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Research Article

Existence of Mild Solutions for the Elastic Systems with Structural Damping in Banach Spaces

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This paper deals with the existence and uniqueness of mild solutions for a second order evolution equation initial value problem in a Banach space, which can model an elastic system with structural damping. The discussion is based on the operator semigroups theory and fixed point theorem. In addition, an example is presented to illustrate our theoretical results.

1. Introduction

Our aim in this paper is to study the existence and uniqueness of mild solutions for the semilinear elastic system with structural damping

$$\begin{gathered} \ddot{u}(t) + \rho\mathscr{A}\dot{u}(t) + \mathscr{A}^2 u(t) = f(t, u(t)), \quad 0 < t < a, \\ u(0) = x_0, \qquad \dot{u}(0) = y_0 \end{gathered} \tag{1}$$

in a Banach space $\mathbb{X}$, where $\cdot$ means d/dt, $\rho \geq 2$ is a constant; $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{X} \to \mathbb{X}$ is a closed linear operator and $-\mathscr{A}$ generates a C_0-semigroup $T(t)$ $(t \geq 0)$ on $\mathbb{X}$; $f \in C([0, a] \times \mathbb{X}, \mathbb{X})$, $x_0 \in D(\mathscr{A})$, $y_0 \in \mathbb{X}$.

In 1982, Chen and Russell [1] investigated the following linear elastic system described by the second order equation

$$\ddot{u}(t) + B\dot{u}(t) + Au(t) = 0 \tag{2}$$

in a Hilbert space H with inner $(\cdot,\cdot)$, where A (the elastic operator) and B (the damping operator) are positive definite selfadjoint operators in H. They reduced (2) to the first order equation in $H \times H$

$$\frac{d}{dt}\begin{pmatrix} A^{1/2}u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -B \end{pmatrix}\begin{pmatrix} A^{1/2}u \\ \dot{u} \end{pmatrix}. \tag{3}$$

Let $V = \mathscr{D}(A^{1/2})$, $\mathscr{H} = V \times H$ with the naturally induced inner products. Then, (2) is equivalent to the first order equation in $\mathscr{H}$

$$\frac{d}{dt}\begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \mathscr{A}_B \begin{pmatrix} u \\ \dot{u} \end{pmatrix}, \tag{4}$$

where

$$\begin{gathered} \mathscr{A}_B = \begin{pmatrix} 0 & I \\ -A & -B \end{pmatrix}, \\ D(\mathscr{A}_B) = D(A) \times \left[D\left(A^{1/2}\right) \cap D(B)\right]. \end{gathered} \tag{5}$$

Chen and Russell [1] conjectured that $\mathscr{A}_B$ is the infinitesimal generator of an analytic semigroup on $\mathscr{H}$ if

$$D\left(A^{1/2}\right) \subset D(B) \tag{6}$$

and either of the following two inequalities holds for some $\beta_1, \beta_2 > 0$:

$$\begin{gathered} \beta_1\left(A^{1/2}v, v\right) \leq (Bv, v) \leq \beta_2\left(A^{1/2}v, v\right), \quad v \in D\left(A^{1/2}\right); \\ \beta_1(Av, v) \leq \left(B^2 v, v\right) \leq \beta_2(Av, v), \quad v \in D(A). \end{gathered} \tag{7}$$

In the same paper they obtained some results in this direction. The complete proofs of the two conjectures were given by

Huang [2, 3]. Then, other sufficient conditions for $\mathscr{A}_B$ or its closure $\overline{\mathscr{A}_B}$ to generate an analytic or differentiable semigroup on $\mathscr{H}$ were discussed in [4–10], by choosing B to be an operator comparable with A^α for $0 < \alpha \le 1$, based on an explicit matrix representation of the resolvent operator of $\mathscr{A}_B$ or $\overline{\mathscr{A}_B}$.

However, so far as we know, among the previous works, little is concerned with an elastic system with structural damping in a Banach space. Motivated by previous works, in this paper, we investigate the existence and uniqueness of mild solutions for the elastic system (1) in a frame of Banach spaces. To this end, we firstly introduce the concept of mild solutions for system (1), which is based on the discussion about associated linear system. Secondly, we prove the existence and uniqueness of mild solutions for the semilinear elastic system (1) in a Banach space $\mathbb{X}$.

