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*Research Article*

# A Note on the Right-Hand Side Identification Problem Arising in Biofluid Mechanics

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The inverse problem of reconstructing the right-hand side (RHS) of a mixed problem for one-dimensional diffusion equation with variable space operator is considered. The well-posedness of this problem in Hölder spaces is established.

## 1. Introduction

It is known that many applied problems in fluid mechanics, other areas of physics, and mathematical biology were formulated as the mathematical model of partial differential equations of the variable types [1–3]. A model for transport across microvessel endothelium was developed to determine the forces and bending moments acting on the structure of the flow over endothelial cells (ECs) [4]. Computational blood flow analysis through glycocalyx on the EC is performed as a direct problem previously under smooth and nonsmooth initial conditions (see [5–7]). But it is known that, due to the lack of some data and/or coefficients, many real-life problems are modeled as inverse problems [8–11].

In this paper, the well-posedness of the inverse problem of reconstructing the right side of a parabolic equation arisen in computational blood flow analysis is investigated. The importance of well-posedness has been widely recognized by the researchers in the field of partial differential equations [12–16]. Moreover, the well-posedness of the RHS identification problems for a parabolic equation where the unknown function $p$ is in space variable and in time variable is well investigated [17–27]. As it is known, well-posedness in the sense of Hadamard means that there is existence and uniqueness of the solution and the solution is stable. In this study, we deal with the stability analysis of the inverse problem of reconstructing the right-hand side. The existence of a solution for two-phase flow in porous media has been studied previously (for instance, see [28]).

### *1.1. Problem Formulation*

Blood flow over the EC inside the arteries is modeled in two regions (see [6]). Core region ($0 < x < l$) flow is defined through the center of capillary and porous region ($l < x < L$) flow is through the glycocalyx. RHS function includes the pressure difference along the microchannels under unsteady fluid flow conditions. When the pressure difference is an unknown function of $t$, we reach a new model, and, by overdetermined (additional) conditions derived from an observation point, the solution of this problem can be obtained. The model can be considered as the mixed problem for one-dimensional diffusion equation with variable space operator:

$$\begin{gathered}
\frac{\partial u(t,x)}{\partial t} = a(x)\frac{\partial^2 u(t,x)}{\partial x^2} + p(t)q(x) + f(t,x), \quad x \in (0,l), \ t \in (0,T], \\
\frac{\partial u(t,x)}{\partial t} = a(x)\frac{\partial^2 u(t,x)}{\partial x^2} + b(t,x)u(t,x) + p(t)q(x) + g(t,x), \quad x \in (l,L), \ t \in (0,T], \\
u(0,x) = \varphi(x), \quad x \in [0,L], \\
u_x(t,0) = 0, \quad u(t,L) = 0, \quad t \in [0,T], \\
u(t,l+) = u(t,l-), \quad u_x(t,l+) = u_x(t,l-), \quad t \in [0,T], \\
u(t,x^*) = \rho(t), \quad 0 \le x^* \le l, \quad 0 \le t \le T.
\end{gathered} \tag{1.1}$$

Here, $u(t,x)$ and $p(t)$ are unknown functions, $a(x)$, $b(t,x)$, $f(t,x)$,$g(t,x)$,$\rho(t)$, and $\varphi(x)$ are given sufficiently smooth functions, and $a(x) \geqslant a > 0$. Also, $q(x)$ is a sufficiently smooth function assuming that $q'(0) = q(L) = 0$ and $q(x^*) \neq 0$.

## 2. Main Results

### *2.1. Differential Case*

To formulate our results, we introduce the Banach space $\overset{\circ}{C}{}^{\alpha}[0,L]$, $\alpha \in (0,1)$, of all continuous functions $\phi(x)$ defined on $[0,L]$ with $\phi'(0) = \phi(L) = 0$ satisfying a Hölder condition for which the following norm is finite:

$$\|\phi\|_{\overset{\circ}{C}{}^{\alpha}[0,L]} = \max_{0\le x\le L}|\phi(x)| + \sup_{0\le x<x+h\le L}\frac{|\phi(x+h)-\phi(x)|}{h^{\alpha}}. \tag{2.1}$$

In a Banach space $E$, with the help of a positive operator $A$ we introduce the fractional spaces $E_\alpha, 0 < \alpha < 1$, consisting of all $v \in E$ for which the following norm is finite:

$$\|v\|_{E_\alpha} = \|v\|_E + \sup_{\lambda>0}\lambda^{1-\alpha}\|A\exp\{-\lambda A\}v\|_E. \tag{2.2}$$

Positive constants will be indicated by $M$ which can be differ in time. On the other hand $M_i$ $(\alpha,\beta,\ldots)$ is used to focus on the fact that the constant depends only on $\alpha,\beta,\ldots$, and the subindex $i$ is used to indicate a different constant.

**Theorem 2.1.** *Let* $\varphi \in \overset{\circ}{C}{}^{2\alpha+2}[0,L]$, $F_1 \in C([0,T], \overset{\circ}{C}{}^{2\alpha}[0,L])$, *and* $\rho' \in C[0,T]$. *Then for the solution of problem* (1.1), *the following coercive stability estimates*

$$\begin{aligned}
&\|u_t\|_{C([0,T],\, \overset{\circ}{C}{}^{2\alpha}[0,L])} + \|u\|_{C([0,T],\, \overset{\circ}{C}{}^{2\alpha+2}[0,L])} \\
&\quad\leq M(x^*, q)\|\rho'\|_{C[0,T]} + M(a, \delta, \sigma, \alpha, x^*, q, T) \\
&\quad\quad\times \left(\|\varphi\|_{\overset{\circ}{C}{}^{2\alpha+2}[0,L]} + \|F_1\|_{C([0,T],\, \overset{\circ}{C}{}^{2\alpha}[0,L])} + \|\rho\|_{C[0,T]}\right), \\
&\|p\|_{C[0,T]} \leq M(x^*, q)\|\rho'\|_{C[0,T]} \\
&\quad+ M(a, \delta, \sigma, \alpha, x^*, q, T)\left[\|\varphi\|_{\overset{\circ}{C}{}^{2\alpha+2}[0,L]} + \|F_1\|_{C([0,T],\overset{\circ}{C}{}^{2\alpha}[0,L])} + \|\rho\|_{C[0,T]}\right]
\end{aligned} \tag{2.3}$$

*hold.*

*Proof.* Let us search for the solution of inverse problem (1.1) in the following form (see [23]):

$$u(t, x) = \eta(t)q(x) + w(t, x), \tag{2.4}$$

where

$$\eta(t) = \int_0^t p(s)ds. \tag{2.5}$$

Taking derivatives from (2.4) with respect to $t$ and $x$, we get

$$\begin{aligned}
\frac{\partial u(t, x)}{\partial t} &= p(t)q(x) + \frac{\partial w(t, x)}{\partial t}, \\
\frac{\partial^2 u(t, x)}{\partial x^2} &= \eta(t)\frac{d^2 q(x)}{dx^2} + \frac{\partial^2 w(t, x)}{\partial x^2}.
\end{aligned} \tag{2.6}$$

Moreover, substituting $x$ by $x^*$ in (2.4), we obtain

$$\begin{aligned}
u(t, x^*) &= \eta(t)q(x^*) + w(t, x^*) = \rho(t), \\
\eta(t) &= \frac{\rho(t) - w(t, x^*)}{q(x^*)}.
\end{aligned} \tag{2.7}$$

