[\cdot]$ is the expectation under the physical measure. The jump size x is stochastic and r, μ, κ, and θ are constants.

The integration form of stock process in (2) is given by

$$\ln \frac{S_T}{S_t} = \int_t^T \left(\mu - \frac{1}{2} V_\tau \right) d\tau - \left[\lambda E (e^x - 1) \right] \tau$$

$$+ \int_t^T \sqrt{V_\tau} dB_\tau^S + \sum_{i=1}^{N_\tau} x_i. \tag{3}$$

We suppose that the portfolio is $(\omega, 1 - \omega)$ which, respectively, means the fraction of wealth invested in the stock and money market; then the wealth process $W(t)$ and the initial wealth $W(0) = w_0 > 0$ satisfy the equations as follows:

$$\frac{dW_t}{W_t} = \left[r + \omega\phi - \omega\lambda E (e^x - 1) \right] dt$$

$$+ \omega \sqrt{V_t} dB_t^S + \omega (e^x - 1) \, dN_t, \tag{4}$$

$$dV_t = \kappa (\theta - V_t) \, dt + \varepsilon \sqrt{V_t} dB_t^V,$$

where $\phi = \mu - r$ is the equity premium.

The representative investor maximizes his/her expected utility:

$$J (W(t), V(t), t) = \max E_t [U (W(T))], \tag{5}$$

where $E_t[\cdot]$ is the conditional expectation and equals $E[\cdot \mid \mathcal{F}_t]$.

For tractability, we concentrate our attention on the case of constant relative risk aversion (CRRA) utility function:

$$U (W_T, T) = \frac{W_T^{1-\gamma}}{1 - \gamma}, \quad \gamma > 0, \ \gamma \neq 1, \tag{6}$$

where the constant γ is the relative risk aversion coefficient.

3. Equity Premium

The equity premium is very important for option pricing in general equilibrium framework. Following the idea of Santa-Clara and Yan (2004) [12] and Zhang et al. (2012) [6], we obtain the equilibrium equity premium by modeling general equilibrium economy in the following proposition.

Proposition 1. *In general equilibrium framework, the equilibrium equity premium is given by*

$$\phi = V_t \left[\gamma - \varepsilon\rho A(t) \right] + \lambda E \left[(1 - e^{-\gamma x}) (e^x - 1) \right], \tag{7}$$

where

$$A(t) = \gamma (1 - \gamma) \frac{1 - e^{-\alpha(t-T)}}{\beta_- - \beta_+ e^{-\alpha(t-T)}}, \tag{8}$$

$$\beta_\pm = \kappa \pm \alpha, \quad \alpha = \sqrt{\kappa^2 - \varepsilon^2 \gamma (1 - \gamma)}.$$

Proof. From the optimal control problem (5), we get the Bellman equation as follows:

$$0 = J_t + \max_{\omega} \{\mathscr{L}(J)\}, \tag{9}$$

where

$$\begin{aligned}
\mathscr{L}(J) = & \left[r + \omega\phi - \omega\lambda E(e^x - 1)\right] WJ_W \\
& + \frac{1}{2}\omega^2 W^2 VJ_{WW} + \kappa(\theta - V) J_V \\
& + \frac{1}{2}\varepsilon^2 VJ_{VV} + \varepsilon\omega WV\rho J_{WV} \\
& + \lambda E\left[J\left(W(1 + \omega(e^x - 1)), V, t\right) - J\right].
\end{aligned} \tag{10}$$

Equating the derivatives of the Bellman equation (9) with respect to ω to zero, we have following equation:

$$\begin{aligned}
0 = & \left[\phi - \lambda E(e^x - 1)\right] WJ_W \\
& + \omega W^2 VJ_{WW} + \varepsilon WV\rho J_{WV} \\
& + \lambda E\left[J_W\left(W(1 + \omega(e^x - 1)), V, t\right) \cdot W(e^x - 1)\right].
\end{aligned} \tag{11}$$

In equilibrium, the money market is in zero net supply. Therefore, the representative investor holds all the wealth in the stock market; that is, $\omega = 1$. Then we can get the expression of ϕ from (11):

$$\begin{aligned}
\phi = & -V\frac{WJ_{WW}}{J_W} + \lambda E(e^x - 1) - \varepsilon V\rho\frac{J_{WV}}{J_W} \\
& - \frac{\lambda}{J_W} E\left[J_W(We^x, V, t) \cdot (e^x - 1)\right].
\end{aligned} \tag{12}$$

Then Bellman equation (9) can be written as follows:

$$\begin{aligned}
0 = & J_t + rWJ_W - \frac{1}{2}W^2 VJ_{WW} + \kappa(\theta - V) J_V \\
& + \frac{1}{2}\varepsilon^2 VJ_{VV} - \lambda WE\left[J_W(We^x, V, t) \cdot (e^x - 1)\right] \\
& + \lambda E\left[J(We^x, V, t)\right] - \lambda J.
\end{aligned} \tag{13}$$

From (6), we conjecture that

$$J(W_t, V_t, t) = \exp\left(A(t)V_t + B(t)\right)\frac{W_t^{1-\gamma}}{1-\gamma}. \tag{14}$$

Then, substituting (11) into (14), we obtain

$$\begin{aligned}
0 = & \dot{A}(t)V_t + \dot{B}(t) + r(1-\gamma) + \frac{1}{2}\gamma(1-\gamma)V \\
& + \kappa(\theta - V)A(t) + \frac{1}{2}\varepsilon^2 VA^2(t) \\
& + \lambda\gamma E\left[e^{(1-\gamma)x}\right] + \lambda(1-\gamma)E\left[e^{-\gamma x}\right] - \lambda.
\end{aligned} \tag{15}$$

This leads to a system of two ordinary differential equations (ODEs):

$$\dot{A}(t) + \frac{1}{2}\varepsilon^2 A^2(t) - \kappa A(t) + \frac{1}{2}\gamma(1-\gamma) = 0, \tag{16}$$
$$A(T) = 0,$$

$$\dot{B}(t) + \kappa\theta A(t) + a = 0, \tag{17}$$
$$B(T) = 0,$$

where

$$a = r(1-\gamma) + \lambda\gamma E\left[e^{(1-\gamma)x}\right] + \lambda(1-\gamma)E\left[e^{-\gamma x}\right] - \lambda. \tag{18}$$

This system (16) and (17) can be solved explicitly. First, we solve the first ODE (16), which is the Riccati differential equation. Making the substitution

$$A(t) = \frac{2}{\varepsilon^2}\frac{C'(t)}{C(t)}, \tag{19}$$

we obtain the second-order differential equation

$$C''(t) - \kappa C'(t) + \frac{\varepsilon^2}{4}\gamma(1-\gamma)C(t) = 0, \tag{20}$$
$$C'(T) = 0.$$

A general solution has the form

$$C(t) = d_+ e^{(1/2)\beta_+ t} + d_- e^{(1/2)\beta_- t}, \tag{21}$$

where

$$\beta_\pm = \kappa \pm \alpha, \quad \alpha = \sqrt{\kappa^2 - \varepsilon^2\gamma(1-\gamma)},$$

$$d_+ = \frac{\beta_- C(T)}{(\beta_- - \beta_+)e^{(1/2)\beta_+ T}}, \quad d_- = \frac{\beta_+ C(T)}{(\beta_+ - \beta_-)e^{(1/2)\beta_- T}}. \tag{22}$$

Thus,

$$A(t) = \gamma(1-\gamma)\frac{1 - e^{-\alpha(t-T)}}{\beta_- - \beta_+ e^{-\alpha(t-T)}}. \tag{23}$$

Then, the solution of the second ODE (17) is

$$B(t) = \frac{2\kappa\theta}{\varepsilon^2}\left[\frac{1}{2}\beta_+(t-T) + \ln\frac{\beta_+ e^{-\alpha(t-T)} - \beta_-}{2\alpha}\right] + a(t-T). \tag{24}$$

Substituting (14) into (12), we get Proposition 1. □

Remark 2. In the special case where there is no stochastic volatility and jumps, $V_t = \sigma^2$ and $\varepsilon = \kappa = 0$, and consequently $\phi = \sigma^2\gamma$ which is constant in Merton (1976) [2]. In the special case where there is no stochastic volatility, $\varepsilon = 0$, and consequently $\phi = \sigma^2\gamma + \lambda E[(1-e^{-\gamma x})(e^x-1)]$ which is constant in Zhang et al. (2012) [6].