The paper is organized as follows. In Section 2, we discuss the associated linear elastic system and give its definition of mild solutions. In Section 3, we study the existence and uniqueness of mild solutions for the semilinear elastic system (1). An example to illustrate our theoretical results is given in Section 4.

2. Preliminaries on Linear Elastic Systems

Let $\mathbb{X}$ be a Banach space, we consider the linear elastic system with structural damping

$$\ddot{u}(t) + \rho\mathscr{A}\dot{u}(t) + \mathscr{A}^2 u(t) = h(t), \quad 0 < t < a, \qquad u(0) = x_0, \quad \dot{u}(0) = y_0, \tag{8}$$

where $\cdot$ means d/dt, $\rho \ge 2$ is a constant; $\mathscr{A} : D(\mathscr{A}) \subset \mathbb{X} \to \mathbb{X}$ is a closed linear operator, and $-\mathscr{A}$ generates a C_0-semigroup $T(t)$ $(t \ge 0)$ on $\mathbb{X}$; $h : [0, a] \to \mathbb{X}$, $x_0 \in D(\mathscr{A})$, $y_0 \in \mathbb{X}$.

For the second order evolution equation

$$\ddot{u}(t) + \rho\mathscr{A}\dot{u}(t) + \mathscr{A}^2 u(t) = h(t), \tag{9}$$

it has the following decomposition

$$\left(\frac{\partial}{\partial t} + \sigma_1\mathscr{A}\right)\left(\frac{\partial}{\partial t} + \sigma_2\mathscr{A}\right)u = h(t). \tag{10}$$

That is,

$$\frac{\partial^2 u}{\partial t^2} + (\sigma_1 + \sigma_2)\mathscr{A}\frac{\partial u}{\partial t} + \sigma_1\sigma_2\mathscr{A}^2 u = h(t). \tag{11}$$

It follows from (9) and (11) that

$$\sigma_1 + \sigma_2 = \rho, \qquad \sigma_1\sigma_2 = 1. \tag{12}$$

By (12), we have

$$\text{(i) if } \rho > 2, \quad \text{then } \sigma_1 = \frac{\rho + \sqrt{\rho^2 - 4}}{2}, \quad \sigma_2 = \frac{\rho - \sqrt{\rho^2 - 4}}{2}, \qquad \text{(ii) if } \rho = 2, \quad \text{then } \sigma_1 = \sigma_2 = 1. \tag{13}$$

Let

$$\frac{\partial u}{\partial t} + \sigma_2\mathscr{A}u = v(t), \quad 0 \le t \le a, \tag{14}$$

which means

$$v_0 := v(0) = y_0 + \sigma_2\mathscr{A}x_0. \tag{15}$$

So we reduce the linear elastic system (8) to the following two abstract Cauchy problems in Banach space $\mathbb{X}$:

$$\frac{\partial v}{\partial t} + \sigma_1\mathscr{A}v = h(t), \quad 0 < t < a, \qquad v(0) = v_0, \tag{16}$$

$$\frac{\partial u}{\partial t} + \sigma_2\mathscr{A}u = v(t), \quad 0 < t < a, \qquad u(0) = x_0. \tag{17}$$

It is clear that (16) and (17) are linear inhomogeneous initial value problems for $-\sigma_1\mathscr{A}$ and $-\sigma_2\mathscr{A}$, respectively. Since $-\mathscr{A}$ is the infinitesimal generator of C_0-semigroup $T(t)$ $(t \ge 0)$. Furthermore, for any $\rho \ge 2$, (13) yield $\sigma_1 > 0$, $\sigma_2 > 0$. Thus, by operator semigroups theory [11], $-\sigma_1\mathscr{A}$ and $-\sigma_2\mathscr{A}$ are infinitesimal generators of C_0-semigroups, which implies initial value problems (16) and (17) are well-posed.

