Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 164876, 17 pages
doi:10.1155/2012/164876

Research Article

On the Study of Local Solutions for a Generalized Camassa-Holm Equation

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Received 23 May 2012; Revised 30 June 2012; Accepted 18 July 2012

Academic Editor: Yong Hong Wu

The pseudoparabolic regularization technique is employed to study the local well-posedness of strong solutions for a nonlinear dispersive model, which includes the famous Camassa-Holm equation. The local well-posedness is established in the Sobolev space $H^s(R)$ with $s > 3/2$ via a limiting procedure.

1. Introduction

In recent years, extensive research has been carried out worldwide to study highly nonlinear equations including the Camassa-Holm (CH) equation and its various generalizations [1–6]. It is shown in [7–9] that the inverse spectral or scattering approach is a powerful technique to handle the Camassa-Holm equation and analyze its dynamics. It is pointed out in [10–12] that the CH equation gives rise to geodesic flow of a certain invariant metric on the Bott-Virasoro group, and this geometric illustration leads to a proof that the Least Action Principle holds. Li and Olver [13] established the local well-posedness to the CH model in the Sobolev space $H^s(R)$ with $s > 3/2$ and gave conditions on the initial data that lead to finite time blow-up of certain solutions. Constantin and Escher [14] proved that the blow-up occurs in the form of breaking waves, namely, the solution remains bounded but its slope becomes unbounded in finite time. Hakkaev and Kirchev [15] investigated a generalized form of the Camassa-Holm equation with high order nonlinear terms and obtained the orbit stability of the traveling wave solutions under certain assumptions. Lai and Wu [16] discussed a generalized Camassa-Holm model and acquired its local existence and uniqueness. Recently, Li et al. [17] investigated the generalized Camassa-Holm equation

$$u_t - u_{txx} + ku^m u_x + (m+3)u^{m+1}u_x = (m+2)u^m u_x u_{xx} + u^{m+1}u_{xxx}, \tag{1.1}$$

where $m \geq 0$ is a natural number and $k \geq 0$. The authors in [17] assume that the initial value satisfies the sign condition and establish the global existence of solutions for (1.1).

In this paper, we will study the following generalization of (1.1):

$$u_t - u_{txx} + ku^m u_x + (m+3)u^{m+1}u_x = (m+2)u^m u_x u_{xx} + u^{m+1}u_{xxx} + \lambda(u - u_{xx}), \tag{1.2}$$

where $m \geq 0$ is a natural number, $k \geq 0$, and λ is a constant.

The objective of this paper is to study the local well-posedness of (1.2). Its local well-posedness of strong solutions in the Sobolev space $H^s(R)$ with $s > 3/2$ is investigated by using the pseudoparabolic regularization method. Comparing with the work by Li et al. [17], (1.2) considered in this paper possesses a conservation law different to that in [17] (see Lemma 3.2 in Section 3). Also (1.2) contains a dissipative term $\lambda(u - u_{xx})$, which causes difficulty to establish its local and global existence in the Sobolev space. It should be mentioned that the existence and uniqueness of local strong solutions for the generalized nonlinear Camassa-Holm models like (1.2) have never been investigated in the literatures.

The organization of this work is as follows. The main result is given in Section 2. Section 3 establishes several lemmas, and the last section gives the proof of the main result.

2. Main Result

Firstly, we introduce several notations.

$L^p = L^p(R)$ $(1 \leq p < +\infty)$ is the space of all measurable functions h such that $\|h\|_{L^p}^p = \int_R |h(t,x)|^p dx < \infty$. We define $L^\infty = L^\infty(R)$ with the standard norm $\|h\|_{L^\infty} = \inf_{m(e)=0} \sup_{x \in R \setminus e} |h(t,x)|$. For any real number s, $H^s = H^s(R)$ denotes the Sobolev space with the norm defined by

$$\|h\|_{H^s} = \left(\int_R \left(1 + |\xi|^2\right)^s \left|\widehat{h}(t,\xi)\right|^2 d\xi \right)^{1/2} < \infty, \tag{2.1}$$

where $\widehat{h}(t,\xi) = \int_R e^{-ix\xi} h(t,x)dx$.

For $T > 0$ and nonnegative number s, $C([0,T); H^s(R))$ denotes the Frechet space of all continuous H^s-valued functions on $[0,T)$. We set $\Lambda = (1 - \partial_x^2)^{1/2}$. For simplicity, throughout this paper, we let c denote any positive constant that is independent of parameter ε.