4. Option Pricing

In this section, we will study the pricing kernel and the option pricing in general equilibrium framework. We first derive the pricing kernel which acts like the physical and risk-neutral densities in the economy and is the key to obtain the PDE of option price as follows.

Proposition 3. *In general equilibrium framework, the pricing kernel is given in differential form by*

$$\frac{d\pi_t}{\pi_t} = -r\,dt - (\gamma - \varepsilon\rho A)\sqrt{V_t}dB_t^S \tag{25}$$
$$+ (e^y - 1)\,dN_t - \lambda E(e^y - 1)\,dt,$$

and the integration is given by

$$\frac{\pi_T}{\pi_t} = \exp\left\{ [-r - \lambda E(e^y - 1)]\,\tau \right.$$
$$- \frac{1}{2}\int_t^T (\gamma - \varepsilon\rho A_\tau)^2 V_\tau d\tau \tag{26}$$
$$\left. - \int_t^T (\gamma - \varepsilon\rho A_\tau)\sqrt{V_\tau}dB_\tau^S + \sum_{i=1}^{N_\tau} y_i \right\},$$

where $\tau = T - t$.

The martingale condition, $\pi_t S_t = E_t[\pi_T S_T]$, requires that the jump size y satisfies the following restriction:

$$E\left[(e^y - e^{-\gamma x})(e^x - 1)\right] = 0. \tag{27}$$

Proof. To satisfy the martingale condition, $\pi_t S_t = E_t[\pi_T S_T]$, from (3) and (26), we have

$$E \exp\left\{ \int_t^T \left[\phi_\tau - \frac{1}{2}V_\tau - \frac{1}{2}(\gamma - \varepsilon\rho A_\tau)^2 V_\tau \right] d\tau \right.$$
$$- \lambda\left[E(e^y - 1)\tau + E(e^x - 1) \right]\tau$$
$$+ \int_t^T \left[1 - (\gamma - \varepsilon\rho A_\tau) \right]\sqrt{V_\tau}dB_\tau^S \tag{28}$$
$$\left. + \sum_{i=1}^{N_\tau} x_i + \sum_{i=1}^{N_\tau} y_i \right\} = 1.$$

Substituting (7) into (28), we have

$$E \exp\left\{ \int_t^T \lambda E\left[(1 - e^{-\gamma x})(e^x - 1)\right]\tau \right.$$
$$- \lambda\left[E(e^y - 1)\tau + E(e^x - 1) \right]\tau \tag{29}$$
$$\left. + \sum_{i=1}^{N_\tau} x_i + \sum_{i=1}^{N_\tau} y_i \right\} = 1.$$

Using the property of Poisson process, $E[\exp(\sum_{i=1}^{N_\tau} x_i)] = \exp[\lambda E(e^x - 1)\tau]$, it is easy to obtain Proposition 3. $\square$

Remark 4. In this market, there is only one tradable asset, a stock with price S_t, but there are at least two dimensions of risk, diffusive risk, and jump risk. Therefore, the market is incomplete and the pricing kernel is not unique. The nonuniqueness of the pricing kernel can be justified by the fact that the distribution of jump size y in the pricing kernel can be arbitrary as long as it satisfies the martingale restriction (27). In a special case, we can choose $y = -\gamma x$, as in Liu et al. (2005) [37].

Remark 5. With Proposition 3, we define a new probability measure Q:

$$\frac{dQ}{dP} = Z_T = e^{rT}\pi_T$$
$$= \exp\left\{ -\lambda E(e^y - 1)T - \frac{1}{2}\int_0^T (\gamma - \varepsilon\rho A_\tau)^2 V_\tau d\tau \right. \tag{30}$$
$$\left. - \int_0^T (\gamma - \varepsilon\rho A_\tau)\sqrt{V_\tau}dB_\tau^s + \sum_{i=1}^{N_T} y_i \right\},$$

since, for any assets P_t at time t, we have

$$E_t^Q[P_T] = \frac{1}{Z_t}E_t[Z_T P_T] = e^{r(T-t)}E_t\left[\frac{\pi_T}{\pi_t}P_T\right] = e^{r(T-t)}P_t, \tag{31}$$

which means Q is a risk-neutral probability measure.

Lemma 6. *Define a new probability measure, Q^*, by the following Radon-Nikodym derivative:*

$$\frac{dQ^*}{dP} = \exp\left\{ [-\lambda E(e^y - 1)]T + \sum_{i=1}^{N_T} y_i \right\}; \tag{32}$$

then the following relation

$$E[e^y f(x)] = E(e^y) E^{Q^*}[f(x)] \tag{33}$$

is true.

Proof. The change of probability measure formula gives

$$E^{Q^*}[f(x)] = E\left[\frac{dQ^*}{dP}f(x)\right]$$
$$= \exp\left\{[-\lambda E(e^y - 1)]T\right\} E\left[f(x)\prod_{i=1}^{N_T} e^{y_i}\right]$$
$$= \exp\left\{[-\lambda E(e^y - 1)]T\right\}$$
$$\times \sum_{n=1}^{+\infty} e^{-\lambda T}\frac{(\lambda T)^n}{n!}E\left[f(x)\prod_{i=1}^{n} e^{y_i}\right]. \tag{34}$$

Since y_i, $i = 1, 2, \ldots i, \ldots, n$, is i.i.d. and y and x are correlated, this means that only one of the y_i is correlated with

x. Without loss of generality, we assume that y_n is correlated with x and other $y_i s$ are independent of x. Then we have

$$
\begin{aligned}
E^{Q^*}[f(x)] &= E\left[\frac{dQ^*}{dP}f(x)\right] \\
&= \exp\{[-\lambda E(e^y - 1)]T\}E\left[f(x)\prod_{i=1}^{N_T}e^{y_i}\right] \\
&= E[f(x)e^y]\exp\{[-\lambda E(e^y - 1)]T\} \\
&\quad \times \sum_{n=1}^{+\infty}e^{-\lambda T}\frac{(\lambda T)^n}{n!}E\left[f(x)\prod_{i=1}^{n-1}e^{y_i}\right] \\
&= \frac{E[e^y f(x)]}{E(e^y)}.
\end{aligned} \tag{35}
$$

Remark 7. These results are also true in Q measure, because the difference between Q and Q^* is the Brownian motion that is independent of the jumps.

Now, we consider a European call written on the stock price S_t at time t. The option has a payoff function $(S_T - K)^+$ at time T. Its price is denoted as $c(S_t, V_t, t)$ at time t. We derive a PDE which $c(S_t, V_t, t)$ has to satisfy in the following proposition.