Throughout this paper, we assume that $-\sigma_1\mathscr{A}$ and $-\sigma_2\mathscr{A}$ generate C_0-semigroups $S_1(t)$ $(t \ge 0)$ and $S_2(t)$ $(t \ge 0)$ on $\mathbb{X}$, respectively. Note that $\sigma_1 > 0$, $\sigma_2 > 0$ and $-\mathscr{A}$ is the infinitesimal generator of C_0-semigroup $T(t)$ $(t \ge 0)$. It follows that

$$S_1(t) = T(\sigma_1 t), \quad S_2(t) = T(\sigma_2 t), \quad t \ge 0. \tag{18}$$

It is well known [12, Chapter 4], when $h \in L^1([0, a], \mathbb{X})$, the linear initial value problem (16) has a mild solution v given by

$$v(t) = S_1(t)v_0 + \int_0^t S_1(t - s)h(s)\,ds. \tag{19}$$

Similarly, if $v \in C([0, a], \mathbb{X})$, then the mild solution of the linear initial value problem (17) expressed by

$$u(t) = S_2(t)x_0 + \int_0^t S_2(t - s)v(s)\,ds. \tag{20}$$

Substituting (19) into (20), we get

$$u(t) = S_2(t)x_0 + \int_0^t S_2(t - s)S_1(s)v_0\,ds + \int_0^t\int_0^s S_2(t - s)S_1(s - \tau)h(\tau)\,d\tau\,ds. \tag{21}$$

From the argument above, we obtain the following corollary.

Corollary 1. *If $h \in L^1([0, a], \mathbb{X})$, then the initial value problem* (8) *has at most one solution. If it has a solution, this solution is given by* (21).

For every $h \in L^1([0,a],\mathbb{X})$, the right-hand side of (21) is a continuous function on $[0,a]$. It is natural to consider it as a generalized solution of (8) even it is not differentiable and dose not strictly satisfy the equation. We therefore define the following.

Definition 2. Let $-\mathscr{A}$ is the infinitesimal generator of C_0-semigroup $T(t)$ $(t \geq 0)$. Then a continuous solution $u(t)$ of the integral equation

$$u(t) = S_2(t)x_0 + \int_0^t S_2(t-s)S_1(s)v_0 ds + \int_0^t\int_0^s S_2(t-s)S_1(s-\tau)h(\tau)\,d\tau\,ds \quad (22)$$

is said to be a mild solution of the initial value problem (8). Where $S_1(t)$ $(t \geq 0)$, $S_2(t)$ $(t \geq 0)$ were defined in (18) and v_0 was specified in (15).

3. Main Results

Let $C(J,\mathbb{X})$ be the Banach space of all continuous functions $u : J \to \mathbb{X}$ with norm $\|u\|_C = \max_{s\in J}\|u(s)\|$, $J = [0,a]$. Let $\mathscr{L}(\mathbb{X})$ be the Banach space of all linear and bounded operators on $\mathbb{X}$. Note that $S_1(t)$ $(t \geq 0)$ and $S_2(t)$ $(t \geq 0)$ are C_0-semigroups on $\mathbb{X}$. Thus, there exist $M_1 \geq 1$ and $M_2 \geq 1$ such that

$$M_1 = \sup_{t\in J}\|S_1(t)\|_{\mathscr{L}(\mathbb{X})}, \qquad M_2 = \sup_{t\in J}\|S_2(t)\|_{\mathscr{L}(\mathbb{X})}. \quad (23)$$

In what follows, we firstly give the definition of a mild solution for the initial value problem (1) below.

Definition 3. Let $-\mathscr{A}$ is the infinitesimal generator of C_0-semigroup $T(t)$ $(t \geq 0)$. Then a continuous solution $u(t)$ of the integral equation

$$u(t) = S_2(t)x_0 + \int_0^t S_2(t-s)S_1(s)v_0 ds + \int_0^t\int_0^s S_2(t-s)S_1(s-\tau)f(\tau,u(\tau))\,d\tau\,ds \quad (24)$$

is said to be a mild solution of the initial value problem (1). Where $S_1(t)$ $(t \geq 0)$, $S_2(t)$ $(t \geq 0)$ were defined in (18) and v_0 was specified in (15).