We consider the Cauchy problem of (1.2)

$$\begin{gathered} u_t - u_{txx} = -\frac{k}{m+1}\left(u^{m+1}\right)_x - \frac{m+3}{m+2}\left(u^{m+2}\right)_x + \frac{1}{m+2}\partial_x^3\left(u^{m+2}\right) \\ - (m+1)\partial_x\left(u^m u_x^2\right) + u^m u_x u_{xx} + \lambda(u - u_{xx}), \quad k \geq 0,\ m \geq 0, \\ u(0,x) = u_0(x). \end{gathered} \tag{2.2}$$

Now, we give our main results for problem of (2.2).

Theorem 2.1. *Suppose that the initial function $u_0(x)$ belongs to the Sobolev space $H^s(R)$ with $s > 3/2$ and λ is a constant. Then, there is a $T > 0$, which depends on $\|u_0\|_{H^s}$, such that problem* (2.2) *has a unique solution $u(t,x)$ satisfying*

$$u(t,x) \in C([0,T); H^s(R)) \bigcap C^1\left([0,T); H^{s-1}(R)\right). \tag{2.3}$$

3. Local Well-Posedness

In order to prove Theorem 2.1, we consider the associated regularized problem

$$\begin{aligned} u_t - u_{txx} + \varepsilon u_{txxxx} &= -\frac{k}{m+1}\left(u^{m+1}\right)_x - \frac{m+3}{m+2}\left(u^{m+2}\right)_x + \frac{1}{m+2}\partial_x^3\left(u^{m+2}\right) \\ &\quad - (m+1)\partial_x\left(u^m u_x^2\right) + u^m u_x u_{xx} + \lambda(u - u_{xx}), \\ u(0,x) &= u_0(x), \end{aligned} \tag{3.1}$$

where the parameter ε satisfies $0 < \varepsilon < 1/4$.

Lemma 3.1. *For $s \geq 1$ and $f(x) \in H^s(R)$ and letting $k_1 > 0$ be an integer such that $k_1 \leq s-1$, $f, f', \ldots, f^{k_1}$ are uniformly continuous bounded functions that converge to 0 at $x = \pm\infty$.*

The proof of Lemma 3.1 was stated on page 559 by Bona and Smith [18].

Lemma 3.2. *If $u(t,x) \in H^s (s > 7/2)$ is a solution to problem* (3.1)*, it holds that*

$$\int_R \left(u^2 + u_x^2 + \varepsilon u_{xx}^2\right)dx = \int_R \left(u_0^2 + u_{0x}^2 + \varepsilon u_{0xx}^2\right)dx + 2\lambda \int_0^t \int_R \left(u^2 + u_x^2\right)dx. \tag{3.2}$$

Proof. Using Lemma 3.1, we have $u(t,\pm\infty) = u_x(t,\pm\infty) = u_{xx}(t,\pm\infty) = u_{xxx}(t,\pm\infty) = 0$. The integration by parts results in

$$\begin{aligned} \int_R u^{m+2} u_{xxx} dx &= \int_R u^{m+2} du_{xx} = u^{m+2}u_{xx}|_{-\infty}^{+\infty} - (m+2)\int_R u^{m+1} u_x u_{xx} dx \\ &= -(m+2)\int_R u^{m+1} u_x u_{xx} dx. \end{aligned} \tag{3.3}$$

Direct calculation and integration by parts give rise to

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_R \left(u^2+u_x^2+\varepsilon u_{xx}^2\right)dx
&= \int_R (uu_t+u_xu_{tx}+\varepsilon u_{xx}u_{txx})dx \\
&= \int_R u(u_t-u_{txx}+\varepsilon u_{txxxx})dx \\
&= \int_R u\Bigg[-\frac{k}{m+1}\left(u^{m+1}\right)_x-\frac{m+3}{m+2}\left(u^{m+2}\right)_x+\frac{1}{m+2}\partial_x^3\left(u^{m+2}\right) \\
&\qquad -(m+1)\partial_x\left(u^m u_x^2\right)+u^m u_x u_{xx}+\lambda(u-u_{xx})\Bigg]dx \\
&= \int_R u\left[(m+2)u^m u_x u_{xx}+u^{m+1}u_{xxx}\right]dx+\lambda\int_R\left(u^2-uu_{xx}\right)dx \\
&= \int_R \left[(m+2)u^{m+1}u_x u_{xx}+u^{m+2}u_{xxx}\right]dx+\lambda\int_R\left(u^2+u_x^2\right)\,dx \\
&= \lambda\int_R\left(u^2+u_x^2\right)dx,
\end{aligned}
\tag{3.4}
$$