Differentiating both sides of (2.7) with respect to $t$, we get

$$p(t) = \frac{\rho'(t) - w_t(t, x^*)}{q(x^*)}. \tag{2.8}$$

From identity (2.8) and the triangle inequality, it follows that

$$\begin{aligned} |p(t)| &= \left| \frac{\rho'(t) - w_t(t,x^*)}{q(x^*)} \right| \le M(x^*,q)\left(|\rho'(t)| + |w_t(t,x^*)|\right) \\ &\le M(x^*,q)\left( \max_{0\le t\le T} |\rho'(t)| + \max_{0\le t\le T}\max_{0\le x\le L} |w_t(t,x)| \right) \\ &\le M(x^*,q)\left( \max_{0\le t\le T} |\rho'(t)| + \max_{0\le t\le T} \|w_t(t)\|_{\overset{\circ}{C}{}^{2\alpha}[0,L]} \right), \end{aligned} \tag{2.9}$$

for any $t, t \in [0,T]$. Using problem (1.1) and (2.4)–(2.7), one can show that $w(t,x)$ is the solution of the following problem:

$$\begin{aligned} &\frac{\partial w(t,x)}{\partial t} = a(x)\frac{\partial^2 w(t,x)}{\partial x^2} + a(x)\frac{\rho(t) - w(t,x^*)}{q(x^*)}\frac{d^2 q(x)}{dx^2} + f(t,x), \quad x \in (0,l),\ t \in (0,T], \\ &\frac{\partial w(t,x)}{\partial t} = a(x)\frac{\partial^2 w(t,x)}{\partial x^2} + b(t,x)w(t,x) \\ &\qquad + \frac{\rho(t) - w(t,x^*)}{q(x^*)}\left( a(x)\frac{d^2 q(x)}{dx^2} + b(t,x)q(x) \right) + g(t,x), \quad x \in (l,L),\ t \in (0,T], \\ &w(0,x) = \varphi(x), \quad x \in [0,L], \\ &w_x(t,0) = 0, \quad w(t,L) = 0, \quad t \in [0,T], \\ &w(t,l+) = w(t,l-), \quad w_x(t,l+) = w_x(t,l-), \quad t \in [0,T], \end{aligned} \tag{2.10}$$

under the same assumptions on $q(x)$. Estimate (2.9) and the following theorem conclude the proof of Theorem 2.1. □

**Theorem 2.2.** *For the solution of problem* (2.10), *the following coercive stability estimate*

$$\begin{aligned} \|w_t\|_{\overset{\circ}{C}{}^{2\alpha}[0,L]} &\le M(a,\delta,\sigma,\alpha,x^*,q,T) \\ &\quad \times \left( \|\varphi\|_{\overset{\circ}{C}{}^{2\alpha+2}[0,L]} + \|F_1\|_{C([0,T],\overset{\circ}{C}{}^{2\alpha}[0,L])} + \|\rho\|_{C[0,T]} \right) \end{aligned} \tag{2.11}$$

*holds.*

*Proof.* Let us rewrite problem (2.10) in the abstract form as an initial-value problem:

$$\begin{aligned} &w_t + Aw + Bw = (aq'' - \sigma q)\frac{\rho(t) - w(t,x^*)}{q^*} + F_1(t) + F_2(t), \quad 0 < t \le T, \\ &w(0) = \varphi \end{aligned} \tag{2.12}$$

in the Banach space $E = \overset{\circ}{C}[0, L]$. Here, the positive operator $A$ is defined by

$$Au = -a(x)\frac{\partial^2 u(t,x)}{\partial x^2} + \sigma u, \tag{2.13}$$

with

$$D(A) = \{u(x) : u, u', u'' \in C[0, L], u_x(0) = u(L) = 0\}, \tag{2.14}$$

and for every fixed $t \in [0, T]$, the differential operator $B$ is given by the formula

$$B(t)u = \begin{cases} -\sigma u^n, & 0 \leq x < l, \\ -(\sigma - b(t))u, \ b(t) = b(t,x), & l < x \leq L. \end{cases} \tag{2.15}$$

Here, $\sigma$ is a positive constant. The right-hand side functions are defined by

$$F_1(t) = \begin{cases} f(t), & 0 \leq x < l, \\ 0, & \\ g(t) + \dfrac{\rho(t)}{q^*} b(t) q, & l < x \leq L, \end{cases}$$
$$F_2(t) = \begin{cases} 0, & 0 \leq x < l, \\ 0, & \\ -\dfrac{w(t,x^*)}{q^*} b(t) q, & l < x \leq L, \end{cases} \tag{2.16}$$

where $f(t) = f(t,x), g(t) = g(t,x), b(t) = b(t,x)$ are known, and $w(t) = w(t,x)$ is unknown abstract functions defined on $[0,T]$ with values in $E = \overset{\circ}{C}[0, L]$, $w(t,x^*)$ is unknown scalar function defined on $[0,T]$, $q = q(x)$, $q'' = q''(x)$, $\varphi = \varphi(x)$, and $a = a(x)$ are elements of $E = \overset{\circ}{C}[0, L]$, and $q^* = q(x^*)$ is a number.

It is known that operator-$A$ generates an analytic semigroup $\exp\{-tA\}(t > 0)$ and the following estimate holds:

$$\left\| A^{\alpha} \exp\{-tA\} \right\|_{E \to E} \leq M e^{-\delta t} t^{-\alpha}, \quad 0 \leq \alpha \leq 1, \tag{2.17}$$

where $t, \delta, M > 0$ [29].

By the Cauchy formula, the solution can be written as

$$
\begin{aligned}
w(t) = e^{-tA}\varphi &- \int_0^t e^{-(t-s)A}\frac{aq''-\sigma q}{q^*}w(s,x^*)ds \\
&+ \int_0^t e^{-(t-s)A}\frac{\rho(s)\left(aq''-\sigma q\right)}{q^*}ds + \int_0^t e^{-(t-s)A}F_1(s)ds \\
&+ \int_0^t e^{-(t-s)A}F_2(s)ds - \int_0^t e^{-(t-s)A}B(s)w(s)ds.
\end{aligned}
\tag{2.18}
$$

Then, the following presentation of the solution of abstract problem (2.12) exists:

$$
\begin{aligned}
Aw(t) = Ae^{-tA}\varphi &- \int_0^t Ae^{-(t-s)A}\frac{aq''-\sigma q}{q^*}w(s,x^*)ds \\
&+ \int_0^t Ae^{-(t-s)A}\frac{\rho(s)\left(aq''-\sigma q\right)}{q^*}ds + \int_0^t Ae^{-(t-s)A}F_1(s)ds \\
&+ \int_0^t Ae^{-(t-s)A}F_2(s)ds + \int_0^t Ae^{-(t-s)A}B(s)w(s)ds = \sum_{k=1}^{6}G_k(t).
\end{aligned}
\tag{2.19}
$$

Here,

$$
\begin{aligned}
G_1(t) &= Ae^{-tA}\varphi, \\
G_2(t) &= -\int_0^t Ae^{-(t-s)A}\frac{aq''-\sigma q}{q^*}w(s,x^*)ds, \\
G_3(t) &= -\int_0^t Ae^{-(t-s)A}\frac{\rho(s)\left(aq''-\sigma q\right)}{q^*}ds, \\
G_4(t) &= \int_0^t Ae^{-(t-s)A}F_1(s)ds, \\
G_5(t) &= \int_0^t Ae^{-(t-s)A}F_2(s)ds, \\
G_6(t) &= \int_0^t Ae^{-(t-s)A}B(s)w(s)ds.
\end{aligned}
\tag{2.20}
$$

From the fact that the operators $R, \exp\{-\lambda A\}$ and $A$ commute, it follows that [29]

$$
\|R\|_{E_\alpha \to E_\alpha} \le \|R\|_{E \to E}.
\tag{2.21}
$$

Now, we estimate $G_k(t)$ for $k = 1, 2, \ldots, 5$ separately. Applying the definition of norm of the spaces $E_\alpha$ and estimate (2.21), we get

$$\|G_1(t)\|_{E_\alpha} = \left\|Ae^{-tA}\varphi\right\|_{E_\alpha} \le \left\|e^{-tA}\right\|_{E_\alpha \to E_\alpha} \|A\varphi\|_{E_\alpha} \le \left\|e^{-tA}\right\|_{E \to E} \|A\varphi\|_{E_\alpha}. \tag{2.22}$$

Then, using estimate (2.17) for $\alpha = 0$, we reach to

$$\|G_1(t)\|_{E_\alpha} \le M_1 \|A\varphi\|_{E_\alpha}, \tag{2.23}$$

for any $t,\ t \in [0, T]$.