Proposition 8. *In general equilibrium framework, the price of European call option satisfies the following PDE:*

$$
\begin{aligned}
0 &= c_t + \frac{1}{2}VS^2 c_{SS} + \frac{1}{2}\varepsilon^2 V c_{VV} + \varepsilon\rho VS c_{SV} \\
&\quad + \left[r - \lambda^Q E^Q(e^x - 1)\right]Sc_S - rc + \left[\kappa^Q(\theta^Q - V)\right]c_V \\
&\quad + \lambda^Q\left\{E^Q[c(Se^x, V, t)] - c(S, V, t)\right\}, \\
&\qquad c(S, V, T) = (S - K)^+,
\end{aligned} \tag{36}
$$

where

$$
E^Q[f(x)] = \frac{E[e^y f(x)]}{E(e^y)}, \qquad \lambda^Q = \lambda E(e^y),
$$
$$
\kappa^Q = \kappa + (\gamma - \varepsilon\rho A)\varepsilon\rho, \qquad \theta^Q = \frac{\kappa\theta}{\kappa + (\gamma - \varepsilon\rho A)\varepsilon\rho}. \tag{37}
$$

Proof. First, we rewrite the stock with continue part and jump part:

$$
dS = d^C S + S(e^x - 1)dN_t, \tag{38}
$$

where

$$
\begin{aligned}
d^C S &= [r + \phi - \lambda E(e^x - 1)]S\,dt + \sqrt{V}S\,dB^S \\
dV &= \kappa(\theta - V)\,dt + \varepsilon\sqrt{V}dB_t^V.
\end{aligned} \tag{39}
$$

Similarly,

$$
d\pi = d^C\pi + (e^y - 1)\pi dN_t, \tag{40}
$$

where

$$
d^C\pi = [-r - \lambda E(e^y - 1)]\pi dt - (\gamma - \varepsilon\rho A)\sqrt{V}\pi dB_t^S, \tag{41}
$$

$$
dc = d^C c + [c(Se^x, V, t) - c(S, V, t)]dN_t, \tag{42}
$$

where

$$
\begin{aligned}
d^C c &= \left\{c_t + (r + \phi - \lambda E(e^x - 1))Sc_S + \frac{1}{2}VS^2 c_{SS}\right. \\
&\quad \left. + \kappa(\theta - V)c_V + \frac{1}{2}\varepsilon^2 V c_{VV} + c_{SV}\varepsilon\rho SV\right\}dt \\
&\quad + \sqrt{V}Sc_S dB^S + \varepsilon\sqrt{V}c_V dB_t^V.
\end{aligned} \tag{43}
$$

From (40) and (42), we get

$$
d(\pi c) = d^C(\pi c) + [\pi e^y c(Se^x, V, t) - \pi c(S, V, t)]dN_t, \tag{44}
$$

where

$$
\begin{aligned}
d^C(\pi c) &= c[-r - \lambda E(e^y - 1)]\pi dt \\
&\quad - c(\gamma - \varepsilon\rho A)\sqrt{V}\pi dB_t^S \\
&\quad + \pi\left\{c_t + (r + \phi - \lambda E(e^x - 1))Sc_S + \frac{1}{2}VS^2 c_{SS}\right. \\
&\quad \left. + \kappa(\theta - V)c_V + \frac{1}{2}\varepsilon^2 V c_{VV} + c_{SV}\varepsilon\rho SV\right\}dt \\
&\quad + \pi\sqrt{V}Sc_S dB^S + \pi\varepsilon\sqrt{V}c_V dB_t^V \\
&\quad - V\pi(\gamma - \varepsilon\rho A)(Sc_S dt + \varepsilon\rho c_V dt).
\end{aligned} \tag{45}
$$

The martingale condition $E[d(\pi c)] = 0$ requires

$$
\begin{aligned}
0 &= c_t + \frac{1}{2}VS^2 c_{SS} + \frac{1}{2}\varepsilon^2 V c_{VV} \\
&\quad + [(r + \phi - V(\gamma - \varepsilon\rho A) - \lambda E(e^x - 1))]Sc_S \\
&\quad - [r + \lambda E(e^y - 1)]c \\
&\quad + [\kappa(\theta - V) - V(\gamma - \varepsilon\rho A)\varepsilon\rho]c_V \\
&\quad + \varepsilon\rho SV c_{SV} + \lambda E[e^y c(Se^x, V, t)] - \lambda c(S, V, t).
\end{aligned} \tag{46}
$$

Using Lemma 6, we have

$$
E^Q[f(x)] = \frac{E[e^y f(x)]}{E(e^y)}. \tag{47}
$$

Denoting $\lambda^Q = \lambda E(e^y)$ and using the restriction condition $E[(e^y - e^{-\gamma x})(e^x - 1)] = 0$, $\phi = V[\gamma - \varepsilon\rho A] + \lambda E[(1 - e^{-\gamma x})(e^x - 1)]$

in Proposition 1 and the terminal payoff function $c(S, V, T) = (S - K)^+$, we obtain

$$0 = c_t + \frac{1}{2} V S^2 c_{SS} + \frac{1}{2} \varepsilon^2 V c_{VV} + \varepsilon \rho V S c_{SV}$$

$$+ \left[r - \lambda^Q E^Q (e^x - 1) \right] S c_S - rc$$

$$+ \left[\kappa (\theta - V) - V (\gamma - \varepsilon \rho A) \varepsilon \rho \right] c_V \qquad (48)$$

$$+ \lambda^Q \left\{ E^Q \left[c (S e^x, V, t) \right] - c (S, V, t) \right\}$$

$$c (S, V, T) = (S - K)^+.$$

If we denote $\kappa^Q = \kappa + (\gamma - \varepsilon \rho A) \varepsilon \rho$ and $\theta^Q = (\kappa \theta / (\kappa + (\gamma - \varepsilon \rho A) \varepsilon \rho))$, then (48) can be written as

$$0 = c_t + \frac{1}{2} V S^2 c_{SS} + \frac{1}{2} \varepsilon^2 V c_{VV} + \varepsilon \rho V S c_{SV}$$

$$+ \left[r - \lambda^Q E^Q (e^x - 1) \right] S c_S$$

$$- rc + \left[\kappa^Q \left(\theta^Q - V \right) \right] c_V \qquad (49)$$

$$+ \lambda^Q \left\{ E^Q \left[c (S e^x, V, t) \right] - c (S, V, t) \right\}$$

$$c (S, V, T) = (S - K)^+. \qquad \square$$

Remark 9. The stock process (2) in a risk-neutral measure Q can be written as

$$\frac{dS_t}{S_t} = r \, dt + \sqrt{V_t} d\widetilde{B}_t^S + (e^x - 1) dN_t - \lambda^Q E^Q (e^x - 1) dt,$$

$$S_0 > 0,$$

$$dV_t = \kappa^Q \left(\theta^Q - V_t \right) dt + \varepsilon \sqrt{V_t} d\widetilde{B}_t^V, \quad V_0 > 0,$$

$$(50)$$

where

$$d\widetilde{B}_t^S = dB_t^S + (\gamma - \varepsilon \rho A) \sqrt{V_t} dt,$$

$$d\widetilde{B}_t^V = dB_t^V + \rho (\gamma - \varepsilon \rho A) \sqrt{V_t} dt,$$

$$\lambda^Q = \lambda E (e^y), \qquad \kappa^Q = \kappa + (\gamma - \varepsilon \rho A) \varepsilon \rho,$$

$$\theta^Q = \frac{\kappa \theta}{\kappa + (\gamma - \varepsilon \rho A) \varepsilon \rho}, \qquad E^Q [f(x)] = \frac{E [e^y f(x)]}{E (e^y)}.$$

$$(51)$$

The proof is very easy. Substituting all equations in Remark 9 to (50) and using the equilibrium equity premium (7) and the restriction (27), we will get (2).

