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*Research Article*

# Bounds of the Neuman-Sándor Mean Using Power and Identric Means

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In this paper we find the best possible lower power mean bounds for the Neuman-Sándor mean and present the sharp bounds for the ratio of the Neuman-Sándor and identric means.

## 1. Introduction

For $p \in \mathbb{R}$ the $p$th power mean $M_p(a,b)$, Neuman-Sándor Mean $M(a,b)$ [1], and identric mean $I(a,b)$ of two positive numbers $a$ and $b$ are defined by

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \tag{1}$$

$$M(a,b) = \begin{cases} \dfrac{a-b}{2\sinh^{-1}((a-b)/(a+b))}, & a \neq b, \\ a, & a = b, \end{cases} \tag{2}$$

$$I(a,b) = \begin{cases} \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & a \neq b, \\ a, & a = b, \end{cases} \tag{3}$$

respectively, where $\sinh^{-1}(x) = \log(x + \sqrt{1+x^2})$ is the inverse hyperbolic sine function.

The main properties for $M_p(a,b)$ and $I(a,b)$ are given in [2]. It is well known that $M_p(a,b)$ is continuously and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Recently, the power, Neuman-Sándor, and identric means have been a subject of intensive research. In particular, many remarkable inequalities for these means can be found in the literature [3–26].

Let $H(a,b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (b-a)/(\log b - \log a)$, $P(a,b) = (a-b)/[4\arctan(\sqrt{a/b}) - \pi]$, $A(a,b) = (a+b)/2$, $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$, and $C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic, geometric, logarithmic, first Seiffert, arithmetic, second Seiffert, quadratic, and contraharmonic means of two positive numbers $a$ and $b$ with $a \neq b$, respectively. Then, it is well known that the inequalities

$$\begin{aligned} H(a,b) &= M_{-1}(a,b) < G(a,b) = M_0(a,b) < L(a,b) \\ &< P(a,b) < I(a,b) < A(a,b) = M_1(a,b) < M(a,b) \\ &< T(a,b) < Q(a,b) = M_2(a,b) < C(a,b), \end{aligned} \tag{4}$$

hold for all $a, b > 0$ with $a \neq b$.

The following sharp bounds for $L$, $I$, $(IL)^{1/2}$, and $(I+L)/2$ in terms of power means are presented in [27–32]:

$$\begin{gathered} M_0(a,b) < L(a,b) < M_{1/3}(a,b), \\ M_{2/3}(a,b) < I(a,b) < M_{\log 2}(a,b), \\ M_0(a,b) < I^{1/2}(a,b)\, L^{1/2}(a,b) < M_{1/2}(a,b), \\ \frac{1}{2}[I(a,b) + L(a,b)] < M_{1/2}(a,b), \end{gathered} \tag{5}$$

for all $a, b > 0$ with $a \neq b$.

Pittenger [31] found the greatest value $r_1$ and the least value $r_2$ such that the double inequality

$$M_{r_1}(a,b) \le L_p(a,b) \le M_{r_2}(a,b), \quad (6)$$

holds for all $a, b > 0$, where $L_r(a,b)$ is the $r$th generalized logarithmic means which is defined by

$$L_r(a,b) = \begin{cases} \left[\dfrac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}\right]^{1/r}, & a \ne b,\ r \ne -1,\ r \ne 0, \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & a \ne b,\ r = 0, \\ \dfrac{b-a}{\log b - \log a}, & a \ne b,\ r = -1, \\ a, & a = b. \end{cases} \quad (7)$$

The following sharp power mean bounds for the first Seiffert mean $P(a,b)$ are given in [10, 33]:

$$M_{\log 2/\log \pi}(a,b) < P(a,b) < M_{2/3}(a,b), \quad (8)$$

for all $a, b > 0$ with $a \ne b$.

In [17], the authors answered the question: for $\alpha \in (0,1)$, what are the greatest value $p$ and the least value $q$ such that the double inequality

$$M_p(a,b) < P^{\alpha}(a,b)\, G^{1-\alpha}(a,b) < M_q(a,b) \quad (9)$$

holds for all $a, b > 0$ with $a \ne b$?