Let us estimate $G_2(t)$:

$$\begin{aligned}\|G_2(t)\|_{E_\alpha} &= \left\| \int_0^t Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} w(s, x^*) ds \right\| \\ &\le \int_0^t \left\| Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_{E_\alpha} |w(s, x^*)| ds.\end{aligned} \tag{2.24}$$

By the definition of norm of the spaces $E_\alpha$, we have that

$$\begin{aligned}\int_0^t \left\| Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_{E_\alpha} ds = &\int_0^t \left\| Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_E ds \\ &+ \sup_{\lambda>0} \int_0^t \left\| \lambda^{1-\alpha} Ae^{-\lambda A} Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_E ds.\end{aligned} \tag{2.25}$$

Let us estimate the first term. From the definition of norm of the spaces $E_\alpha$ it follows that

$$\begin{aligned}\int_0^t \left\| Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_E ds &= \int_0^t (t-s)^{\alpha-1} \left\| (t-s)^{1-\alpha} Ae^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_E ds \\ &\le \int_0^t (t-s)^{\alpha-1} ds \left\| \frac{aq'' - \sigma q}{q^*} \right\|_{E_\alpha} \\ &\le \frac{T^\alpha}{\alpha} \left\| \frac{aq'' - \sigma q}{q^*} \right\|_{E_\alpha} = M_2(a, \sigma, \alpha, x^*, q, T).\end{aligned} \tag{2.26}$$

Using estimate (2.17), we obtain

$$\begin{aligned}\int_0^t \left\| \lambda^{1-\alpha} A e^{-\lambda A} A e^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_E ds &\le \int_0^t \frac{2^{2-\alpha}\lambda^{1-\alpha}}{(\lambda + t - s)^{2-\alpha}} ds \left\| \frac{\lambda + t - s}{2} A e^{-((\lambda+t-s)/2)A} \right\|_{E\to E} \\ &\quad \times \left\| \left( \frac{\lambda + t - s}{2} \right)^{1-\alpha} A e^{-((\lambda+t-s)/2)A} \frac{aq'' - \sigma q}{q^*} \right\|_E \\ &\le M_3(\alpha) \left\| \frac{aq'' - \sigma q}{q^*} \right\|_{E_\alpha} \int_0^t \frac{\lambda^{1-\alpha}}{(\lambda + t - s)^{2-\alpha}} ds \\ &\le M_3(\alpha) \left\| \frac{aq'' - \sigma q}{q^*} \right\|_{E_\alpha} \left( \frac{\lambda^{1-\alpha}}{(1-\alpha)(\lambda + t)^{1-\alpha}} \right),\end{aligned} \tag{2.27}$$

for any $\lambda > 0$. From that it follows

$$\begin{aligned}&\sup_{\lambda>0} \int_0^t \left\| \lambda^{1-\alpha} A e^{-\lambda A} A e^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_E ds \\ &\quad \le M_3(\alpha) \left\| \frac{aq'' - \sigma q}{q^*} \right\|_{E_\alpha} \frac{1}{(1-\alpha)} = M_4(a, \sigma, \alpha, x^*, q).\end{aligned} \tag{2.28}$$

Then, we get

$$\int_0^t \left\| A e^{-(t-s)A} \frac{aq'' - \sigma q}{q^*} \right\|_{E_\alpha} ds \le M_5(a, \sigma, \alpha, x^*, q, T) \tag{2.29}$$

$$\|G_2(t)\|_{E_\alpha} \le M_6(a, \sigma, \alpha, x^*, q, T) \int_0^t |w(s, x^*)| ds. \tag{2.30}$$

Using definitions of norm of spaces $E$ and $E_\alpha$ and estimate (2.21), we obtain that

$$|w(s, x^*)| \le \|w\|_E \le \|w\|_{E_\alpha} = \left\| A^{-1} A w \right\|_{E_\alpha} \le \left\| A^{-1} \right\|_{E\to E} \|Aw\|_{E_\alpha} \le M \|Aw\|_{E_\alpha}, \tag{2.31}$$

$$\|G_2(t)\|_{E_\alpha} \le M_7(a, \sigma, \alpha, x^*, q, T) \|Aw\|_{E_\alpha}, \tag{2.32}$$

for any $t \in [0, T]$.

From estimate (2.29), the estimate of $G_3(t)$ is as follows:

$$\begin{aligned}\|G_3(t)\|_{E_\alpha} &= \left\|\int_0^t Ae^{-(t-s)A}\rho(s)\frac{aq''-\sigma q}{q^*}ds\right\|_{E_\alpha} \\ &\le \int_0^t \left\|Ae^{-(t-s)A}\frac{aq''-\sigma q}{q^*}\right\|_{E_\alpha} ds\|\rho\|_{C[0,T]} \\ &\le M_8(a,\sigma,\alpha,x^*,q,T)\|\rho\|_{C[0,T]}.\end{aligned} \tag{2.33}$$

Now, let us estimate $G_4(t)$. By the definition of the norm of the spaces $E_\alpha$, we get

$$\begin{aligned}\|G_4(t)\|_{E_\alpha} &= \left\|\int_0^t Ae^{-(t-s)A}F_1(s)ds\right\|_{E_\alpha} \\ &= \left\|\int_0^t Ae^{-(t-s)A}F_1(s)ds\right\|_E + \sup_{\lambda>0}\lambda^{1-\alpha}\left\|Ae^{-\lambda A}\int_0^t Ae^{-(t-s)A}F_1(s)ds\right\|_E.\end{aligned} \tag{2.34}$$

Equation (2.2) yields that

$$\begin{aligned}\left\|\int_0^t Ae^{-(t-s)A}F_1(s)ds\right\|_E &= \int_0^t (t-s)^{\alpha-1}\left\|(t-s)^{1-\alpha}Ae^{-(t-s)A}F_1(s)\right\|_E ds \\ &\le \int_0^t (t-s)^{\alpha-1}ds\|F_1\|_{C(E_\alpha)} = \frac{t^\alpha}{\alpha}\|F_1\|_{C(E_\alpha)} \le M_9(\alpha,T)\|F_1\|_{C(E_\alpha)}.\end{aligned} \tag{2.35}$$

Now, we consider the second term. Using (2.2), we get

$$\begin{aligned}\lambda^{1-\alpha}\left\|Ae^{-\lambda A}\int_0^t Ae^{-(t-s)A}F_1(s)ds\right\|_E &\le \lambda^{1-\alpha}\int_0^t\left(\frac{t-s+\lambda}{2}\right)^{\alpha-1}\left(\frac{t-s+\lambda}{2}\right)^{-1} \\ &\quad\times\left\|\frac{t-s+\lambda}{2}Ae^{-((t-s+\lambda)/2)A}\right\|_{E\to E} \\ &\quad\times\left\|\left(\frac{t-s+\lambda}{2}\right)^{1-\alpha}Ae^{-((t-s+\lambda)/2)A}F_1(s)\right\|_E ds \\ &\le M_{10}\lambda^{1-\alpha}\int_0^t\left(\frac{t-s+\lambda}{2}\right)^{\alpha-2}\|F_1\|_{E_\alpha}ds \\ &\le M_{11}\lambda^{1-\alpha}\int_0^t\left(\frac{t-s+\lambda}{2}\right)^{\alpha-2}ds\|F_1\|_{C(E_\alpha)},\end{aligned} \tag{2.36}$$

for any $\lambda > 0$. Then,

$$\sup_{\lambda>0} \lambda^{1-\alpha} \left\| Ae^{-\lambda A} \int_0^t Ae^{-(t-s)A} F_1(s) ds \right\|_E \le \frac{M_9 2^{1-\alpha}}{1-\alpha} \|F_1\|_{C(E_\alpha)} = M_{11}(\alpha) \|F_1\|_{C(E_\alpha)}. \tag{2.37}$$

Combining estimates (2.35) and (2.37), we obtain

$$\|G_4(t)\|_{E_\alpha} \le M_{12}(\alpha, T) \|F_1\|_{C(E_\alpha)}. \tag{2.38}$$

The estimate of $G_5(t)$ is as follows. Since operators $A$ and $e^{-tA}$ commute, we can write that

$$\|G_5(t)\|_{E_\alpha} \le \left\| \int_0^t Ae^{-(t-s)A} F_2(s) ds \right\|_{E_\alpha} \le M_{13}(\alpha, T) \|Aw\|_{E_\alpha}. \tag{2.39}$$