Secondly, we consider the existence and uniqueness of mild solutions for (1). To this end, we make the following assumptions:

$(H1)$ $f : [0,a]\times\mathbb{X} \to \mathbb{X}$ be continuous and there exists $L > 0$, such that

$$\|f(t,u_2) - f(t,u_1)\| \leq L\|u_2 - u_1\|, \quad t \in [0,a], \ u_1,u_2 \in \mathbb{X}. \quad (25)$$

$(H2)$ $f : [0,a]\times\mathbb{X} \to \mathbb{X}$ be continuous and there exists a positive function $\mu \in L^\infty(J,\mathbb{R}^+)$ $(\mathbb{R}^+ = [0,+\infty))$ such that

$$\|f(t,u)\| \leq \mu(t), \quad t \in [0,a], \ u \in \mathbb{X}. \quad (26)$$

$(H3)$ The C_0-semigroup $T(t)$ $(t \geq 0)$ is compact for $t > 0$.

Theorem 4. *Assume that $(H1)$ holds, $-\mathscr{A}$ is the infinitesimal generator of C_0-semigroup $T(t)$ $(t \geq 0)$. Then for every $x_0 \in D(\mathscr{A})$, $y_0 \in \mathbb{X}$ and $\rho \geq 2$, the initial value problem (1) has a unique mild solution $u \in C([0,a],\mathbb{X})$.*

Proof. Define the operator $Q : C(J,\mathbb{X}) \to C(J,\mathbb{X})$ by

$$(Qu)(t) = S_2(t)x_0 + \int_0^t S_2(t-s)S_1(s)v_0 ds + \int_0^t\int_0^s S_2(t-s)S_1(s-\tau)f(\tau,u(\tau))\,d\tau\,ds. \quad (27)$$

It is obvious that the mild solution of the initial value problem (1) is equivalent to the fixed point of Q.

For any $u_1,u_2 \in C(J,\mathbb{X})$, (23), (27), and $(H1)$ yield

$$\begin{aligned}\|(Qu_2)(t) - (Qu_1)(t)\| &\leq \int_0^t\int_0^s \|S_2(t-s)\|_{\mathscr{L}(\mathbb{X})}\|S_1(s-\tau)\|_{\mathscr{L}(\mathbb{X})} \times \|f(\tau,u_2(\tau)) - f(\tau,u_1(\tau))\|\,d\tau\,ds \\ &\leq LM_1M_2\int_0^t\int_0^s \|u_2(\tau) - u_1(\tau)\|\,d\tau\,ds \\ &\leq LM_1M_2\int_0^t\int_0^s \|u_2 - u_1\|_C d\tau\,ds \\ &\leq \frac{LM_1M_2a^2}{2}\|u_2 - u_1\|_C.\end{aligned} \quad (28)$$

Using (27), (28), and induction on n it follows easily that

$$\|Q^n u_2(t) - Q^n u_1(t)\| \leq \frac{(LM_1M_2a^2)^n}{(2n)!}\|u_2 - u_1\|_C. \quad (29)$$

Hence

$$\|Q^n u_2 - Q^n u_1\|_C \leq \frac{(LM_1M_2a^2)^n}{(2n)!}\|u_2 - u_1\|_C. \quad (30)$$

Since

$$\frac{(LM_1M_2a^2)^n}{(2n)!} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (31)$$

Thus, for n large enough $(LM_1M_2a^2)^n/(2n)! < 1$ and by well known extension of the contraction mapping principle, Q has a unique fixed point $u \in C([0,a],\mathbb{X})$. This fixed point is the desired solution of the integral equation (24). □

Theorem 5. *Suppose that assumptions $(H2)$ and $(H3)$ hold. Then for every $x_0 \in D(\mathscr{A})$, $y_0 \in \mathbb{X}$ and $\rho \geq 2$, the initial value problem (1) has at least one mild solution $u \in C([0,a],\mathbb{X})$.*

Proof. Define the operator $Q : C(J, \mathbb{X}) \to C(J, \mathbb{X})$ as (27) and choose $r > 0$ such that

$$r \geq M_2 \|x_0\| + M_1 M_2 a \|v_0\| + M_1 M_2 a^2 \|\mu\|_{L^\infty(J,\mathbb{R}^+)}. \quad (32)$$

Let $B_r = \{u \in C(J, \mathbb{X}) : \|u\|_C \leq r\}$. We proceed in two main steps.