Furthermore, we also can understand $\lambda^Q = \lambda E(e^y)$ by

$$E_0^Q [N_T] = E_0 [Z_T N_T] = E_0 \left[e^{-\lambda E(e^y - 1)T + \sum_{i=1}^{N_T} y_i} N_T \right]$$

$$= \sum_{n=0}^{+\infty} e^{-\lambda E(e^y) T} \frac{\left(\lambda E (e^y) T \right)^n}{n!} n = \lambda E (e^y) T := \lambda^Q T.$$

$$(52)$$

Employing the Feynman-Kac theorem to (38), we also can obtain PDE (36) in Proposition 8.

Remark 10. $\kappa^Q = \kappa + (\gamma - \varepsilon \rho A) \varepsilon \rho$ and $\theta^Q = (\kappa \theta / (\kappa + (\gamma - \varepsilon \rho A) \varepsilon \rho))$ are similar forms in Heston (1993) [40].

Following the approach of Lewis (2000) [8], we will derive the pricing formula in a risk-neutral measure Q from the PDE.

Proposition 11. *In general equilibrium framework, the pricing formula is given by*

$$c (S, V, \tau)$$

$$= S - \frac{e^{-r\tau}}{2\pi} \int_{(i/2)-\infty}^{(i/2)+\infty} \frac{K^{ik+1}}{k^2 - ik} e^{-ik(r\tau + \ln S) + \Gamma^Q(\tau) V + \Lambda^Q(\tau)} dk,$$

$$(53)$$

where

$$\Gamma^Q (\tau) = \left(k^2 - ik \right) \frac{1 - e^{-\zeta^Q \tau}}{\xi_-^Q - \xi_+^Q e^{-\zeta^Q \tau}},$$

$$\Lambda^Q (\tau) = -\frac{2\kappa^Q \theta^Q}{\varepsilon^2} \left[\frac{1}{2} \xi_+^Q \tau + \ln \left(\frac{\xi_+^Q e^{-\zeta^Q \tau} - \xi_-^Q}{2\zeta^Q} \right) \right] + b^Q \tau,$$

$$\xi_\pm^Q = \left(ik\varepsilon\rho + \kappa^Q \right) \pm \zeta^Q,$$

$$\zeta^Q = \sqrt{\left(ik\varepsilon\rho + \kappa^Q \right)^2 + \varepsilon^2 \left(k^2 - ik \right)},$$

$$b^Q = ik\lambda^Q E^Q (e^x - 1) + \lambda^Q E^Q \left(e^{-ikx} - 1 \right).$$

$$(54)$$

Proof. We denote $X = \ln S$ and $g(X, V, t) := c(e^X, V, t) = c(S, V, t)$; then, the PDE (36) can be rewritten as

$$-g_t = -rg + \frac{1}{2} V g_{XX} + \frac{1}{2} \varepsilon^2 V g_{VV} + \varepsilon \rho V g_{XV}$$

$$+ \left[r - \lambda^Q E^Q (e^x - 1) - \frac{1}{2} V \right] g_X + \left[\kappa^Q \left(\theta^Q - V \right) \right] g_V$$

$$+ \lambda^Q \left\{ E^Q \left[g (X + x, V, t) \right] - g (X, V, t) \right\},$$

$$g (X, V, T) = \left(e^X - K \right)^+.$$

$$(55)$$

Let $G(k, V, t)$ be Fourier transform of $g(X, V, t)$:

$$G (k, V, t) = \int_{-\infty}^{+\infty} e^{ikX} g (X, V, t) dX \qquad (56)$$

with $G(k, V, T) = -(K^{ik+1} / (k^2 - ik))$.

Denote $k = k_r + ik_i$; the inverse Fourier transform is given by

$$g(X, V, t) = \frac{1}{2\pi} \int_{ik_i - \infty}^{ik_i + \infty} e^{-ikX} G(k, V, t)\, dk. \quad (57)$$

Then, the PDE (55) can be rewritten as

$$-G_t = -(1 + ik)\, rG + ik\lambda^Q E^Q (e^x - 1)\, G - \frac{1}{2} V\left(k^2 - ik\right) G$$

$$+ \frac{1}{2}\varepsilon^2 V G_{VV} - ik\varepsilon\rho V G_V$$

$$+ \left[\kappa^Q\left(\theta^Q - V\right)\right] G_V + \lambda^Q E^Q \left[e^{-ikx} - 1\right] G,$$

$$G(k, V, T) = -\frac{K^{ik+1}}{k^2 - ik}. \quad (58)$$

Denote $h \equiv e^{(1+ik)r\tau} G$; the PDE (58) can be rewritten as

$$h_\tau = \left[ik\lambda^Q E^Q (e^x - 1) - \frac{1}{2} V\left(k^2 - ik\right)\right] h$$

$$+ \frac{1}{2}\varepsilon^2 V h_{VV} + \left[\kappa^Q\left(\theta^Q - V\right) - ik\varepsilon\rho V\right] h_V \quad (59)$$

$$+ \lambda^Q E^Q \left[e^{-ikx} - 1\right] h$$

$$\text{with } h(k, V, 0) = -\frac{K^{ik+1}}{k^2 - ik}. \quad (60)$$

According to the study of Lewis (2000) [8], to solve the PDE (59) with the initial condition (60), it is enough to solve the same equation with the initial value being equal to one. We call this solution *fundamental transform* and denote it $h(k, V, \tau)$ which satisfies following equation by:

$$h_\tau = \left[ik\lambda^Q E^Q (e^x - 1) - \frac{1}{2} V\left(k^2 - ik\right)\right] h + \frac{1}{2}\varepsilon^2 V h_{VV}$$

$$+ \left[\kappa^Q\left(\theta^Q - V\right) - ik\varepsilon\rho V\right] h_V + \lambda^Q E^Q \left[e^{-ikx} - 1\right] h,$$

$$h(k, V, 0) = 1,$$

$$\quad (61)$$

and the option price satisfies

$$c(S, V, \tau)$$

$$= S - \frac{1}{2\pi} \int_{(i/2) - \infty}^{(i/2) + \infty} e^{-ik \ln S} e^{-(1+ik)r\tau} \frac{K^{ik+1}}{k^2 - ik} h(k, V, \tau)\, dk.$$

$$\quad (62)$$

Denoting $H(V, \tau) := h(k, V, \tau)$, then, the PDE (61) can be rewritten as

$$H_\tau = \left[ik\lambda^Q E^Q (e^x - 1) - \frac{1}{2} V\left(k^2 - ik\right)\right] H + \frac{1}{2}\varepsilon^2 V H_{VV}$$

$$+ \left[\kappa^Q\left(\theta^Q - V\right) - ik\varepsilon\rho V\right] H_V + \lambda^Q E^Q \left[e^{-ikx} - 1\right] H,$$

$$H(V, 0) = 1.$$

$$\quad (63)$$

We guess that a solution of (63) is as the following form:

$$H(V, \tau) = e^{\Gamma^Q(\tau)V + \Lambda^Q(\tau)}, \quad (64)$$

with $\Gamma^Q(0) = 0$ and $\Lambda^Q(0) = 0$.

This leads to a system of two ODEs:

$$\dot{\Gamma}^Q(\tau) - \frac{1}{2}\varepsilon^2\left(\Gamma^Q(\tau)\right)^2 + \left(ik\varepsilon\rho + \kappa^Q\right)\Gamma^Q(\tau)$$

$$+ \frac{1}{2}\left(k^2 - ik\right) = 0, \quad (65)$$

$$\Gamma^Q(0) = 0,$$

$$\dot{\Lambda}^Q(\tau) - \kappa^Q\theta^Q\Gamma^Q(\tau) - b^Q = 0,$$

$$\Lambda^Q(0) = 0, \quad (66)$$

where $b^Q = ik\lambda^Q E^Q(e^x - 1) + \lambda^Q E^Q(e^{-ikx} - 1)$.