Let us estimate $G_6(t)$ :

$$\begin{aligned} \|G_6(t)\|_{E_\alpha} &= \left\| \int_0^t Ae^{-(t-s)A} B(s) w(s) ds \right\|_{E_\alpha} \\ &= \left\| \int_0^t Ae^{-(t-s)A} B(s) A^{-1} Aw(s) ds \right\|_{E_\alpha} \\ &\le \int_0^t \left\| Ae^{-(t-s)A} B(s) A^{-1} \right\|_{E_\alpha \to E_\alpha} \|Aw(s)\|_{E_\alpha} ds. \end{aligned} \tag{2.40}$$

Since

$$\begin{gathered} \left\| e^{-tA} \right\|_{E_\alpha \to E_\alpha} \le \left\| e^{-tA} \right\|_{E \to E} \le Me^{-\delta t}, \\ \left\| AB(s) A^{-1} \right\|_{E_\alpha \to E_\alpha} \le M, \end{gathered} \tag{2.41}$$

we get

$$\|G_6(t)\|_{E_\alpha} \le M_{14} \int_0^t \|Aw(s)\|_{E_\alpha} ds. \tag{2.42}$$

Finally combining estimates (2.23), (2.32), (2.33), (2.38), (2.39), and (2.42), we get

$$\begin{aligned}\|Aw\|_{E_\alpha} \le\ & M_1\|A\varphi\|_{E_\alpha} + M_8(a,\sigma,\alpha,x^*,q,T)\|\rho\|_{C[0,T]} \\ & + M_{12}(\alpha,T)\|F_1\|_{C(E_\alpha)} + M_{15}\int_0^t \|Aw(s)\|_{E_\alpha} ds,\end{aligned} \tag{2.43}$$

where $M_{15} = M_7 + M_{13} + M_{14}$.

Using Gronwall's inequality, we can write

$$\begin{aligned}\|Aw\|_{E_\alpha} \le e^{M_{15}T}\Big[& M_1\|A\varphi\|_{E_\alpha} + M_8(a,\sigma,\alpha,x^*,q,T)\|\rho\|_{C[0,T]} \\ & + M_{12}(\alpha,T)\|F_1\|_{C(E_\alpha)}\Big].\end{aligned} \tag{2.44}$$

Applying the formulas

$$\begin{aligned} & w(t,x^*) = w(0,x^*) + \int_0^t w_z(z,x^*)dz = \varphi(x^*) + \int_0^t w_z(z,x^*)dz, \\ & \left|\varphi(x^*)\right| \le \max_{0\le x\le L}\left|\varphi(x)\right| = \|\varphi\|_E \le \|\varphi\|_{E_\alpha} \\ & \qquad \le \left\|A^{-1}\right\|_{E_\alpha\to E_\alpha}\|A\varphi\|_{E_\alpha} \le M\|A\varphi\|_{E_\alpha}\end{aligned} \tag{2.45}$$

and the triangle inequality, we can write

$$\begin{aligned} & \left\|\frac{(aq''-\sigma q)}{q^*}(\rho(t) - w(t,x^*))\right\|_{E_\alpha} \\ & \qquad \le \left\|\frac{(aq''-\sigma q)}{q^*}\right\|_{E_\alpha}\left(\|\rho\|_{C[0,T]} + M_{13}\|A\varphi\|_{E_\alpha} + \int_0^t \|w_z\|_{E_\alpha}dz\right).\end{aligned} \tag{2.46}$$

Using boundedness of $B$, problem (2.12), and estimate (2.46), we have

$$\begin{aligned}\|w_t\|_{E_\alpha} \le\ & \|Aw\|_{E_\alpha} + \|Bw\|_{E_\alpha} + \|F_1\|_{C(E_\alpha)} \\ & + \left\|\frac{(aq''-\sigma q)}{q^*}\right\|_{E_\alpha}\left(\|\rho\|_{C[0,T]} + M_{13}\|A\varphi\|_{E_\alpha} + \int_0^t \|w_z\|_{E_\alpha}dz\right).\end{aligned} \tag{2.47}$$

So, Gronwall's inequality and the following theorem finish the proof of Theorem 2.3. □

**Theorem 2.3** (see [29]). *For $0 < \alpha < 1/2$, the spaces $E_\alpha$ $(C[0,L], A)$ and $C^{2\alpha}[0,L]$ coincide and their norms are equivalent.*

## 2.2. Difference Case

For the approximate solution of problem (1.1), the Rothe difference scheme

$$\begin{gathered}
\frac{u_n^k - u_n^{k-1}}{\tau} = a(x_n)\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + p^k q_n + f(t_k, x_n), \\
1 \le k \le N, \quad 1 \le n \le M_l - 1, \quad M_l h = l, \quad N\tau = T, \\
\frac{u_n^k - u_n^{k-1}}{\tau} = a(x_n)\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + b(t_k, x_n)u_n^k + p^k q_n + g(t_k, x_n), \\
1 \le k \le N, \quad M_l + 1 \le n \le M, \quad Mh = L, \quad N\tau = T, \\
u_n^0 = \varphi(x_n), \quad 0 \le n \le M, \\
u_1^k - u_0^k = u_M^k = 0, \quad 0 \le k \le N, \\
u_{M_l+1}^k - u_{M_l}^k = u_{M_l}^k - u_{M_l-1}^k, \quad 0 \le k \le N, \\
u_{\lfloor x^*/h \rfloor}^k = u_s^k = \rho(t_k), \quad 0 \le k \le N, \quad 0 \le s \le \mathrm{M},
\end{gathered} \tag{2.48}$$

where $p^k = p(t_k), q_n = q(x_n), x_n = nh$, and $t_k = k\tau$ is constructed. Here, $q_s \neq 0$ and $q_1 - q_0 = q_M = 0$ are assumed. $\lfloor x \rfloor$ represents the floor function of $x$.

With the help of a positive operator $A$, we introduce the fractional spaces $E'_\alpha, 0 < \alpha < 1$, consisting of all $v \in E$ for which the following norm is finite:

$$\|v\|_{E'_\alpha} = \|v\|_E + \sup_{\lambda > 0} \lambda^\alpha \left\| A(\lambda + A)^{-1} v \right\|_E. \tag{2.49}$$

To formulate our results, we introduce the Banach space $\overset{\circ}{C}{}_h^{\alpha} = \overset{\circ}{C}{}^{\alpha}[0, L]_h$, $\alpha \in (0, 1)$, of all grid functions $\phi^h = \{\phi_n\}_{n=1}^{M-1}$ defined on

$$[0, L]_h = \{x_n = nh, 0 \le n \le M, Mh = L\}, \tag{2.50}$$

with $\phi_1 - \phi_0 = \phi_M = 0$ equipped with the norm

$$\begin{gathered}
\|\phi_h\|_{\overset{\circ}{C}{}_h^{\alpha}} = \|\phi_h\|_{C_h} + \sup_{1 \le n < n+r \le M} |\phi_{n+r} - \phi_n| (rh)^{-\alpha}, \\
\|\phi_h\|_{C_h} = \max_{1 \le n \le M} |\phi_n|.
\end{gathered} \tag{2.51}$$

Moreover, $C_\tau(E) = C([0,\mathrm{T}]_\tau, E)$ is the Banach space of all grid functions $\phi^\tau = \{\phi(t_k)\}_{k=1}^{N-1}$ defined on $[0,T]_\tau = \{t_k = k\tau, 0 \le k \le N, Nh = T\}$ with values in $E$ equipped with the norm

$$\|\phi^\tau\|_{C_\tau(E)} = \max_{1\le k\le N} \|\phi(t_k)\|_E. \tag{2.52}$$

Then, the following theorem on well-posedness of problem (2.48) is established.