in which we have used (3.3). From (3.4), we obtain the conservation law (3.2). □

Lemma 3.3. *Let $s \geq 7/2$. The function $u(t,x)$ is a solution of problem* (3.1) *and the initial value $u_0(x) \in H^s$. Then, the following inequality holds:*

$$
\begin{aligned}
\|u\|_{H^1}^2 &\leq \int_R\left(u_0^2+u_{0x}^2+\varepsilon u_{0xx}^2\right)dx, \quad \text{if } \lambda \leq 0, \\
\|u\|_{H^1}^2 &\leq e^{2\lambda t}\int_R\left(u_0^2+u_{0x}^2+\varepsilon u_{0xx}^2\right)dx, \quad \text{if } \lambda > 0.
\end{aligned}
\tag{3.5}
$$

For $q \in (0, s-1]$, there is a constant c independent of ε such that

$$
\begin{aligned}
\int_R\left(\Lambda^{q+1}u\right)^2dx &\leq \int_R\left[\left(\Lambda^{q+1}u_0\right)^2+\varepsilon(\Lambda^q u_{0xx})^2\right]dx \\
&\quad + c\int_0^t \|u\|_{H^{q+1}}^2\left(|\lambda|+\left(\|u\|_{L^\infty}^{m-1}+\|u\|_{L^\infty}^m\right)\|u_x\|_{L^\infty}+\|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}^2\right)d\tau.
\end{aligned}
\tag{3.6}
$$

For $q \in [0, s-1]$, there is a constant c independent of ε such that

$$
(1-2\varepsilon)\|u_t\|_{H^q} \leq c\|u\|_{H^{q+1}}\left(|\lambda|+\left(\|u\|_{L^\infty}^{m-1}+\|u\|_{L^\infty}^m\right)\|u\|_{H^1}+\|u\|_{L^\infty}^m\|u_x\|_{L^\infty}+\|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}^2\right).
\tag{3.7}
$$

The proof of this lemma is similar to that of Lemma 3.5 in [17]. Here we omit it.

Lemma 3.4. *Let r and q be real numbers such that $-r < q \le r$. Then,*

$$\begin{aligned} \|uv\|_{H^q} &\le c\|u\|_{H^r}\|v\|_{H^q}, \quad \text{if } r > \frac{1}{2}, \\ \|uv\|_{H^{r+q-1/2}} &\le c\|u\|_{H^r}\|v\|_{H^q}, \quad \text{if } r < \frac{1}{2}. \end{aligned} \tag{3.8}$$

This lemma can be found in [19] or [20].

Lemma 3.5. *Let $u_0(x) \in H^s(R)$ with $s > 3/2$. Then, the Cauchy problem* (3.1) *has a unique solution $u(t,x) \in C([0,T]; H^s(R))$, where $T > 0$ depends on $\|u_0\|_{H^s(R)}$. If $s \ge 7/2$, the solution $u \in C([0,+\infty); H^s)$ exists for all time.*

Proof. Letting $D = (1-\partial_x^2+\varepsilon\partial_x^4)^{-1}$, we know that $D : H^s \to H^{s+4}$ is a bounded linear operator. Applying the operator D on both sides of the first equation of system (3.1) and then integrating the resultant equation with respect to t over the interval $(0,t)$, we get

$$\begin{aligned} u(t,x) = u_0(x) + \int_0^t D\Big[&-\frac{k}{m+1}\left(u^{m+1}\right)_x - \frac{m+3}{m+2}\left(u^{m+2}\right)_x + \frac{1}{m+2}\partial_x^3\left(u^{m+2}\right) \\ &-(m+1)\partial_x\left(u^m u_x^2\right) + u^m u_x u_{xx} + \lambda(u-u_{xx})\Big]dt. \end{aligned} \tag{3.9}$$

Suppose that both u and v are in the closed ball $B_{M_0}(0)$ of radius M_0 about the zero function in $C([0,T]; H^s(R))$ and A is the operator in the right-hand side of (3.9). For any fixed $t \in [0,T]$, we obtain