Neuman and Sándor [1] established that

$$\begin{gathered} A(a,b) < M(a,b) < \frac{A(a,b)}{\log\left(1+\sqrt{2}\right)}, \\ \frac{\pi}{4} T(a,b) < M(a,b) < T(a,b), \\ M(a,b) < \frac{2A(a,b) + Q(a,b)}{3}, \end{gathered} \quad (10)$$

for all $a, b > 0$ with $a \ne b$.

Let $0 < a, b \le 1/2$ with $a \ne b$, $a' = 1 - a$ and $b' = 1 - b$. Then, the Ky Fan inequalities

$$\frac{G(a,b)}{G(a',b')} < \frac{L(a,b)}{L(a',b')} < \frac{P(a,b)}{P(a',b')} < \frac{A(a,b)}{A(a',b')} < \frac{M(a,b)}{M(a',b')} < \frac{T(a,b)}{T(a',b')} \quad (11)$$

were presented in [1].

In [24], Li et al. found the best possible bounds for the Neuman-Sándor mean $M(a,b)$ in terms of the generalized logarithmic mean $L_r(a,b)$. Neuman [25] and Zhao et al. [26] proved that the inequalities

$$\begin{aligned} &\alpha Q(a,b) + (1-\alpha) A(a,b) \\ &\quad < M(a,b) < \beta Q(a,b) + (1-\beta) A(a,b), \\ &\lambda C(a,b) + (1-\lambda) A(a,b) < M(a,b) \\ &\quad < \mu C(a,b) + (1-\mu) A(a,b), \\ &\alpha_1 H(a,b) + (1-\alpha_1) Q(a,b) < M(a,b) \\ &\quad < \beta_1 H(a,b) + (1-\beta_1) Q(a,b), \\ &\alpha_2 G(a,b) + (1-\alpha_2) Q(a,b) < M(a,b) \\ &\quad < \beta_2 G(a,b) + (1-\beta_2) Q(a,b) \end{aligned} \quad (12)$$

hold for all $a, b > 0$ with $a \ne b$ if and only if $\alpha \le [1 - \log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})]$, $\beta \ge 1/3$, $\lambda \le [1 - \log(1+\sqrt{2})]/\log(1+\sqrt{2})$, $\mu \ge 1/6$, $\alpha_1 \ge 2/9$, $\beta_1 \le 1 - 1/[\sqrt{2}\log(1+\sqrt{2})]$, $\alpha_2 \ge 1/3$, and $\beta_2 \le 1 - 1/[\sqrt{2}\log(1+\sqrt{2})]$.

In [7], Sándor and Trif proved that the inequalities

$$\begin{gathered} e^{((a-b)^2/6(a+b)^2)} < \frac{A(a,b)}{I(a,b)} < e^{((a-b)^2/24ab)}, \\ e^{((a-b)^2/3(a+b)^2)} < \frac{I(a,b)}{G(a,b)} < e^{((a-b)^2/12ab)}, \\ e^{((a-b)^4/30(a+b)^4)} < \frac{I(a,b)}{A^{2/3}(a,b)\, G^{1/3}(a,b)} < e^{((a-b)^4/120ab(a+b)^4)} \end{gathered} \quad (13)$$

hold for all $a, b > 0$ with $a \ne b$.

Neuman and Sándor [15] and Gao [20] proved that $\alpha_1 = 1$, $\beta_1 = e/2$, $\alpha_2 = 1$, $\beta_2 = 2\sqrt{2}/e$, $\alpha_3 = 1$, $\beta_3 = 3/e$, $\alpha_4 = e/\pi$, $\beta_4 = 1$, $\alpha_5 = 1$, and $\beta_5 = 2e/\pi$ are the best possible constants such that the double inequalities $\alpha_1 < A(a,b)/I(a,b) < \beta_1$, $\alpha_2 < I(a,b)/M_{2/3}(a,b) < \beta_2$, $\alpha_3 < I(a,b)/He(a,b) < \beta_3$, $\alpha_4 < P(a,b)/I(a,b) < \beta_4$, and $\alpha_5 < T(a,b)/I(a,b) < \beta_5$ hold for all $a, b > 0$ with $a \ne b$, where $He(a,b) = (a + \sqrt{ab} + b)/3 = (2A(a,b) + G(a,b))/3$ is the Heronian mean of $a$ and $b$.

In [34], Sándor established that

$$He(a,b) < M_{2/3}(a,b), \quad (14)$$

for all $a, b > 0$ with $a \ne b$.