**Theorem 2.4.** *For the solution of problem* (2.48)*, the following coercive stability estimates*

$$\begin{aligned}
&\left\|\left\{\frac{u_k^h - u_{k-1}^h}{\tau}\right\}_{k=1}^{N}\right\|_{C_\tau(\overset{\circ}{C}{}_h^{2\alpha})} + \left\|\left\{D_h^2 u_k^h\right\}_{k=1}^{N}\right\|_{C_\tau(\overset{\circ}{C}{}_h^{2\alpha})} \\
&\quad\le M(q,s)\left\|\left\{\frac{\rho(t_k)-\rho(t_{k-1})}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau} \\
&\qquad + M(\tilde{a},\phi,\alpha,T)\left(\left\|D_h^2\varphi^h\right\|_{\overset{\circ}{C}{}_h^{2\alpha}} + \left\|\left\{F_1^h(t_k)\right\}_{k=1}^{N}\right\|_{C_\tau(\overset{\circ}{C}{}_h^{2\alpha})} + \|\rho^\tau\|_{C[0,T]_\tau}\right), \\
&\|p^\tau\|_{C[0,T]_\tau} \le M(q,s)\left\|\left\{\frac{\rho(t_k)-\rho(t_{k-1})}{\tau}\right\}_{k=1}^{N}\right\|_{C[0,T]_\tau} \\
&\qquad + M(\tilde{a},\phi,\alpha,T)\left[\left\|D_h^2\varphi^h\right\|_{\overset{\circ}{C}{}_h^{2\alpha}} + \left\|\left\{F_1^h(t_k)\right\}_{k=1}^{N}\right\|_{C_\tau(\overset{\circ}{C}{}_h^{2\alpha})} + \|\rho^\tau\|_{C[0,T]_\tau}\right]
\end{aligned} \tag{2.53}$$

*hold. Here,*

$$\begin{gathered}
F_1^h(t_k) = \left\{\begin{matrix} f(t_k,x_n) \\ 0 \\ b(t_k,x_n)\dfrac{\rho(t_k)}{q_s}q_n + g(t_k,x_n)\end{matrix}\right\}_{n=1}^{M-1}, \qquad \varphi^h = \{\varphi(x_n)\}_{n=1}^{M-1}, \\
\rho^\tau = \{\rho(t_k)\}_{k=0}^{N}, \qquad D_h^2 u^h = \left\{\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2}\right\}_{n=1}^{M-1} \\
\tilde{a} = \frac{1}{q_s}\left(aD_h^2 q^h - \sigma q^h\right).
\end{gathered} \tag{2.54}$$

*Proof.* The solution of problem (2.48) is searched in the following form:

$$u_n^k = \eta^k q_n + w_n^k, \tag{2.55}$$

where

$$\eta^k = \sum_{i=1}^{k} p^i \tau, \quad 1 \le k \le N, \quad \eta^0 = 0. \tag{2.56}$$

Difference derivatives of (2.55) can be written as

$$\begin{gathered}
\frac{u_n^k - u_n^{k-1}}{\tau} = \frac{\eta^k - \eta^{k-1}}{\tau} q_n + \frac{w_n^k - w_n^{k-1}}{\tau} = p^k q_n + \frac{w_n^k - w_n^{k-1}}{\tau}, \\
\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} = \eta^k \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} + \frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2},
\end{gathered} \tag{2.57}$$

for any $n, 1 \le n \le M-1$. At the interior grid point $s = \lfloor x^*/h \rfloor$, we have that

$$\begin{gathered}
u_s^k = \eta^k q_s + w_s^k = \rho(t_k), \\
\eta^k = \frac{\rho(t_k) - w_s^k}{q_s}.
\end{gathered} \tag{2.58}$$

Taking the difference derivative of the last equality and using the triangle inequality, we obtain

$$p^k = \frac{1}{q_s}\left(\frac{\rho(t_k) - \rho(t_{k-1})}{\tau} - \frac{w_s^k - w_s^{k-1}}{\tau}\right), \tag{2.59}$$

$$\begin{aligned}
\left|p^k\right| &\le M(q,s)\left(\left|\frac{\rho(t_k) - \rho(t_{k-1})}{\tau}\right| + \left|\frac{w_s^k - w_s^{k-1}}{\tau}\right|\right) \\
&\le M(q,s)\left(\max_{1\le k\le N}\left|\frac{\rho(t_k) - \rho(t_{k-1})}{\tau}\right| + \max_{1\le k\le N}\max_{0\le s\le M}\left|\frac{w_s^k - w_s^{k-1}}{\tau}\right|\right) \\
&\le M(q,s)\left(\max_{1\le k\le N}\left|\frac{\rho(t_k) - \rho(t_{k-1})}{\tau}\right| + \max_{1\le k\le N}\left\|\frac{w_k^h - w_{k-1}^h}{\tau}\right\|_{\overset{\circ}{C}{}_h^{2\alpha}}\right),
\end{aligned} \tag{2.60}$$

for any $k, 1 \le k \le N$.

In estimate (2.60), $\{w_k^h\}_{k=0}^N$ is the solution of the following difference scheme:

$$\begin{gathered}
\frac{w_n^k - w_n^{k-1}}{\tau} = a(x_n)\frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2} + a(x_n)\frac{\rho(t_k) - w_s^k}{q_s}\frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} \\
+ f(t_k, x_n),\ 1 \le k \le N,\ 1 \le n \le M_l - 1,\ M_l h = l,\ N\tau = T, \\
\frac{w_n^k - w_n^{k-1}}{\tau} = a(x_n)\frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2} + a(x_n)\frac{\rho(t_k) - w_s^k}{q_s}\frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} \\
b(t_k, x_n)w_n^k \ + b(t_k, x_n)\frac{\rho(t_k) - w_s^k}{q_s}q_n + g(t_k, x_n), \\
1 \le k \le N,\ M_l + 1 \le n \le M - 1,\ Mh = L,\ N\tau = T, \\
w_n^0 = \varphi(x_n), \quad 0 \le n \le M, \\
w_1^k - w_0^k = w_M^k = 0, \quad 0 \le k \le N, \\
w_{M_l+1}^k - w_{M_l}^k = w_{M_l}^k - w_{M_l-1}^k, \quad 0 \le k \le N,
\end{gathered} \tag{2.61}$$

where $x_n = nh, t_k = k\tau$. Therefore, estimate (2.60) and the following theorem finish the proof of Theorem 2.5. □

**Theorem 2.5.** *For the solution of problem* (2.61), *the following coercive stability estimate*

$$\begin{aligned}
\left\| \left\{ \frac{w_k^h - w_{k-1}^h}{\tau} \right\}_{k=1}^N \right\|_{C_\tau(\overset{\circ}{C}{}_h^{2\alpha})} &\le M(\tilde{a}, \phi, \alpha, T) \\
&\quad \times \left( \left\|\varphi^h\right\|_{\overset{\circ}{C}{}_h^{2\alpha}} + \left\| \left\{ F_1^h(t_k) \right\}_{k=1}^N \right\|_{C_\tau(\overset{\circ}{C}{}_h^{2\alpha})} + \left\|\rho^\tau\right\|_{C[0,T]_\tau} \right)
\end{aligned} \tag{2.62}$$

*holds.*

*Proof.* We can rewrite difference scheme (2.61) in the abstract form:

$$\begin{gathered}
\frac{w_k^h - w_{k-1}^h}{\tau} + A_h^x w_k^h + B_h^x w_k^h = \left( a\frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \sigma q \right)\frac{\rho(t_k) - w_s^k}{q_s} \\
+ F_1^h(t_k) + F_2^h(t_k), \quad t_k = k\tau,\ 1 \le k \le N,\ N\tau = T, \\
w_0^h = \varphi^h,
\end{gathered} \tag{2.63}$$

in a Banach space $E = \overset{\circ}{C}[0, l]_h$ with the positive operator $A_h^x$ defined by

$$A_h^x u^h = \left\{ -a(x_n)\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + \sigma u \right\}_{n=1}^{M-1}, \tag{2.64}$$

acting on grid functions $u^h$ such that it satisfies the condition

$$u_1 - u_0 = u_M = 0. \tag{2.65}$$

For every fixed $t \in [0, T]$, the difference operators $B_h^x(t)$ are given by the formula

$$B_h^x(t)u^h = \left\{ \begin{matrix} -\sigma u^n, \ 1 \le n \le M_l, \\ -(\sigma - b^n(t))u^n, \ b^n(t) = b(t, x_n), \\ x_n = nh, \ M_l + 1 \le n \le M - 1, \end{matrix} \right\}_{n=1}^{M-1}, \tag{2.66}$$

where $\sigma$ is a positive constant and the right-hand side functions are

$$\begin{gathered} F_1^h(t_k) = \left\{ \begin{matrix} f(t_k), \ 1 \le n \le M_l, \\ b(t_k)q_n \dfrac{\rho(t_k)}{q_s} + g(t_k), \ M_l + 1 \le n \le M - 1, \end{matrix} \right\}_{n=1}^{M-1}, \\ F_2^h(t_k) = \left\{ \begin{matrix} 0, \ 1 \le n \le M_l, \\ -b(t_k)q_n \dfrac{w_s^k}{q_s}, \ M_l + 1 \le n \le M - 1, \end{matrix} \right\}_{n=1}^{M-1}. \end{gathered} \tag{2.67}$$