$$\begin{aligned} &\Bigg\| \int_0^t D\Big[-\frac{k}{m+1}\left(u^{m+1}\right)_x - \frac{m+3}{m+2}\left(u^{m+2}\right)_x + \frac{1}{m+2}\partial_x^3\left(u^{m+2}\right) \\ &\qquad -(m+1)\partial_x\left(u^m u_x^2\right) + u^m u_x u_{xx} + \lambda(u-u_{xx})\Big]dt \\ &\quad - \int_0^t D\Big[-\frac{k}{m+1}\left(v^{m+1}\right)_x - \frac{m+3}{m+2}\left(v^{m+2}\right)_x + \frac{1}{m+2}\partial_x^3\left(v^{m+2}\right) \\ &\qquad -(m+1)\partial_x\left(v^m v_x^2\right) + v^m v_x v_{xx} + \lambda(v-v_{xx})\Big]dt \Bigg\|_{H^s} \\ &\le TC_1\Bigg(\sup_{0\le t\le T}\|u-v\|_{H^s} + \sup_{0\le t\le T}\left\|u^{m+1}-v^{m+1}\right\|_{H^s} + \sup_{0\le t\le T}\left\|u^{m+2}-v^{m+2}\right\|_{H^s} \\ &\qquad + \sup_{0\le t\le T}\left\|D\partial_x\left[u^m u_x^2 - v^m v_x^2\right]\right\|_{H^s} + \sup_{0\le t\le T}\|D[u^m u_x u_{xx} - v^m v_x v_{xx}]\|_{H^s}\Bigg), \end{aligned} \tag{3.10}$$

where C_1 may depend on ε. The algebraic property of $H^{s_0}(R)$ with $s_0 > 1/2$ derives

$$\begin{aligned}
\left\|u^{m+2}-v^{m+2}\right\|_{H^s} &= \left\|(u-v)\left(u^{m+1}+u^m v+\cdots+uv^m+v^{m+1}\right)\right\|_{H^s} \\
&\le \|(u-v)\|_{H^s}\sum_{j=0}^{m+1}\|u\|_{H^s}^{m+1-j}\|v\|_{H^s}^{j}, \\
&\le M_0^{m+1}\|u-v\|_{H^s}, \\
\left\|u^{m+1}-v^{m+1}\right\|_{H^s} &\le M_0^m\|u-v\|_{H^s}, \\
\left\|D\partial_x\left(u^m u_x^2-v^m v_x^2\right)\right\|_{H^s} &\le \left\|D\partial_x\left[u^m\left(u_x^2-v_x^2\right)\right]\right\|_{H^s}+\left\|D\partial_x\left[v_x^2(u^m-v^m)\right]\right\|_{H^s} \\
&\le C\left(\left\|u^m\left(u_x^2-v_x^2\right)\right\|_{H^{s-1}}+\left\|v_x^2(u^m-v^m)\right\|_{H^{s-1}}\right) \\
&\le CM_0^{m+1}\|u-v\|_{H^s}.
\end{aligned} \tag{3.11}$$

Using the first inequality of Lemma 3.4 gives rise to

$$\begin{aligned}
\|D[u^m u_x u_{xx}-v^m v_x v_{xx}]\|_{H^s} &= \left\|\frac{1}{2}D\left[u^m\left(u_x^2\right)_x-v^m\left(v_x^2\right)_x\right]\right\|_{H^s} \\
&\le \frac{1}{2}\left(\left\|D\left[u^m\left(u_x^2-v_x^2\right)_x\right]\right\|_{H^s}+\left\|D\left[\left(v_x^2\right)_x(u^m-v^m)\right]\ \right\|_{H^s}\right) \\
&\le C\left(\left\|u^m\left(u_x^2-v_x^2\right)_x\right\|_{H^{s-2}}+\left\|\left(v_x^2\right)_x(u^m-v^m)\right\|_{H^{s-2}}\right) \\
&\le C\left(\|u^m\|_{H^s}\left\|u_x^2-v_x^2\right\|_{H^{s-1}}+\left\|v_x^2\right\|_{H^{s-1}}\|u^m-v^m\|_{H^s}\right) \\
&\le CM_0^{m+1}\|u-v\|_{H^s},
\end{aligned} \tag{3.12}$$

where C may depend on ε. From (3.11)-(3.12), we obtain

$$\|Au-Av\|_{H^s} \le \theta\|u-v\|_{H^s}, \tag{3.13}$$

where $\theta = TC_2(M_0^m + M_0^{m+1})$ and C_2 is independent of $0 < t < T$. Choosing T sufficiently small such that $\theta < 1$, we know that A is a contraction. Similarly, it follows from (3.10) that

$$\|Au\|_{H^s} \le \|u_0\|_{H^s} + \theta\|u\|_{H^s}. \tag{3.14}$$

Choosing T sufficiently small such that $\theta M_0 + \|u_0\|_{H^s} < M_0$, we deduce that A maps $B_{M_0}(0)$ to itself. It follows from the contraction-mapping principle that the mapping A has a unique fixed point u in $B_{M_0}(0)$. It completes the proof. □