It is not difficult to verify that the inequality

$$\frac{2A(a,b) + Q(a,b)}{3} < \left[He\left(a^2, b^2\right)\right]^{1/2} \quad (15)$$

holds for all $a, b > 0$ with $a \ne b$.

From inequalities (10), (14), and (15), one has

$$M(a,b) < \left[M_{2/3}\left(a^2, b^2\right)\right]^{1/2} = M_{4/3}(a,b), \quad (16)$$

for all $a, b > 0$ with $a \ne b$.

It is the aim of this paper to find the best possible lower power mean bound for the Neuman-Sándor mean $M(a,b)$ and to present the sharp constants $\alpha$ and $\beta$ such that the double inequality

$$\alpha < \frac{M(a,b)}{I(a,b)} < \beta \tag{17}$$

holds for all $a, b > 0$ with $a \neq b$.

## 2. Main Results

**Theorem 1.** $p_0 = (\log 2)/\log[2\log(1+\sqrt{2})] = 1.224\ldots$ *is the greatest value such that the inequality*

$$M(a,b) > M_{p_0}(a,b) \tag{18}$$

*holds for all $a, b > 0$ with $a \neq b$.*

*Proof.* From (1) and (2), we clearly see that both $M(a,b)$ and $M_p(a,b)$ are symmetric and homogenous of degree one. Without loss of generality, we assume that $b = 1$ and $a = x > 1$.

Let $p_0 = (\log 2)/\log[2\log(1+\sqrt{2})]$, then from (1) and (2) one has

$$\log M(x,1) - \log M_{p_0}(x,1) = \log \frac{x-1}{2\sinh^{-1}((x-1)/(x+1))} - \frac{1}{p_0}\log\frac{x^{p_0}+1}{2}. \tag{19}$$

Let

$$f(x) = \log \frac{x-1}{2\sinh^{-1}((x-1)/(x+1))} - \frac{1}{p_0}\log\frac{x^{p_0}+1}{2}. \tag{20}$$

Then, simple computations lead to

$$\lim_{x\to 1^+} f(x) = 0, \tag{21}$$

$$\lim_{x\to+\infty} f(x) = \frac{1}{p_0}\log 2 - \log\left[2\sinh^{-1}(1)\right] = 0, \tag{22}$$

$$f'(x) = \frac{\left(1+x^{p_0-1}\right) f_1(x)}{(x-1)(x^{p_0}+1)\sinh^{-1}((x-1)/(x+1))}, \tag{23}$$

where

$$f_1(x) = -\frac{\sqrt{2}(x-1)(x^{p_0}+1)}{(x+1)(x^{p_0-1}+1)\sqrt{1+x^2}} + \sinh^{-1}\left(\frac{x-1}{x+1}\right),$$
$$f_1(1) = 0, \tag{24}$$

$$\lim_{x\to+\infty} f_1(x) = -\sqrt{2} + \sinh^{-1}(1) = -0.5328\cdots < 0, \tag{25}$$

$$f_1'(x) = \frac{\sqrt{2}(x-1) f_2(x)}{(x+1)^2\left(x^{p_0-1}+1\right)^2\left(1+x^2\right)^{3/2}}, \tag{26}$$

where

$$f_2(x) = 1 + x + 2x^2 + (p_0-1)x^{p_0-2} - x^{p_0-1} + x^{p_0+1} - (p_0-1)x^{p_0+2} - 2x^{2p_0-2} - x^{2p_0-1} - x^{2p_0}, \tag{27}$$

$$f_2(1) = 0,$$
$$\lim_{x\to+\infty} f_2(x) = -\infty, \tag{28}$$

$$\begin{aligned} f_2'(x) = {} & 1 + 4x + (p_0-1)(p_0-2)x^{p_0-3} - (p_0-1)x^{p_0-2} \\ & + (p_0+1)x^{p_0} - (p_0-1)(p_0+2)x^{p_0+1} \\ & - 4(p_0-1)x^{2p_0-3} - (2p_0-1)x^{2p_0-2} - 2p_0 x^{2p_0-1}, \end{aligned}$$
$$f_2'(1) = 4(4-3p_0) > 0, \tag{29}$$