This system can be solved explicitly. First, we solve the first ODE (65), which is the Riccati differential equation. Making the substitution

$$\Gamma^Q(\tau) = -\frac{2}{\varepsilon^2}\frac{I'(\tau)}{I(\tau)}, \quad (67)$$

we obtain the second-order differential equation

$$I''(\tau) + \left(ik\varepsilon\rho + \kappa^Q\right)I'(\tau) - \frac{\varepsilon^2}{4}\left(k^2 - ik\right)I(\tau) = 0,$$

$$I'(0) = 0. \quad (68)$$

A general solution of (68) has the form

$$I(\tau) = D_+^Q e^{(1/2)\xi_+^Q\tau} + D_-^Q e^{(1/2)\xi_-^Q\tau}, \quad (69)$$

where

$$\xi_\pm^Q = \left(ik\varepsilon\rho + \kappa^Q\right) \pm \zeta^Q,$$

$$\zeta^Q = \sqrt{\left(ik\varepsilon\rho + \kappa^Q\right)^2 + \varepsilon^2\left(k^2 - ik\right)}$$

$$D_+^Q = \frac{\xi_-^Q I(0)}{\xi_-^Q - \xi_+^Q}, \qquad D_-^Q = \frac{\xi_+^Q I(0)}{\xi_+^Q - \xi_-^Q}. \quad (70)$$

Thus,

$$\Gamma^Q(\tau) = \left(k^2 - ik\right)\frac{1 - e^{-\zeta^Q\tau}}{\xi_-^Q - \xi_+^Q e^{-\zeta^Q\tau}}. \quad (71)$$

Then, the solution of the second ODE (66) is

$$\Lambda^Q(\tau) = -\frac{2\kappa^Q\theta^Q}{\varepsilon^2}\left[\frac{1}{2}\xi_+^Q\tau + \ln\left(\frac{\xi_+^Q e^{-\zeta^Q\tau} - \xi_-^Q}{2\zeta^Q}\right)\right] + b^Q\tau.$$

$$\quad (72)$$

$\square$

Note. Although the pricing formula (40) contains a complex integral, the result is real.

With the European option price being well understood, we study the relations between the physical and risk-neutral densities and moments which can help us to study the negative variance risk premium, the implied volatility smirk, and the prediction of realized skewness. In this paper, we extend the results about those relations in [6] to a stochastic volatility case.

Proposition 12. *The risk-neutral density of the continuously compounded return within $\tau = T - t$, $R_\tau = \ln S_T / S_t$, S_t given in (38), is given by*

$$\varphi^Q (R_\tau \mid V, \tau) = \frac{e^{-r\tau}}{2\pi} \int_{i(i/2)-\infty}^{(i/2)+\infty} e^{-ik(R-r\tau)+\Gamma^Q(\tau)V+\Lambda^Q(\tau)} dk. \tag{73}$$

The first moment, second, third, and fourth central moments of the continuously compounded return in the risk-neutral measure are given by

$$E_t^Q (R_\tau) = \left[\mu - \frac{1}{2} E_t^Q (V_\tau) - \lambda^Q E^Q (e^x - 1 - x) \right] \tau,$$

$$E_t^Q \left[R_\tau - E_t^Q (R_\tau) \right]^2 = \tau E_t^Q (V_\tau) + \lambda^Q \tau \left[\left(\mu_x^Q \right)^2 + \mathrm{Var}^Q (x) \right],$$

$$E_t^Q \left[R_\tau - E_t^Q (R_\tau) \right]^3$$
$$= \lambda^Q \tau \left[\left(\mu_x^Q \right)^3 + 3\mu_x^Q \mathrm{Var}^Q (x) + 3rd^Q (x) \right],$$

$$E_t^Q \left[R_\tau - E_t^Q (R_\tau) \right]^4$$
$$= 3 \left(\lambda^Q \tau \left[\left(\mu_x^Q \right)^2 + \mathrm{Var}^Q (x) \right] + \tau E_t^Q [V_\tau] \right)^2$$
$$+ \lambda^Q \tau \left[\left(\mu_x^Q \right)^4 + 6 \left(\mu_x^Q \right)^2 \mathrm{Var}^Q (x) \right.$$
$$\left. + 4\mu_x^Q 3rd^Q (x) + 4th^Q (x) \right], \tag{74}$$

where $E_t^Q(V_\tau) = e^{-\kappa^Q \tau} V_0 + \theta^Q (1 - e^{-\kappa^Q \tau})$, $\mu_x^Q = E^Q(x)$ is the first moment, $\mathrm{Var}^Q(x) = E^Q[(x - \mu_x^Q)^2]$, $3rd^Q(x) = E^Q[(x - \mu_x^Q)^3]$, and $4th^Q(x) = E^Q[(x - \mu_x^Q)^4]$ are the second, third, and fourth central moments in the risk-neutral measure of random number x.

The physical density is given by

$$\varphi (R_\tau \mid V, \tau) = \frac{e^{-\mu\tau}}{2\pi} \int_{(i/2)-\infty}^{(i/2)+\infty} e^{-ik(R-\mu\tau)+\Gamma(\tau)V+\Lambda(\tau)} dk, \tag{75}$$

where

$$\Gamma (\tau) = \left(k^2 - ik \right) \frac{1 - e^{-\zeta\tau}}{\xi_- - \xi_+ e^{-\zeta\tau}},$$

$$\Lambda (\tau) = -\frac{2\kappa\theta}{\varepsilon^2} \left[\frac{1}{2} \xi_+ \tau + \ln \left(\frac{\xi_+ e^{-\zeta\tau} - \xi_-}{2\zeta} \right) \right] + b\tau,$$

$$\xi_\pm = (ik\varepsilon\rho + \kappa) \pm \zeta,$$

$$\zeta = \sqrt{(ik\varepsilon\rho + \kappa)^2 + \varepsilon^2 (k^2 - ik)},$$

$$b = ik\lambda E (e^x - 1) + \lambda E (e^{-ikx} - 1). \tag{76}$$

The first moment, second, third, and fourth central moments of the continuously compounded return in the physical measure are given by

$$E_t (R_\tau) = \left[\mu - \frac{1}{2} E_t (V_\tau) - \lambda E (e^x - 1 - x) \right] \tau,$$

$$E_t [R_\tau - E_t (R_\tau)]^2 = \tau E_t (V_\tau) + \lambda\tau \left[\mu_x^2 + \mathrm{Var} (x) \right],$$

$$E_t [R_\tau - E_t (R_\tau)]^3 = \lambda\tau \left[\mu_x^3 + 3\mu_x \mathrm{Var} (x) + 3rd (x) \right],$$

$$E_t [R_\tau - E_t (R_\tau)]^4 = 3 \left(\lambda\tau \left[\mu_x^2 + \mathrm{Var} (x) \right] + \tau E_t [V_\tau] \right)^2$$
$$+ \lambda\tau \left[\mu_x^4 + 6\mu_x^2 \mathrm{Var} (x) \right.$$
$$\left. + 4\mu_x 3rd (x) + 4th (x) \right], \tag{77}$$

where $E_t(V_\tau) = e^{-\kappa\tau} V_0 + \theta(1 - e^{-\kappa\tau})$, $\mu_x = E(x)$ is the first moment, $\mathrm{Var}(x) = E[(x - \mu_x)^2]$, $3rd(x) = E[(x - \mu_x)^3]$, and $4th(x) = E[(x - \mu_x)^4]$ are second, third, and fourth central moments in the physical measure of random number x.