Let us denote $R = (I + \tau A_h^x)^{-1}$. In problem (2.63), we have that

$$w_k^h = Rw_{k-1}^h + R\tau \left( a \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} \frac{\rho(t_k) - w_s^k}{q_s} - B_h^x(t)w_k^h + F_1^h(t_k) + F_2^h(t_k) \right), \tag{2.68}$$

for all $k, 1 \le k \le N$. By recurrence relations, we get

$$\begin{aligned} w_k^h = R^k \varphi^h &+ \sum_{m=1}^{k} R^{k-m+1} \frac{\tau}{q_s} a \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} \rho(t_m) \\ &- \sum_{m=1}^{k} R^{k-m+1} \frac{\tau}{q_s} a \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} w_s^m \\ &- \sum_{m=1}^{k} R^{k-m+1} B_h^x(t) \tau w_s^m + \sum_{m=1}^{k} R^{k-m+1} \tau F_1^h(t_m) + \sum_{m=1}^{k} R^{k-m+1} \tau F_2^h(t_m). \end{aligned} \tag{2.69}$$

Then, the following presentation of the solution of problem (2.63)

$$
\begin{aligned}
A_h^x w_k^h = {} & A_h^x R^k \varphi^h + \sum_{m=1}^{k} A_h^x R^{k-m+1} \frac{\tau}{q_s} a \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} \rho(t_m) \\
& - \sum_{m=1}^{k} A_h^x R^{k-m+1} \frac{\tau}{q_s} a \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} w_s^m \\
& - \sum_{m=1}^{k} A_h^x R^{k-m+1} B_h^x(t) \tau w_s^m + \sum_{m=1}^{k} A_h^x R^{k-m+1} \tau F_1^h(t_m) \\
& + \sum_{m=1}^{k} A_h^x R^{k-m+1} \tau F_2^h(t_m) = \sum_{k=1}^{6} J_k
\end{aligned}
\tag{2.70}
$$

is obtained. Here,

$$
\begin{aligned}
J_1^k &= A_h^x R^k \varphi^h, \\
J_2^k &= \sum_{m=1}^{k} A_h^x R^{k-m+1} \frac{\tau}{q_s} a \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} \rho(t_m), \\
J_3^k &= -\sum_{m=1}^{k} A_h^x R^{k-m+1} \frac{\tau}{q_s} a \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} w_s^m, \\
J_4^k &= -\sum_{m=1}^{k} A_h^x R^{k-m+1} B_h^x(t) \tau w_s^m, \\
J_5^k &= \sum_{m=1}^{k} A_h^x R^{k-m+1} \tau F_1^h(t_m), \\
J_6^k &= \sum_{m=1}^{k} A_h^x R^{k-m+1} \tau F_2^h(t_m).
\end{aligned}
\tag{2.71}
$$

Now, let us estimate $J_r^k$ for $r = 1, 2, \ldots, 6$ separately. We start with $J_1^k$. Applying the definition of norm of the spaces $E'_\alpha$, we get

$$
\begin{aligned}
\left\| J_1^k \right\|_{E'_\alpha} &= \left\| R^k A_h^x \varphi^h \right\|_{E'_\alpha} \le \left\| R^k \right\|_{E'_\alpha \to E'_\alpha} \left\| A_h^x \varphi^h \right\|_{E'_\alpha} \\
&\le \left\| R^k \right\|_{E \to E} \left\| A_h^x \varphi^h \right\|_{E'_\alpha}.
\end{aligned}
\tag{2.72}
$$

Using estimate

$$
\left\| R^k \right\|_{E \to E} \le M,
\tag{2.73}
$$

we get

$$\left\| J_1^k \right\|_{E'_\alpha} \le M_1 \left\| A_h^x \varphi^h \right\|_{E'_\alpha}, \tag{2.74}$$

for any $k,\ 1 \le k \le N$.

Let us estimate $J_2^k$ :

$$\begin{aligned} \left\| J_2^k \right\|_{E'_\alpha} &= \left\| \sum_{m=1}^{k} A_h^x R^{k-m+1} \frac{\tau}{q_s} a \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} \rho(t_m) \right\|_{E'_\alpha} \\ &\le \max_{1\le m\le N} \rho(t_m) \sum_{m=1}^{k} \left\| A_h^x R^{k-m+1} \tau \tilde{a} \right\|_{E'_\alpha}, \end{aligned} \tag{2.75}$$

where

$$\tilde{a} = a \frac{q_{n+1} - 2q_n + q_{n-1}}{q_s h^2}. \tag{2.76}$$

From the definition of norm of the spaces $E'_\alpha$, it follows that

$$\begin{aligned} \left\| \sum_{m=1}^{k} A_h^x R^{k-m+1} \tau \tilde{a} \right\|_{E'_\alpha} \le{}& \sum_{m=1}^{k} \left\| A_h^x R^{k-m+1} \tau \tilde{a} \right\|_E \\ &+ \sup_{\lambda>0} \sum_{m=1}^{k} \left\| \lambda^\alpha A_h^x (\lambda + A_h^x)^{-1} A_h^x R^{k-m+1} \tau \tilde{a} \right\|_E. \end{aligned} \tag{2.77}$$

Let us estimate each term separately. We divide first term into two parts:

$$\sum_{m=1}^{k} \left\| A_h^x R^{k-m+1} \tau \tilde{a} \right\|_E = \sum_{m=1}^{k-1} \left\| A_h^x R^{k-m+1} \tau \tilde{a} \right\|_E + \left\| A_h^x R \tau \tilde{a} \right\|_E. \tag{2.78}$$

In the first part, by the definition of norm of the spaces $E'_\alpha$ and the identity (see [29])

$$(I + \tau A)^{-k} = \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-t} \exp\{-\tau t A\} dt, \quad k \ge 2, \tag{2.79}$$

we deduce that

$$\begin{aligned} \sum_{m=1}^{k-1} \left\| A_h^x R^{k-m+1} \tau \tilde{a} \right\|_E &\le \sum_{m=1}^{k-1} \frac{\tau}{(k-m)!} \int_0^\infty \frac{t^{k-m}}{(\tau t)^{1-\alpha}} e^{-t} \left\| (\tau t)^{1-\alpha} A_h^x e^{-\tau t A} \tilde{a} \right\|_E dt \\ &\le \|\tilde{a}\|_{E'_\alpha} \sum_{m=1}^{k-1} \frac{\tau}{(k-m)!} \int_0^\infty \frac{t^{k-m}}{(\tau t)^{1-\alpha}} e^{-t} dt \end{aligned}$$

$$
\begin{aligned}
&= \|\tilde{a}\|_{E'_\alpha} \sum_{m=1}^{k-1} \frac{\tau^\alpha}{(k-m)!} \int_0^\infty t^{k-m-1+\alpha} e^{-t} dt \\
&= \|\tilde{a}\|_{E'_\alpha} \sum_{m=1}^{k-1} \frac{\tau^\alpha}{(k-m)!} \int_0^\infty t^{(k-m-1)\alpha+\alpha} e^{-\alpha t} t^{(k-m-1)(1-\alpha)} e^{-(1-\alpha)t} dt.
\end{aligned} \tag{2.80}
$$