Step 1. We show that $Q(B_r) \subset B_r$. For that, let $u \in B_r$. Then for $t \in J$, we have

$$\|(Qu)(t)\| \leq \|S_2(t) x_0\| + \left\| \int_0^t S_2(t-s) S_1(s) v_0 ds \right\| + \left\| \int_0^t \int_0^s S_2(t-s) S_1(s-\tau) f(\tau, u(\tau)) d\tau\, ds \right\|, \quad (33)$$

which according to $(H2)$ and (23) gives

$$\|(Qu)(t)\| \leq M_2 \|x_0\| + M_1 M_2 a \|v_0\| + M_1 M_2 a^2 \|\mu\|_{L^\infty(J,\mathbb{R}^+)}. \quad (34)$$

In view of the choice of r, we obtain

$$\|Qu\|_C \leq r. \quad (35)$$

Step 2. We prove that Q is completely continuous. Note that $f : u \to f(\cdot, u(\cdot))$ is a continuous mapping from B_r to $C(J, \mathbb{X})$. Thus, $Q : B_r \to B_r$ is continuous. Next, we show that Q is compact. To this end, we use the Ascoli-Arzela's theorem. For that, we first prove that $\{(Qu)(t) : u \in B_r\}$ is relatively compact in $\mathbb{X}$, for $t \in J$. Obviously, $\{(Qu)(0) : u \in B_r\}$ is compact.

Let $t \in (0, a]$. For each $\epsilon \in (0, t)$ and $u \in B_r$, we define the operator Q_ϵ by

$$\begin{aligned}
(Q_\epsilon u)(t) &= S_2(t) x_0 + \int_0^{t-\epsilon} S_2(t-s) S_1(s) v_0 ds \\
&\quad + \int_0^{t-\epsilon} \int_0^s S_2(t-s) S_1(s-\tau) f(\tau, u(\tau)) d\tau ds \\
&= S_2(t) x_0 + S_2(\epsilon) \int_0^{t-\epsilon} S_2(t-\epsilon-s) S_1(s) v_0 ds \\
&\quad + S_2(\epsilon) \int_0^{t-\epsilon} \int_0^s S_2(t-\epsilon-s) S_1(s-\tau) \\
&\quad \times f(\tau, u(\tau)) d\tau\, ds.
\end{aligned} \quad (36)$$

Then the sets $\{(Q_\epsilon u)(t) : u \in B_r\}$ are relatively compact in $\mathbb{X}$ since by $(H3)$ and (18), the semigroup $S_2(t)$ $(t \geq 0)$ is compact for $t > 0$ on $\mathbb{X}$. Moreover, using (23) and $(H2)$ we have

$$\begin{aligned}
&\|(Qu)(t) - (Q_\epsilon u)(t)\| \\
&\quad \leq \left\| \int_{t-\varepsilon}^t S_2(t-s) S_1(s) v_0 ds \right\| \\
&\qquad + \left\| \int_{t-\varepsilon}^t \int_0^s S_2(t-s) S_1(s-\tau) f(\tau, u(\tau)) d\tau\, ds \right\| \\
&\quad \leq M_1 M_2 \|v_0\| \epsilon + M_1 M_2 a \|\mu\|_{L^\infty(J,\mathbb{R}^+)} \epsilon.
\end{aligned} \quad (37)$$

Therefore, the set $\{(Qu)(t) : u \in B_r\}$ is relatively compact in $\mathbb{X}$ for all $t \in (0, a]$ and since it is compact at $t = 0$ we have the relatively compactness in $\mathbb{X}$ for all $t \in J$.