$$\lim_{x\to+\infty} f_2'(x) = -\infty, \tag{30}$$

$$\begin{aligned} f_2''(x) = {} & 4 + (p_0-1)(p_0-2)(p_0-3)x^{p_0-4} \\ & - (p_0-1)(p_0-2)x^{p_0-3} + p_0(p_0+1)x^{p_0-1} \\ & - (p_0-1)(p_0+2)(p_0+1)x^{p_0} \\ & - 4(p_0-1)(2p_0-3)x^{2p_0-4} \\ & - 2(2p_0-1)(p_0-1)x^{2p_0-3} \\ & - 2p_0(2p_0-1)x^{2p_0-2}, \end{aligned}$$
$$f_2''(1) = 4(2p_0-1)(4-3p_0) > 0, \tag{31}$$

$$\lim_{x\to+\infty} f_2''(x) = -\infty, \tag{32}$$

$$f_2'''(x) = (p_0-1)x^{p_0-5} f_3(x), \tag{33}$$

where

$$\begin{aligned} f_3(x) = {} & -(2-p_0)(3-p_0)(4-p_0) - (2-p_0)(3-p_0)x \\ & + p_0(p_0+1)x^3 - p_0(p_0+1)(p_0+2)x^4 \\ & - 8(3-2p_0)(2-p_0)x^{p_0} + 2(2p_0-1)(3-2p_0)x^{p_0+1} \\ & - 4p_0(2p_0-1)x^{p_0+2} \\ < {} & -(2-p_0)(3-p_0)(4-p_0) \\ & - (2-p_0)(3-p_0)x + p_0(p_0+1)x^4 \\ & - p_0(p_0+1)(p_0+2)x^4 - 8(3-2p_0)(2-p_0)x^{p_0} \\ & + 2(2p_0-1)(3-2p_0)x^{p_0+2} - 4p_0(2p_0-1)x^{p_0+2} \\ = {} & -(2-p_0)(3-p_0)(4-p_0) - (2-p_0)(3-p_0)x \\ & - p_0(p_0+1)^2x^4 - 8(3-2p_0)(2-p_0)x^{p_0} \\ & - 2(2p_0-1)(4p_0-3)x^{p_0+2} < 0, \end{aligned} \tag{34}$$

for $x > 1$.

Equation (33) and inequality (34) imply that $f_2''(x)$ is strictly decreasing on $[1,+\infty)$. Then, the inequality (31) and (32) lead to the conclusion that there exists $x_1 > 1$, such that $f_2'(x)$ is strictly increasing on $[1,x_1]$ and strictly decreasing on $[x_1,+\infty)$.

From (29) and (30) together with the piecewise monotonicity of $f_2'(x)$, we clearly see that there exists $x_2 > x_1 > 1$, such that $f_2(x)$ is strictly increasing on $[1,x_2]$ and strictly decreasing on $[x_2,+\infty)$.

It follows from (26)–(28) and the piecewise monotonicity of $f_2(x)$ that there exists $x_3 > x_2 > 1$, such that $f_1(x)$, is strictly increasing on $[1,x_3]$ and strictly decreasing on $[x_3,+\infty)$.

From (23)–(25) and the piecewise monotonicity of $f_1(x)$ we see that there exists $x_4 > x_3 > 1$, such that $f(x)$ is strictly increasing on $(1,x_4]$ and strictly decreasing on $[x_4,+\infty)$.

Therefore, $M(x,1) > M_{p_0}(x,1)$ for $x > 1$ follows easily from (19)–(22) and the piecewise monotonicity of $f(x)$.

Next, we prove that $p_0 = (\log 2)/\log[2\log(1+\sqrt{2})] = 1.224\ldots$ is the greatest value such that $M(x,1) > M_{p_0}(x,1)$ for all $x > 1$.