Proof. We denote the corresponding probability function $\Phi = \mathrm{Prob}(S_T \leq K)$, and Φ also satisfies the PDE (36) with a different boundary condition $\Phi|_{t=T} = 1_{\{S_T \leq K\}}$. Under the Fourier transform, this boundary condition becomes (K^{ik}/ik). Solving the PDE (36) under this new boundary condition, we have

$$\Phi = \frac{e^{-r\tau}}{2\pi} \int_{i(i/2)-\infty}^{(i/2)+\infty} \frac{e^{ik(\ln(K/S)-r\tau)}}{ik} e^{\Gamma^Q(\tau)V+\Lambda^Q(\tau)} dk. \tag{78}$$

We denote the stock return between time t and T by R so that $K = Se^R$; then we can differentiate Φ with respect to R to get the density function $\varphi^Q(R_\tau \mid V, \tau)$. We find that the parameters of the physical stock price process in (2) are the same those of risk-neutral stock price process in (50); then the parameters of $\varphi(R_\tau \mid V, \tau)$ are just removed by the superscript $.^Q$.

Now we compute the first moment, second, third, and fourth central moments of the continuously compounded return in the physical measure. There are at least two methods to solve the problem of how to compute central moments.

One method is using the R_τ's characteristic function to calculate the nth moment of R_τ. Since

$$f (k, V, t) = \int_{-\infty}^{+\infty} e^{ikR} \varphi (R \mid V, \tau) dR,$$

$$\frac{\partial^n f}{\partial k^n} = (i)^n \int_{-\infty}^{+\infty} R^n e^{ikR} \varphi (R \mid V, \tau) dR, \tag{79}$$

$$E_t (R_\tau^n) = (-i)^n \frac{\partial^n f}{\partial k^n} \bigg|_{k=0},$$

then

$$E_t\left(R_\tau^n\right) = \int_{-\infty}^{+\infty} R^n e^{ikR} \varphi\left(R \mid V, \tau\right) dR \bigg|_{k=0}. \tag{80}$$

Although this method is very straightforward, it is too complicated to calculate the integration. We introduce another way to obtain the central moments by a direct calculation.

From (3), we have

$$R_\tau = \left[\mu - \lambda E\left(e^x - 1\right)\right]\tau - \frac{1}{2}\int_t^T V_\tau d\tau + \int_t^T \sqrt{V_\tau} dB_\tau^S + \sum_{i=1}^{N_\tau} x_i. \tag{81}$$

Then, the first moment of the continuously compounded return in the physical measure can be easy to be obtained:

$$E_t\left(R_\tau\right) = \left[\mu - \lambda E\left(e^x - 1 - x\right)\right]\tau$$

$$- \frac{1}{2}\int_t^T E_t\left[V_\tau\right] d\tau \tag{82}$$

$$= \left[\mu - \frac{1}{2}E_t\left[V_\tau\right] - \lambda E\left(e^x - 1 - x\right)\right]\tau.$$

Define the first central moment, second, third, and fourth central moments in the physical measure of random number x as $\mu_x = E(x)$, $\mathrm{Var}(x) = E[(x-\mu_x)^2]$, $3\mathrm{rd}(x) = E[(x-\mu_x)^3]$, and $4\mathrm{th}(x) = E[(x-\mu_x)^4]$.

From (2) we have

$$d\left(e^{\kappa\tau}V_\tau\right) = \kappa\theta e^{\kappa\tau}dt + \varepsilon e^{\kappa\tau}\sqrt{V_\tau}dB_\tau^V. \tag{83}$$

The integration is given by

$$e^{\kappa\tau}V_\tau = V_0 + \theta\left(e^{\kappa\tau} - 1\right)\tau + \varepsilon\int_0^\tau e^{\kappa\tau}\sqrt{V_\tau}dB_\tau^V. \tag{84}$$

Then

$$E_t\left(V_\tau\right) = e^{-\kappa\tau}V_0 + \theta\left(1 - e^{-\kappa\tau}\right). \tag{85}$$

Using (81), we have

$$E_t[R_\tau - E_t\left(R_\tau\right)]^2 = E_t\left[\int_t^T \sqrt{V_\tau}dB_\tau^S \cdot \int_t^T \sqrt{V_\tau}dB_\tau^S\right]$$

$$+ E_t\left[\left(\sum_{i=1}^{N_\tau} x_i - \lambda\tau\mu_x\right)^2\right]$$

$$= \int_t^T E_t\left[V_\tau\right] d\tau + E\left(N_\tau\right)E\left[\left(x - \mu_x\right)^2\right]$$

$$+ 2E\left[N_\tau\left(N_\tau - \lambda\tau\right)\right]E\left(x - \mu_x\right)\mu_x$$

$$+ E\left[\left(x - \mu_x\right)^2\right]\mu_x^2$$

$$= E_t\left[V_\tau\right]\tau + \lambda\tau\left[\mu_x^2 + \mathrm{Var}(x)\right],$$

$$E_t[R_\tau - E_t\left(R_\tau\right)]^3 = E_t\left[\left(\sum_{i=1}^{N_\tau} x_i - \lambda\tau\mu_x\right)^3\right]$$

$$= \lambda\tau\left[\mu_x^3 + 3\mu_x \mathrm{Var}(x) + 3\mathrm{rd}(x)\right],$$

$$E_t[R_\tau - E_t\left(R_\tau\right)]^4 = 3!!\left(E_t\left[\int_t^T V_\tau d\tau\right]\right)^2$$

$$+ 6\left[\int_t^T E_t\left[V_\tau\right]d\tau\right]$$

$$\cdot E_t\left[\left(\sum_{i=1}^{N_\tau} x_i - \lambda\tau\mu_x\right)^2\right]$$

$$+ E_t\left[\left(\sum_{i=1}^{N_\tau} x_i - \lambda\tau\mu_x\right)^4\right]$$

$$= 3\tau^2\left(E_t\left[V_\tau\right]\right)^2$$

$$+ 6\lambda\tau E_t\left[V_\tau\right]\left(\mu_x^2 + \mathrm{Var}(x)\right)$$

$$+ \lambda\tau\left[\mu_x^4 + 6\mu_x^2 \mathrm{Var}(x)\right.$$

$$\left. + 4\mu_x 3\mathrm{rd}(x) + 4\mathrm{th}(x)\right]$$

$$+ 3\lambda^2\tau^2\left[\mu_x^2 + \mathrm{Var}(x)\right]^2$$

$$= 3\left(\lambda\tau[\mu_x^2 + \mathrm{Var}(x)] + \tau E_t[V_\tau]\right)^2$$

$$+ \lambda\tau\left[\mu_x^4 + 6\mu_x^2 \mathrm{Var}(x)\right.$$

$$\left. + 4\mu_x 3\mathrm{rd}(x) + 4\mathrm{th}(x)\right]. \tag{86}$$

The results of risk-neutral moments can be obtained with the same procedure in the risk-neutral probability measure. □

Remark 13. If $E_t[V_\tau] = E_t^Q[V_\tau] = \sigma^2$, where σ^2 is constant, our results will degenerate into Proposition 3.19 by Zhang et al. (2012) [6].

We denote $\overline{\mathrm{Skewness}}$, $\overline{\mathrm{Kurtosis}}$ as the skewness and kurtosis in risk-neutral measure, respectively. Skewness and kurtosis are very important for asset pricing. For example, Bakshi et al. (2003) [13] concluded that variations in the risk-neutral skew were instrumental in explaining the differential pricing of individual equity options and found that less negatively skewed stocks have flatter smiles. From Proposition 12, we easily obtain the following three corollaries about skewness and kurtosis.