Now, let us prove that $Q(B_r)$ is equicontinuous. For $0 \leq t_1 < t_2 \leq a$, we have

$$\begin{aligned}
&\|(Qu)(t_2) - (Qu)(t_1)\| \\
&\quad \leq \|S_2(t_2) x_0 - S_2(t_1) x_0\| \\
&\qquad + \left\| \int_0^{t_1} [S_2(t_2-s) - S_2(t_1-s)] S_1(s) v_0 ds \right\| \\
&\qquad + \left\| \int_{t_1}^{t_2} S_2(t_2-s) S_1(s) v_0 ds \right\| \\
&\qquad + \left\| \int_0^{t_1} \int_0^s [S_2(t_2-s) - S_2(t_1-s)] \right. \\
&\qquad\qquad \left. \times S_1(s-\tau) f(\tau, u(\tau)) d\tau\, ds \right\| \\
&\qquad + \left\| \int_{t_1}^{t_2} \int_0^s S_2(t_2-s) S_1(s-\tau) f(\tau, u(\tau)) d\tau\, ds \right\| \\
&\quad := I_1 + I_2 + I_3 + I_4 + I_5,
\end{aligned} \quad (38)$$

where

$$\begin{aligned}
I_1 &= \|S_2(t_2) x_0 - S_1(t_1) x_0\|, \\
I_2 &= \left\| \int_0^{t_1} [S_2(t_2-s) - S_2(t_1-s)] S_1(s) v_0 ds \right\|, \\
I_3 &= \left\| \int_{t_1}^{t_2} S_2(t_2-s) S_1(s) v_0 ds \right\|, \\
I_4 &= \left\| \int_0^{t_1} \int_0^s [S_2(t_2-s) - S_2(t_1-s)] \right. \\
&\qquad \left. \times S_1(s-\tau) f(\tau, u(\tau)) d\tau\, ds \right\|, \\
I_5 &= \left\| \int_{t_1}^{t_2} \int_0^s S_2(t_2-s) S_1(s-\tau) f(\tau, u(\tau)) d\tau\, ds \right\|.
\end{aligned} \quad (39)$$

In fact, I_1, I_2, I_3, I_4 and I_5 tend to 0 independently of $u \in B_r$ when $t_2 - t_1 \to 0$.

Note that the function $S_2(t)x_0$ is continuous for $t \geq 0$. Thus, $S_2(t)x_0$ is uniformly continuous on $[0, a]$ and thus $\lim_{t_2-t_1 \to 0} I_1 = 0$.

From (23) and $(H2)$, we have

$$\begin{aligned} I_2 \leq & \int_0^{t_1} \left\| S_2(t_2 - s) - S_2(t_1 - s) \right\|_{\mathscr{L}(\mathbb{X})} \\ & \times \left\| S_1(s) \right\|_{\mathscr{L}(\mathbb{X})} \left\| v_0 \right\| ds \\ \leq & M_1 \left\| v_0 \right\| \int_0^a \left\| S_2(t_2 - t_1 + \tau) - S_2(\tau) \right\|_{\mathscr{L}(\mathbb{X})} d\tau. \\ I_4 \leq & \int_0^{t_1} \int_0^s \left\| S_2(t_2 - s) - S_2(t_1 - s) \right\|_{\mathscr{L}(\mathbb{X})} \\ & \times \left\| S_1(s - \tau) \right\|_{\mathscr{L}(\mathbb{X})} \left\| f(\tau, u(\tau)) \right\| d\tau \, ds \\ \leq & M_1 a \|\mu\|_{L^\infty(J, \mathbb{R}^+)} \\ & \times \int_0^a \left\| S_2(t_2 - t_1 + \tau) - S_2(\tau) \right\|_{\mathscr{L}(\mathbb{X})} d\tau. \end{aligned} \tag{40}$$

Let $\phi(\tau) = S_2(t_2 - t_1 + \tau) - S_2(\tau)$. By the compactness of $T(\cdot)$ and (18), we can easily conclude that $S_2(\cdot)$ is compact and therefore $S_2(t)$ is continuous in the uniform operator topology for $t > 0$. Then, $\phi(\tau)$ is also continuous in the uniform operator topology on $(0, a]$. Thus $\|S_2(t_2 - t_1 + \tau) - S_2(\tau)\|_{\mathscr{L}(\mathbb{X})} \to 0$ as $t_2 - t_1 \to 0$. Meanwhile, $\phi(\tau)$ is bounded on $[0, a]$. Hence, using Lebesgue dominated convergence theorem we deduce that $\lim_{t_2-t_1 \to 0} I_2 = \lim_{t_2-t_1 \to 0} I_4 = 0$.