For any $\varepsilon > 0$ and $x > 1$, from (1) and (2), one has

$$\begin{aligned}\lim_{x\to+\infty}&\frac{M_{p_0+\varepsilon}(x,1)}{M(x,1)}\\&=\lim_{x\to+\infty}\left[\left(\frac{1+x^{p_0+\varepsilon}}{2}\right)^{1/(p_0+\varepsilon)}\frac{2\sinh^{-1}((x-1)/(x+1))}{x-1}\right]\\&=2^{-1/(p_0+\varepsilon)}\times 2\sinh^{-1}(1)\\&=2^{\varepsilon/p_0(p_0+\varepsilon)}>1.\end{aligned}\tag{35}$$

Inequality (35) implies that for any $\varepsilon > 0$, there exists $X = X(\varepsilon) > 1$, such that $M(x,1) < M_{p_0+\varepsilon}(x,1)$ for $x \in (X,+\infty)$. □

*Remark 2.* 4/3 is the least value such that inequality (16) holds for all $a,b > 0$ with $a \neq b$, namely, $M_{4/3}(a,b)$ is the best possible upper power mean bound for the Neuman-Sándor mean $M(a,b)$.

In fact, for any $\varepsilon \in (0,4/3)$ and $x > 0$, one has

$$\begin{aligned}M_{4/3-\varepsilon}&(1+x,1)-M(1+x,1)\\&=\left[\frac{(1+x)^{4/3-\varepsilon}+1}{2}\right]^{1/(4/3-\varepsilon)}-\frac{x}{2\sinh^{-1}(x/(2+x))}.\end{aligned}\tag{36}$$

Letting $x \to 0$ and making use of Taylor expansion, we get

$$\begin{aligned}&\left[\frac{(1+x)^{4/3-\varepsilon}+1}{2}\right]^{1/(4/3-\varepsilon)}-\frac{x}{2\sinh^{-1}(x/(2+x))}\\&=\left[1+\frac{4-3\varepsilon}{6}x+\frac{(4-3\varepsilon)(1-3\varepsilon)}{36}x^2+o(x^2)\right]^{1/(4/3-\varepsilon)}\\&\quad-\frac{x}{x-(1/2)x^2+(5/24)x^3+o(x^3)}\\&=\left[1+\frac{1}{2}x+\frac{1-3\varepsilon}{24}x^2+o(x^2)\right]\\&\quad-\left[1+\frac{1}{2}x+\frac{1}{24}x^2+o(x^2)\right]=-\frac{\varepsilon}{8}x^2+o(x^2).\end{aligned}\tag{37}$$

Equations (36) and (37) imply that for any $\varepsilon \in (0,4/3)$ there exists $\delta = \delta(\varepsilon) > 0$, such that $M(1+x,1) > M_{(4/3)-\varepsilon}(1+x,1)$ for $x \in (0,\delta)$.

**Theorem 3.** *For all $a,b > 0$ with $a \neq b$, one has*

$$1<\frac{M(a,b)}{I(a,b)}<\frac{e}{2\log(1+\sqrt{2})},\tag{38}$$

*with the best possible constants 1 and $e/[2\log(1+\sqrt{2})] = 1.5419\ldots$.*

*Proof.* From (2) and (3), we clearly see that both $M(a,b)$ and $I(a,b)$ are symmetric and homogenous of degree one. Without loss of generality, we assume that $b = 1$ and $a = x > 1$. Let

$$f(x)=\frac{M(x,1)}{I(x,1)}=\frac{e(x-1)}{2x^{x/(x-1)}\sinh^{-1}((x-1)/(x+1))}.\tag{39}$$

Then, simple computations lead to

$$\frac{f'(x)}{f(x)}=\frac{\log x}{(x-1)^2\sinh^{-1}((x-1)/(x+1))}f_1(x),\tag{40}$$

where

$$\begin{gathered}f_1(x)=\sinh^{-1}\left(\frac{x-1}{x+1}\right)-\frac{\sqrt{2}(x-1)^2}{(x+1)\sqrt{1+x^2}\log x},\\\lim_{x\to1^+}f_1(x)=0,\end{gathered}\tag{41}$$

$$f_1'(x)=\frac{\sqrt{2}f_2(x)}{x(x+1)^2(1+x^2)^{3/2}\log^2 x},\tag{42}$$

where

$$\begin{aligned}f_2(x)=\;&x(x+1)(1+x^2)\log^2 x\\&-x(3x^3-x^2+x-3)\log x\\&+(x-1)^2(x+1)(1+x^2),\end{aligned}\tag{43}$$

$$f_2(1)=0,$$

$$f_2'(x) = \left(4x^3+3x^2+2x+1\right)\log^2 x + 5\left(-2x^3+x^2+1\right)\log x + 5x^4 - 7x^3 + x^2 - x + 2,$$
$$f_2'(1) = 0, \tag{44}$$