The Hölder inequality with $p = 1/\alpha, q = 1/(1-\alpha)$ and the definition of the gamma function yield that

$$
\begin{aligned}
\sum_{m=1}^{k-1} \left\| A_h^x R^{k-m+1} \tau \tilde{a} \right\|_E &\le \|\tilde{a}\|_{E'_\alpha} \sum_{m=1}^{k-1} \frac{\tau^\alpha}{(k-m)!} \left( \int_0^\infty \left( t^{(k-m-1)\alpha+\alpha} e^{-\alpha t} \right)^{1/\alpha} dt \right)^\alpha \\
&\quad \times \left( \int_0^\infty \left( t^{(k-m-1)(1-\alpha)} e^{-(1-\alpha)t} \right)^{1/(1-\alpha)} dt \right)^{1-\alpha} \\
&= \|\tilde{a}\|_{E'_\alpha} \sum_{m=1}^{k-1} \frac{\tau^\alpha}{(k-m)!} \left( \int_0^\infty t^{k-m} e^{-t} dt \right)^\alpha \left( \int_0^\infty t^{k-m-1} e^{-t} dt \right)^{1-\alpha} \\
&= \|\tilde{a}\|_{E'_\alpha} \sum_{m=1}^{k-1} \frac{\tau^\alpha}{(k-m)!} (\Gamma(k-m+1))^\alpha (\Gamma(k-m))^{1-\alpha}.
\end{aligned} \tag{2.81}
$$

By the fact that $\Gamma(n) = (n-1)!$ and $\Gamma(n) = (n-1)\Gamma(n-1)$, we get

$$
\begin{aligned}
\sum_{m=1}^{k-1} \left\| A_h^x R^{k-m+1} \tau \tilde{a} \right\|_E &\le \|\tilde{a}\|_{E'_\alpha} \sum_{m=1}^{k-1} \frac{\tau^\alpha}{(k-m)!} (k-m)^\alpha \Gamma(k-m) \\
&= \|\tilde{a}\|_{E'_\alpha} \sum_{m=1}^{k-1} \frac{\tau^\alpha}{(k-m)^{1-\alpha}} = \|\tilde{a}\|_{E'_\alpha} \sum_{m=1}^{k-1} \frac{\tau}{((k-m)\tau)^{1-\alpha}} \\
&\le M_2 \|\tilde{a}\|_{E'_\alpha} \int_0^{k\tau} \frac{1}{(k\tau - s)^{1-\alpha}} ds = M_2 \|\tilde{a}\|_{E'_\alpha} \left( \left[ -\frac{(k\tau - s)^\alpha}{\alpha} \right]_0^{k\tau} \right).
\end{aligned} \tag{2.82}
$$

So, we have that

$$
\sum_{m=1}^{k-1} \left\| A_h^x R^{k-m+1} \tau \tilde{a} \right\|_E \le M_2 \|\tilde{a}\|_{E'_\alpha} \frac{(k\tau)^\alpha}{\alpha} \le M_3(\alpha, T) \|\tilde{a}\|_{E'_\alpha}. \tag{2.83}
$$

In the second part, we have that

$$
\left\| A_h^x R \tau \tilde{a} \right\|_E \le \left\| A_h^x R \tau \right\|_{E \to E} \|\tilde{a}\|_E \le M_4 \|\tilde{a}\|_{E'_\alpha}. \tag{2.84}
$$

Combining estimates (2.83) and (2.84), we obtain

$$\sum_{m=1}^{k}\left\|A_h^x R^{k-m+1}\tau\widetilde{a}\right\|_E \le M_5(\alpha,T)\|\widetilde{a}\|_{E'_\alpha}. \tag{2.85}$$

Let us estimate the second term. From the Cauchy-Riesz formula (see [29])

$$f(A)=\frac{1}{2\pi i}\int_\Gamma f(z)(z-A)^{-1}dz, \tag{2.86}$$

it follows that

$$\begin{aligned}
&\sum_{m=1}^{k}\lambda^\alpha A_h^x(\lambda+A_h^x)^{-1}R^{k-m+1}A_h^x\tau\widetilde{a}\\
&\quad=\frac{1}{2\pi i}\int_{S_1\cup S_2}\sum_{m=1}^{k}\frac{z}{(1+z)^{k-m+1}}\frac{\lambda^\alpha}{\lambda+z\tau^{-1}}A_h^x\left(z-\tau A_h^x\right)^{-1}\widetilde{a}dz\\
&\quad=\frac{1}{2\pi i}\int_{S_1\cup S_2}\sum_{m=1}^{k}\frac{(z/\tau)^{-\alpha}}{(1+z)^{k-m+1}}\frac{\lambda^\alpha}{\lambda\tau+z}\left(\frac{z}{\tau}\right)^\alpha A_h^x\left(\frac{z}{\tau}-A_h^x\right)^{-1}\widetilde{a}dz.
\end{aligned} \tag{2.87}$$

Since $z=\rho e^{\pm i\phi}$, with $|\phi|\le\pi/2$, the estimate (see [29])

$$\left\|(\lambda-A)^{-1}\right\|_{E\to E}\le\frac{M(\phi)}{1+|\lambda|} \tag{2.88}$$

yields

$$\begin{gathered}
\left\|\left(\frac{z}{\tau}\right)^\alpha A_h^x\left(\frac{z}{\tau}-A_h^x\right)^{-1}\widetilde{a}\right\|_E\le M_6\left\|\left(\frac{\rho}{\tau}\right)^\alpha A_h^x\left(\frac{\rho}{\tau}+A_h^x\right)^{-1}\widetilde{a}\right\|_E,\\
\frac{1}{|\lambda\tau+z|}\le\frac{M_6}{\lambda\tau+\rho}.
\end{gathered} \tag{2.89}$$

Hence,

$$\begin{aligned}
&\left\|\sum_{m=1}^{k}\lambda^\alpha A_h^x(\lambda+A_h^x)^{-1}R^{k-m+1}A_h^x\tau\widetilde{a}\right\|_E\\
&\quad\le M_6\int_0^\infty\sum_{m=1}^{k}\frac{\rho^{1-\alpha}}{\left[1+2\rho\cos\phi+\rho^2\right]^{(k-m+1)/2}}\frac{(\lambda\tau)^\alpha d\rho}{\lambda\tau+\rho}\|\widetilde{a}\|_{E'_\alpha}.
\end{aligned} \tag{2.90}$$

Summing the geometric progression, we get

$$\left\| \sum_{m=1}^{k} \lambda^{\alpha} A_h^x \left(\lambda + A_h^x\right)^{-1} R^{k-m+1} A_h^x \tau \widetilde{a} \right\|_E \leq M_6 \int_0^{\infty} \sum_{m=1}^{k} \frac{\rho^{1-\alpha}}{\left[1 + 2\rho\cos\phi + \rho^2\right]^{1/2}} \times \left(1 - \frac{1}{\left[1 + 2\rho\cos\phi + \rho^2\right]^{1/2}}\right)^{-1} \frac{(\lambda\tau)^{\alpha} d\rho}{\lambda\tau + \rho} \|\widetilde{a}\|_{E'_\alpha} \leq M_6 \int_0^{\infty} \frac{(\lambda\tau)^{\alpha} \varkappa(\rho) d\rho}{(\lambda\tau + \rho)\rho^{\alpha}} \|\widetilde{a}\|_{E'_\alpha}. \tag{2.91}$$

Since the function

$$\varkappa(\rho) = \frac{\rho}{\left[1 + 2\rho\cos\phi + \rho^2\right]^{1/2} - 1} = \frac{1 + \left[1 + 2\rho\cos\phi + \rho^2\right]^{1/2}}{2\cos\phi + \rho} \tag{2.92}$$

does not increase for $\rho \geq 0$, we have $\varkappa(0) = 1/\cos\phi \geq \varkappa(\rho)$ for all $\rho > 0$. Consequently,

$$\left\| \sum_{m=1}^{k} \lambda^{\alpha} A_h^x \left(\lambda + A_h^x\right)^{-1} R^{k-m+1} A_h^x \tau \widetilde{a} \right\|_E \leq \frac{M_6}{\cos\phi} \int_0^{\infty} \frac{(\lambda\tau)^{\alpha} d\rho}{(\lambda\tau + \rho)\rho^{\alpha}} \|\widetilde{a}\|_{E'_\alpha} \tag{2.93}$$

for any $\lambda > 0$. Hence,

$$\sup_{\lambda > 0} \left\| \sum_{m=1}^{k} \lambda^{\alpha} A_h^x \left(\lambda + A_h^x\right)^{-1} R^{k-m+1} A_h^x \tau \widetilde{a} \right\|_E \leq M_7(\phi, \alpha) \|\widetilde{a}\|_{E'_\alpha}. \tag{2.94}$$