Corollary 14. *The skewness and kurtosis in physical measure are given by*

$$\mathrm{Skewness} = \frac{E_t[R_\tau - E_t\left(R_\tau\right)]^3}{\left[E_t[R_\tau - E_t\left(R_\tau\right)]^2\right]^{3/2}}$$

$$= \frac{\lambda\tau\left[\mu_x^3 + 3\mu_x \mathrm{Var}(x) + 3rd(x)\right]}{\left(E_t[V_\tau]\tau + \lambda\tau\left[\mu_x^2 + \mathrm{Var}(x)\right]\right)^{3/2}},$$

$$\text{Kurtosis} = \frac{E_t[R_\tau - E_t(R_\tau)]^4}{\left[E_t[R_\tau - E_t(R_\tau)]^2\right]^2} - 3$$

$$= \left(\left(3\left(\lambda\tau\left[\mu_x^2 + \text{Var}(x)\right] + \tau E_t[V_\tau]\right)^2\right.\right.$$

$$+ \lambda\tau\left[\mu_x^4 + 6\mu_x^2\,\text{Var}(x)\right.$$

$$\left.\left. + 4\mu_x 3rd(x) + 4th(x)\right]\right)$$

$$\left.\times\left(\left(E_t[V_\tau]\tau + \lambda\tau\left[\mu_x^2 + \text{Var}(x)\right]\right)^2\right)^{-1}\right)$$

$$- 3.$$

$$(87)$$

And the skewness and kurtosis in risk-neutral measure are given by

$$\overline{\text{Skewness}} = \frac{E_t^Q[R_\tau - E_t^Q(R_\tau)]^3}{\left[E_t^Q[R_\tau - E_t^Q(R_\tau)]^2\right]^{3/2}}$$

$$= \frac{\lambda^Q\tau\left[\left(\mu_x^Q\right)^3 + 3\mu_x^Q\text{Var}^Q(x) + 3rd^Q(x)\right]}{\left(\tau E_t^Q(V_\tau) + \lambda^Q\tau\left[\left(\mu_x^Q\right)^2 + \text{Var}^Q(x)\right]\right)^{(3/2)}},$$

$$\overline{\text{Kurtosis}} = \frac{E_t^Q[R_\tau - E_t^Q(R_\tau)]^4}{\left[E_t^Q[R_\tau - E_t^Q(R_\tau)]^2\right]^2} - 3$$

$$= \left(\left(3\left(\lambda\tau\left[\left(\mu_x^Q\right)^2 + \text{Var}^Q(x)\right] + \tau E_t^Q[V_\tau]\right)^2\right.\right.$$

$$+ \lambda^Q\tau\left[\left(\mu_x^Q\right)^4 + 6\left(\mu_x^Q\right)^2\text{Var}^Q(x)\right.$$

$$\left.\left. + 4\mu_x^Q 3rd^Q(x) + 4th^Q(x)\right]\right)$$

$$\left.\times\left(\left(\tau E_t^Q(V_\tau) + \lambda^Q\tau\left[\left(\mu_x^Q\right)^2 + \text{Var}^Q(x)\right]\right)^2\right)^{-1}\right)$$

$$- 3.$$

$$(88)$$

Corollary 15. *For jump size x, the first moment and the second, third, and fourth central moments in the physical measure are given by*

$$E_t(R_\tau) = \mu - \frac{1}{2}E_t(V_\tau) - \lambda E[(e^x - 1 - x)],$$

$$E_t[R_\tau - E_t(R_\tau)]^2 = E_t(V_\tau)\tau + \lambda\tau E[x^2],$$

$$E_t[R_\tau - E_t(R_\tau)]^3 = \lambda\tau E[x^3],$$

$$E_t[R_\tau - E_t(R_\tau)]^4 = \lambda\tau E[x^4] + 3\left(\lambda\tau E[x^2] + \tau E_t[V_\tau]\right)^2.$$

$$(89)$$

The first moment and the second, third, and fourth central moments in the risk-neutral are given by

$$E_t^Q(R_\tau) = \mu - \frac{1}{2}E_t^Q(V_\tau) - \lambda E\left[e^{-\gamma x}(e^x - 1 - x)\right],$$

$$E_t^Q[R_\tau - E_t^Q(R_\tau)]^2 = \tau E_t^Q(V_\tau) + \lambda\tau E\left[e^{-\gamma x}x^2\right],$$

$$E_t^Q[R_\tau - E_t^Q(R_\tau)]^3 = \lambda\tau E\left[e^{-\gamma x}x^3\right], \qquad (90)$$

$$E_t^Q[R_\tau - E_t^Q(R_\tau)]^4$$

$$= \lambda\tau E\left[e^{-\gamma x}x^4\right] + 3\left(\lambda\tau E\left[e^{-\gamma x}x^2\right] + \tau E_t^Q[V_\tau]\right)^2,$$

where $E_t(V_\tau) = \theta - e^{-\kappa\tau}(\theta - V_0)$,

$$E_t^Q(V_\tau) = \frac{\kappa\theta}{\kappa + (\gamma - \varepsilon\rho A)\varepsilon\rho}$$

$$- e^{-[\kappa+(\gamma-\varepsilon\rho A)\varepsilon\rho]\tau}\left(\frac{\kappa\theta}{\kappa + (\gamma - \varepsilon\rho A)\varepsilon\rho} - V_0\right).$$

$$(91)$$

Proof. To compute central moments in the physical measure, we have $E_t(V_\tau) = \theta - e^{-\kappa\tau}(\theta - V_0)$, $\mu_x = E(x)$, $\text{Var}(x) = E[(x - \mu_x)^2] = E[x^2] - \mu_x^2$, $3rd(x) = E[x^3] - 3\mu_x^2 E[x^2] + 2\mu_x^3$, and $4th(x) = E[(x - \mu_x)^4] = E[x^4] - 4\mu_x E[x^3] + 6\mu_x^2 E[x^2] - 3\mu_x^4$. □

Then, from Proposition 12, we have

$$E_t(R_\tau) = \mu - \frac{1}{2}E_t(V_\tau) - \lambda E[(e^x - 1 - x)],$$

$$E_t[R_\tau - E_t(R_\tau)]^2 = E_t(V_\tau)\tau + \lambda\tau E[x^2],$$

$$E_t[R_\tau - E_t(R_\tau)]^3 = \lambda\tau E[x^3],$$

$$E_t[R_\tau - E_t(R_\tau)]^4 = \lambda\tau E[x^4] + 3\left(\lambda\tau E[x^2] + \tau E_t[V_\tau]\right)^2.$$

$$(92)$$

In order to compute central moments in the physical measure, from Proposition 8, we have

$$E_t^Q(V_\tau) = \frac{\kappa\theta}{\kappa + (\gamma - \varepsilon\rho A)\varepsilon\rho}$$

$$- e^{-[\kappa+(\gamma-\varepsilon\rho A)\varepsilon\rho]\tau}\left(\frac{\kappa\theta}{\kappa + (\gamma - \varepsilon\rho A)\varepsilon\rho} - V_0\right),$$

$$\mu_x^Q = E^Q(x) = \frac{E\left[e^{-\gamma x}x\right]}{E(e^{-\gamma x})},$$

$$\text{Var}^Q(x) = \frac{E\left[e^{-\gamma x}x^2\right]}{E(e^{-\gamma x})} - \left(\frac{E[e^{-\gamma x}x]}{E(e^{-\gamma x})}\right)^2$$

$$3\text{rd}^Q(x) = \frac{E\left[e^{-\gamma x}x^3\right]}{E(e^{-\gamma x})} - 3\frac{E\left[e^{-\gamma x}x\right]}{E(e^{-\gamma x})}\frac{E\left[e^{-\gamma x}x^2\right]}{E(e^{-\gamma x})}$$