$$f_2''(x) = 2\left(6x^2+3x+1\right)\log^2 x + 2\left(-11x^2+8x+2+x^{-1}\right)\log x + 20x^3 - 31x^2 + 7x - 1 + 5x^{-1},$$
$$f_2''(1) = 0, \tag{45}$$

$$f_2'''(x) = 6(4x+1)\log^2 x + 2\left(-10x+14+2x^{-1}-x^{-2}\right)\log x + 60x^2 - 84x + 23 + 4x^{-1} - 3x^{-2},$$
$$f_2'''(1) = 0, \tag{46}$$

$$f_2^{(4)}(x) = 24\log^2 x + 4\left(7+3x^{-1}-x^{-2}+x^{-3}\right)\log x + 120x - 104 + 28x^{-1} + 4x^{-3} > 0 \tag{47}$$

for $x > 1$.

From (46) and (47), we clearly see that $f_2''(x)$ is strictly increasing on $[1,+\infty)$. Then, (45) leads to the conclusion that $f_2'(x)$ is strictly increasing on $[1,+\infty)$.

Equations (43) and (44) together with the monotonicity of $f_2'(x)$ impliy that $f_2(x) > 0$ for $x > 1$. Then, (42) leads to the conclusion that $f_1(x)$ is strictly increasing on $[1,+\infty)$.

It follows from equations (40) and (41) together with the monotonicity of $f_1(x)$ that $f(x)$ is strictly increasing on $(1,+\infty)$.

Therefore, Theorem 3 follows from (39) and the monotonicity of $f(x)$ together with the facts that

$$\lim_{x\to+\infty} f(x) = \frac{e}{2\log\left(1+\sqrt{2}\right)}, \qquad \lim_{x\to 1^+} f(x) = 1. \tag{48}$$

□

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## References


[1] E. Neuman and J. Sándor, "On the Schwab-Borchardt mean," *Mathematica Pannonica*, vol. 14, no. 2, pp. 253–266, 2003.

[2] P. S. Bullen, D. S. Mitrinović, and P. M. Vasić, *Means and Their Inequalities*, vol. 31 of *Mathematics and its Applications*, D. Reidel Publishing, Dordrecht, The Netherlands, 1988.

[3] J. Sándor, "On the identric and logarithmic means," *Aequationes Mathematicae*, vol. 40, no. 2-3, pp. 261–270, 1990.

[4] J. E. Pečarić, "Generalization of the power means and their inequalities," *Journal of Mathematical Analysis and Applications*, vol. 161, no. 2, pp. 395–404, 1991.

[5] J. Sándor, "A note on some inequalities for means," *Archiv der Mathematik*, vol. 56, no. 5, pp. 471–473, 1991.

[6] J. Sándor and I. Raşa, "Inequalities for certain means in two arguments," *Nieuw Archief voor Wiskunde*, vol. 15, no. 1-2, pp. 51–55, 1997.

[7] J. Sándor and T. Trif, "Some new inequalities for means of two arguments," *International Journal of Mathematics and Mathematical Sciences*, vol. 25, no. 8, pp. 525–532, 2001.

[8] T. Trif, "On certain inequalities involving the identric mean in *n* variables," *Universitatis Babeş-Bolyai*, vol. 46, no. 4, pp. 105–114, 2001.

[9] H. Alzer and S.-L. Qiu, "Inequalities for means in two variables," *Archiv der Mathematik*, vol. 80, no. 2, pp. 201–215, 2003.

[10] P. A. Hästö, "Optimal inequalities between Seiffert's mean and power means," *Mathematical Inequalities & Applications*, vol. 7, no. 1, pp. 47–53, 2004.

[11] S. Wu, "Generalization and sharpness of the power means inequality and their applications," *Journal of Mathematical Analysis and Applications*, vol. 312, no. 2, pp. 637–652, 2005.

[12] O. Kouba, "New bounds for the identric mean of two arguments," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 9, no. 3, article 71, 6 pages, 2008.

[13] L. Zhu, "New inequalities for means in two variables," *Mathematical Inequalities & Applications*, vol. 11, no. 2, pp. 229–235, 2008.