Then, using estimates (2.85) and (2.94), we get

$$\left\| \sum_{m=1}^{k} \left(R^{k-m+1} - R^{k-m}\right) \widetilde{a} \right\|_{E'_\alpha} \leq M_8(\phi, \alpha, T) \|\widetilde{a}\|_{E'_\alpha}, \tag{2.95}$$

$$\left\| J_2^k \right\|_{E'_\alpha} \leq \max_{1 \leq m \leq N} \rho(t_m) M_8(\phi, \alpha, T) \|\widetilde{a}\|_{E'_\alpha}. \tag{2.96}$$

Now, let us estimate $J_3^k$ :

$$\left\| J_3^k \right\|_{E'_\alpha} = \left\| - \sum_{m=1}^{k} A_h^x R^{k-m+1} \frac{\tau}{q_s} a \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} w_s^m \right\|_{E'_\alpha} \leq \sum_{m=1}^{k} \left\| A_h^x R^{k-m+1} \frac{\tau}{q_s} \left( a \frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} - \sigma q \right) \right\|_{E'_\alpha} |w_s^m|. \tag{2.97}$$

Since

$$\begin{aligned} |w_s^m| &\le \max_{0\le s\le M} |w_s^m| = \left\|w^h\right\|_E \le \left\|w^h\right\|_{E'_\alpha} \\ &\le \left\|\left(A_h^x\right)^{-1}\right\|_{E'_\alpha\to E'_\alpha}\left\|A_h^x w^h\right\|_{E'_\alpha} \le M\left\|A_h^x w^h\right\|_{E'_\alpha}, \end{aligned} \tag{2.98}$$

and using estimate (2.95), we obtain

$$\|J_3\|_{E'_\alpha} \le M_9(\phi,\alpha,T,\tau)\|\tilde{a}\|_{E'_\alpha}\sum_{m=1}^{k}\left\|A_h^x w^h\right\|_{E'_\alpha}\tau. \tag{2.99}$$

$J_4^k$ can be estimated as follows:

$$\begin{aligned} \left\|J_4^k\right\|_{E'_\alpha} &= \left\|-\sum_{m=1}^{k} A_h^x R^{k-m+1} B_h^x(t)\tau w_s^m\right\|_{E'_\alpha} \\ &= \left\|\sum_{m=1}^{k} A_h^x R^{k-m+1} B_h^x(t)\left(A_h^x\right)^{-1} A_h^x \tau w_s^m\right\|_{E'_\alpha} \\ &\le \sum_{m=1}^{k}\left\|A_h^x R^{k-m+1} B_h^x(t)\left(A_h^x\right)^{-1}\right\|_{E'_\alpha\to E'_\alpha}\left\|A_h^x w_s^m\right\|_{E'_\alpha}\tau. \end{aligned} \tag{2.100}$$

From

$$\begin{aligned} &\left\|R^k\right\|_{E'_\alpha\to E'_\alpha} \le \left\|R^k\right\|_{E\to E} \le M, \\ &\left\|A_h^x B_h^x(t)\left(A_h^x\right)^{-1}\right\|_{E'_\alpha\to E'_\alpha} \le M \end{aligned} \tag{2.101}$$

it follows that

$$\left\|J_4^k\right\|_{E'_\alpha} \le M_{10}\sum_{m=1}^{k}\left\|A_h^x w_s^m\right\|_{E'_\alpha}\tau. \tag{2.102}$$

The estimations of $J_5^k$ and $J_6^k$ are as follows. By the definition of the norm of the spaces $E'_\alpha$ and (2.95), we get

$$\begin{aligned} \left\|J_5^k\right\|_{E'_\alpha} &\le \left\|\sum_{m=1}^{k} A_h^x R^{k-m+1}\tau F_1^h(t_m)\right\|_{E'_\alpha} \\ &\quad + M_{11}(\phi,\alpha,T)\left\|\left\{F_1^h(t_k)\right\}_{k=1}^{N}\right\|_{C_\tau(E'_\alpha)}, \end{aligned} \tag{2.103}$$

$$\|J_6\|_{E'_\alpha} \le M_{12}(\phi,\alpha)\sum_{m=1}^{k}\left\|A_h^x w^h\right\|_{E'_\alpha}\tau. \tag{2.104}$$

Combining estimates (2.74), (2.96), (2.99), and (2.102)–(2.104), we get

$$\begin{aligned}\left\|A_h^x w_k^h\right\|_{E'_\alpha} \le{}& M_1\left\|A_h^x\varphi^h\right\|_{E'_\alpha} + \max_{1\le m\le N}\rho(t_m)M_8(\phi,\alpha,T)\|\tilde{a}\|_{E'_\alpha}\\ &+\left(M_9(\phi,\alpha,T,\tau)\|\tilde{a}\|_{E'_\alpha}+M_{10}+M_{12}(\phi,\alpha)\right)\sum_{m=1}^{k}\left\|A_h^x w_k^h\right\|_{E_\alpha}\tau\\ &+M_{11}(\phi,\alpha,T)\left\|\left\{F_1^h(t_k)\right\}_{k=1}^{N}\right\|_{C_\tau(E'_\alpha)}.\end{aligned} \tag{2.105}$$

Using the discrete analogue of Gronwall's inequality, we get

$$\begin{aligned}\left\|A_h^x w_k^h\right\|_{E'_\alpha} \le{}& e^{M_{13}(\tilde{a},\phi,\alpha,T,\tau)}\\ &\times\left[M_1\left\|A_h^x\varphi^h\right\|_{E'_\alpha}+M_{14}(\tilde{a},\phi,\alpha,T)\|\rho^\tau\|_{C[0,T]_\tau}\right.\\ &\left.+M_{11}(\phi,\alpha,T)\left\|\left\{F_1^h(t_k)\right\}_{k=1}^{N}\right\|_{C_\tau(E'_\alpha)}\right].\end{aligned} \tag{2.106}$$

It follows from (2.63) and the triangle inequality that

$$\begin{aligned}\left\|\frac{w_k^h-w_{k-1}^h}{\tau}\right\|_{E'_\alpha} \le{}& e^{M_{12}(\tilde{a},\phi,\alpha,T)}\\ &\times\left[M_1\left\|A_h^x\varphi^h\right\|_{E'_\alpha}+M_{13}(\tilde{a},\phi,\alpha,T)\|\rho^\tau\|_{C[0,T]_\tau}\right.\\ &\left.+M_{11}(\phi,\alpha,T)\left\|\left\{F_1^h(t_k)\right\}_{k=1}^{N}\right\|_{C_\tau(E'_\alpha)}\right],\end{aligned} \tag{2.107}$$

for every $k,\ 1\le k\le N$. Then, we have that

$$\begin{aligned}\left\|\left\{\frac{w_k^h-w_{k-1}^h}{\tau}\right\}_{k=1}^{N}\right\|_{C_\tau(E'_\alpha)} \le{}& M_{14}(\tilde{a},\phi,\alpha,T)\\ &\times\left(\left\|A_h^x\varphi^h\right\|_{E'_\alpha}+\left\|\left\{F_1^h(t_k)\right\}_{k=1}^{N}\right\|_{C_\tau(E'_\alpha)}+\|\rho^\tau\|_{C[0,T]_\tau}\right).\end{aligned} \tag{2.108}$$

The following theorem finishes the proof of Theorem 2.6. □

**Theorem 2.6** (see [30]). *For $0 < \alpha < 1/2$, the spaces $E'_\alpha(C[0,L]_h, A_h^x)$ and $C^{2\alpha}[0,L]_h$ coincide and their norms are equivalent.*

## 3. Conclusion

Since artery disease caused by atherosclerosis is one of the most important causes of the death in the world, investigation of the effect of flow over the glycocalyx takes an important place. The flow equations can be formulated as an inverse problem. Here, our aim is to give more detailed understanding of the flow phenomena. Therefore, the well-posedness of the inverse problem of reconstructing the right side of a parabolic equation was investigated. Further, a new computer code regarding the flow analysis for the unknown pressure difference will be written.

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