Moreover, from (23) we have

$$\begin{aligned} I_3 &\leq M_1 M_2 \left\| v_0 \right\| \left| t_2 - t_1 \right|, \\ I_5 &\leq M_1 M_2 a \|\mu\|_{L^\infty(J, \mathbb{R}^+)} \left| t_2 - t_1 \right|. \end{aligned} \tag{41}$$

Hence, $\lim_{t_2-t_1 \to 0} I_3 = \lim_{t_2-t_1 \to 0} I_5 = 0$.

In short, we have show, that $Q(B_r)$ is relatively compact for $t \in J$, $\{Qu : u \in B_r\}$ is a family of equicontinuous functions. It follows from Ascoli-Arzela's theorem that Q is compact. By Schauder fixed point theorem Q has a fixed point $u \in B_r$, which obviously is a mild solution to (1). □

4. An Example

In order to illustrate our main results, we consider the following initial-boundary value problem, which is a model for elastic system with structural damping

$$\begin{gathered} \frac{\partial^2 u(x,t)}{\partial t^2} - \rho \frac{\partial^3 u(x,t)}{\partial x^2 \partial t} + \frac{\partial^4 u(x,t)}{\partial x^4} \\ = f(x, t, u(x,t)), \quad (x,t) \in [0,1] \times [0,a], \\ u(0,t) = u(1,t) = 0, \quad t \in [0,a], \\ u(x,0) = \varphi(x), \quad \frac{\partial}{\partial t} u(x,0) = \psi(x), \quad x \in [0,1], \end{gathered} \tag{42}$$

where $a > 0$, $\rho \geq 2$ are all constants, $f : [0,1] \times [0,a] \times \mathbb{R} \to \mathbb{R}$ is continuous.

Let $\mathbb{X} = L^p([0,1], \mathbb{R})$ $(1 < p < +\infty)$, we define the linear operator $\mathscr{A}$ in $\mathbb{X}$ by

$$\mathscr{A}u = -\frac{\partial^2 u}{\partial x^2}, \quad u \in D(\mathscr{A}) = W^{2,p}(0,1) \cap W_0^{1,p}(0,1). \tag{43}$$

It is well known from [13] that $-\mathscr{A}$ is the infinitesimal generator of a C_0-semigroup $T(t)$ $(t \geq 0)$ on $\mathbb{X}$.

Let $u(t) = u(\cdot, t)$, $f(t, u(t)) = f(\cdot, t, u(\cdot, t))$, then the initial-boundary value problem (42) can be reformulated as the following abstract second order evolution equation initial value problem in $\mathbb{X}$:

$$\begin{gathered} \ddot{u}(t) + \rho \mathscr{A} \dot{u}(t) + \mathscr{A}^2 u(t) = f(t, u(t)), \quad 0 < t < a, \\ u(0) = \varphi, \qquad \dot{u}(0) = \psi. \end{gathered} \tag{44}$$

In order to solve the initial-boundary value problem (42), we also need the following assumptions:

$(b1)$ $\varphi \in W^{2,p}(0,1) \cap W_0^{1,p}(0,1)$, $\psi \in L^p([0,1], \mathbb{R})$.

$(b2)$ The partial derivative $f'_u(x,t,u)$ is continuous.

Theorem 6. *If the assumptions* $(b1)$ *and* $(b2)$ *are satisfied, then for any* $\rho \geq 2$, *the initial-boundary value problem* (42) *has a unique mild solution* $u \in C([0,a], \ L^p([0,1], \mathbb{R}))$.

Proof. From the assumptions $(b1)$ and $(b2)$, it is easily seen that the conditions in Theorem 4 are satisfied. Hence, by Theorem 4, for any $\rho \geq 2$, the problem (44) has a unique mild solution $u \in C([0,a], \mathbb{X})$, which means u is a mild solution for initial-boundary value problem (42). □

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