[14] L. Zhu, "Some new inequalities for means in two variables," *Mathematical Inequalities & Applications*, vol. 11, no. 3, pp. 443–448, 2008.

[15] E. Neuman and J. Sándor, "Companion inequalities for certain bivariate means," *Applicable Analysis and Discrete Mathematics*, vol. 3, no. 1, pp. 46–51, 2009.

[16] Y.-M. Chu and W.-F. Xia, "Two sharp inequalities for power mean, geometric mean, and harmonic mean," *Journal of Inequalities and Applications*, vol. 2009, Article ID 741923, 6 pages, 2009.

[17] Y.-M. Chu, Y.-F. Qiu, and M.-K. Wang, "Sharp power mean bounds for the combination of Seiffert and geometric means," *Abstract and Applied Analysis*, vol. 2010, Article ID 108920, 12 pages, 2010.

[18] B.-Y. Long and Y.-M. Chu, "Optimal power mean bounds for the weighted geometric mean of classical means," *Journal of Inequalities and Applications*, vol. 2010, Article ID 905679, 6 pages, 2010.

[19] M.-K. Wang, Y.-M. Chu, and Y.-F. Qiu, "Some comparison inequalities for generalized Muirhead and identric means," *Journal of Inequalities and Applications*, vol. 2010, Article ID 295620, 10 pages, 2010.

[20] S. Gao, "Inequalities for the Seiffert's means in terms of the identric mean," *Journal of Mathematical Sciences*, vol. 10, no. 1-2, pp. 23–31, 2011.

[21] Y.-M. Chu, M.-K. Wang, and Z.-K. Wang, "A sharp double inequalities between harmonic and identric means," *Abstract and Applied Analysis*, vol. 2011, Article ID 657935, 7 pages, 2011.

[22] Y.-F. Qiu, M.-K. Wang, Y.-M. Chu, and G.-D. Wang, "Two sharp inequalities for Lehmer mean, identric mean and logarithmic

mean," *Journal of Mathematical Inequalities*, vol. 5, no. 3, pp. 301–306, 2011.

[23] M.-K. Wang, Z.-K. Wang, and Y.-M. Chu, "An optimal double inequality between geometric and identric means," *Applied Mathematics Letters*, vol. 25, no. 3, pp. 471–475, 2012.

[24] Y. M. Li, B. Y. Long, and Y. M. Chu, "Sharp bounds for the Neuman-Sándor mean in terms of generalized logarithmic mean," *Journal of Mathematical Inequalities*, vol. 6, no. 4, pp. 567–577, 2012.

[25] E. Neuman, "A note on a certain bivariate mean," *Journal of Mathematical Inequalities*, vol. 6, no. 4, pp. 637–643, 2012.

[26] T. H. Zhao, Y. M. Chu, and B. Y. Liu, "Optimal bounds for Neuman-Sándor mean in terms of the convex combinations of harmonic, geometric, quadratic, and contraharmonic means," *Abstract and Applied Analysis*, vol. 2012, Article ID 302635, 9 pages, 2012.

[27] K. Baumgartner, "Zur Unauflösbarkeit für $(e/a)^a(b/e)^b$," *Elemente der Mathematik*, vol. 40, no. 5, p. 123, 1985.

[28] H. Alzer, "Ungleichungen für Mittelwerte," *Archiv der Mathematik*, vol. 47, no. 5, pp. 422–426, 1986.

[29] F. Burk, "Notes: the geometric, logarithmic, and arithmetic mean inequality," *The American Mathematical Monthly*, vol. 94, no. 6, pp. 527–528, 1987.

[30] T. P. Lin, "The power mean and the logarithmic mean," *The American Mathematical Monthly*, vol. 81, pp. 879–883, 1974.

[31] A. O. Pittenger, "Inequalities between arithmetic and logarithmic means," *Univerzitet u Beogradu*, no. 678–715, pp. 15–18, 1980.

[32] A. O. Pittenger, "The symmetric, logarithmic and power means," *Univerzitet u Beogradu*, no. 678–715, pp. 19–23, 1980.

[33] A. A. Jagers, "Solution of problem 887," *Nieuw Archief voor Wiskunde*, vol. 12, pp. 230–231, 1994.

[34] J. Sándor, "On certain identities for means," *Universitatis Babeş-Bolyai*, vol. 38, no. 4, pp. 7–14, 1993.