$$+ 2\left(\frac{E[e^{-\gamma x}x]}{E(e^{-\gamma x})}\right)^3,$$

$$4\text{th}^Q(x) = \frac{E\left[e^{-\gamma x}x^4\right]}{E(e^{-\gamma x})} - 4\frac{E\left[e^{-\gamma x}x^3\right]}{E(e^{-\gamma x})}\frac{E\left[e^{-\gamma x}x\right]}{E(e^{-\gamma x})}$$

$$+ 6\frac{E\left[e^{-\gamma x}x^2\right]}{E(e^{-\gamma x})}\left(\frac{E\left[e^{-\gamma x}x\right]}{E(e^{-\gamma x})}\right)^2 - 3\left(\frac{E\left[e^{-\gamma x}x\right]}{E(e^{-\gamma x})}\right)^4.$$

$$(93)$$

Then, from Proposition 12, we have

$$E_t^Q(R_\tau) = \mu - \frac{1}{2}E_t^Q(V_\tau) - \lambda E\left[e^{-\gamma x}\left(e^x - 1 - x\right)\right],$$

$$E_t^Q\left[R_\tau - E_t^Q(R_\tau)\right]^2 = E_t^Q(V_\tau)\tau + \lambda\tau E\left[e^{-\gamma x}x^2\right],$$

$$E_t^Q\left[R_\tau - E_t^Q(R_\tau)\right]^3 = \lambda\tau E\left[e^{-\gamma x}x^3\right],$$

$$(94)$$

$$E_t^Q\left[R_\tau - E_t^Q(R_\tau)\right]^4$$

$$= \lambda\tau E\left[e^{-\gamma x}x^4\right] + 3\left(\lambda\tau E\left[e^{-\gamma x}x^2\right] + \tau E_t^Q[V_\tau]\right)^2.$$

Corollary 16. *For small risk aversion coefficient, one has the following relationship between the third central moments in the neutral-risk measure and in the physical measure:*

$$E_t^Q\left[R_\tau - E_t^Q(R_\tau)\right]^3 \approx E_t[R_\tau - E_t(R_\tau)]^3$$

$$- \gamma\left[E_t[R_\tau - E_t(R_\tau)]^4\right.$$

$$(95)$$

$$\left. -3\left(E_t[R_\tau - E_t(R_\tau)]^2\right)^2\right].$$

Proof. From Corollary 15, for small risk aversion coefficient x, we have

$$E_t^Q\left[R_\tau - E_t^Q(R_\tau)\right]^3 = \lambda\tau E\left[e^{-\gamma x}x^3\right]$$

$$\approx \lambda\tau E\left[\left(1 - \gamma x\right)x^3\right]$$

$$= \lambda\tau E\left[x^3\right] - \gamma\lambda\tau E\left[x^4\right]$$

$$= E_t[R_\tau - E_t(R_\tau)]^3$$

$$(96)$$

$$- \gamma\left[E_t[R_\tau - E_t(R_\tau)]^4\right.$$

$$\left. -3\left(E_t[R_\tau - E_t(R_\tau)]^2\right)^2\right].\qquad\square$$
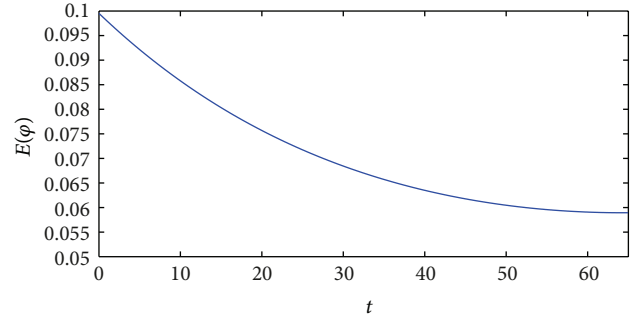

FIGURE 1: The expectation of ϕ.

5. Numerical Results

In this section, we use our model to solve the equity premium puzzle. First, we get the long equilibrium risk premium $\overline{\phi}$ for $t \to T$, where T is sufficiently large. In this case, $V_t \to \theta$; $A(t) \to 0$; then

$$\overline{\phi} = \mu - r = \theta\gamma + \lambda\left(1 - e^{-\gamma x}\right)\left(e^x - 1\right). \qquad (97)$$

To compare with the results from the model in [6], we take nonrandom constant jump size $x = -0.08$ and $r = 0.05$, $\lambda = 0.48$, $\mu = 0.11$, $\theta = 0.16^2$. Then, we obtain the relative risk aversion coefficient $\gamma = 2.13$ which is the same as the numerical value in [6]. However, the risk premium ϕ is stochastic. We get the expectation of ϕ as follows:

$$E[\phi] = \left[e^{-\kappa t}V_0 + \theta\left(1 - e^{-\kappa t}\right)\right]$$

$$\times\left[\gamma - \varepsilon\rho A(t)\right] + \lambda\left(1 - e^{-\gamma x}\right)\left(e^x - 1\right). \qquad (98)$$

If we assume that other parameter values in our model are $k = 0.03$, $V_0 = 0.1$, $\varepsilon = 0.1$, and $\rho = -0.25$, we will find that the expectation of ϕ tends to 0.6 from Figure 1, which is consistent with above analysis.

As we know that the risk premium ϕ is stochastic and it has the same stochastic characteristic with volatility V_t, the risk premium ϕ can be generated by a discrete scheme of system as follows:

$$V(t + \Delta t) = V(t) + \kappa\left(\theta - V(t)\right)\Delta t$$

$$+ \varepsilon\sqrt{V(t)}\rho\omega\sqrt{\Delta t}, \quad V_0 > 0, \qquad (99)$$

where Δt is the time interval and ω is a sample from the standard normal distribution.

One path of ϕ is given in Figure 2.

In realistic world, the risk premium is stochastic. For example, Figure 1a and Figure 5 in Graham and Harvey (2012) show that risk premium follows mean-reverting stochastic process in [41]. Thus, our model is more realistic than that in [6].

6. Conclusions

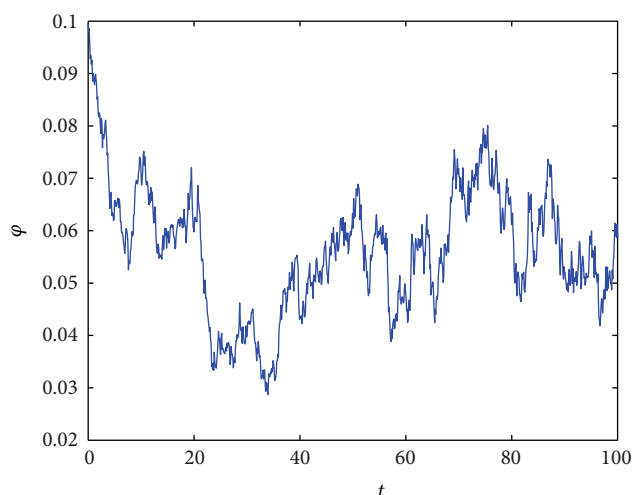We successfully extend the model in [6] and obtain more general results. First, the equilibrium equity premium in

Figure 2: One path of ϕ.

general equilibrium framework links not only the jump risk but also the stochastic volatility risk. Moreover, the pricing kernel with stochastic volatility is shown in this paper. We employ the Fourier transformation method to obtain the exact expression of option price. Next, we get the relationship of central moments between the physical measure and risk-neutral measure. It is a vitally crucial problem in financial area. Our numerical results show that our model is more realistic than the one in [6].

Acknowledgment

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