From the above and Lemma 3.2, we have

$$\int_R \left(u^2 + u_x^2 + \varepsilon u_{xx}^2\right) dx \le e^{2|\lambda|t} \int_R \left(u_0^2 + u_{0x}^2 + \varepsilon u_{0xx}^2\right) dx. \tag{3.15}$$

Therefore,

$$\|u_x\|_{L^\infty} \le C_\varepsilon e^{2|\lambda|t} \int_R \left(u_0^2 + u_{0x}^2 + \varepsilon u_{0xx}^2\right) dx, \tag{3.16}$$

which together with Lemma 3.3 completes the proof of the global existence.

Setting $\phi_\varepsilon(x) = \varepsilon^{-1/4}\phi(\varepsilon^{-1/4}x)$ with $0 < \varepsilon < 1/4$ and $u_{\varepsilon 0} = \phi_\varepsilon \star u_0$, we know that $u_{\varepsilon 0} \in C^\infty$ for any $u_0 \in H^s, s > 0$. From Lemma 3.5, it derives that the Cauchy problem

$$\begin{aligned} u_t - u_{txx} + \varepsilon u_{txxxx} = &-\frac{k}{m+1}\left(u^{m+1}\right)_x - \frac{m+3}{m+2}\left(u^{m+2}\right)_x + \frac{1}{m+2}\partial_x^3\left(u^{m+2}\right) \\ &- (m+1)\partial_x\left(u^m u_x^2\right) + u^m u_x u_{xx} + \lambda(u - u_{xx}), \\ u(0,x) = &\, u_{\varepsilon 0}(x), \quad x \in R, \end{aligned} \tag{3.17}$$

has a unique solution $u_\varepsilon(t,x) \in C^\infty([0,\infty); H^\infty)$.

Furthermore, we have the following.

Lemma 3.6. *For $s > 0, u_0 \in H^s$, it holds that*

$$\|u_{\varepsilon 0x}\|_{L^\infty} \le c\|u_{0x}\|_{L^\infty}, \tag{3.18}$$

$$\|u_{\varepsilon 0}\|_{H^q} \le c, \quad \text{if } q \le s, \tag{3.19}$$

$$\|u_{\varepsilon 0}\|_{H^q} \le c\varepsilon^{(s-q)/4}, \quad \text{if } q > s, \tag{3.20}$$

$$\|u_{\varepsilon 0} - u_0\|_{H^q} \le c\varepsilon^{(s-q)/4}, \quad \text{if } q \le s, \tag{3.21}$$

$$\|u_{\varepsilon 0} - u_0\|_{H^s} = o(1), \tag{3.22}$$

where c is a constant independent of ε.

The proof of Lemma 3.6 can be found in [16].

Remark 3.7. For $s \geq 1$, using $\|u_\varepsilon\|_{L^\infty} \leq c\|u_\varepsilon\|_{H^{1/2+}} \leq c\|u_\varepsilon\|_{H^1}$, $\|u_\varepsilon\|_{H^1}^2 \leq c\int_R (u_\varepsilon^2 + u_{\varepsilon x}^2)dx$, (3.5), (3.19), and (3.20), we know that,

$$
\begin{aligned}
\|u_\varepsilon\|_{L^\infty}^2 \leq c\|u_\varepsilon\|_{H^1}^2 &\leq ce^{2|\lambda|t}\int_R \left(u_{\varepsilon 0}^2 + u_{\varepsilon 0x}^2 + \varepsilon u_{\varepsilon 0xx}^2\right)dx \\
&\leq ce^{2|\lambda|t}\left(\|u_{\varepsilon 0}\|_{H^1}^2 + \varepsilon\|u_{\varepsilon 0}\|_{H^2}^2\right) \\
&\leq ce^{|2\lambda|t}\left(c + c\varepsilon \times \varepsilon^{(s-2)/2}\right) \\
&\leq c_0 e^{2|\lambda|t},
\end{aligned}
\tag{3.23}
$$

where c_0 is independent of ε and t.

Lemma 3.8. *Suppose $u_0(x) \in H^s(R)$ with $s \geq 1$ such that $\|u_{0x}\|_{L^\infty} < \infty$. Let $u_{\varepsilon 0}$ be defined as in system* (3.17). *Then, there exist two positive constants T and c, which are independent of ε, such that the solution u_ε of problem* (3.17) *satisfies $\|u_{\varepsilon x}\|_{L^\infty} \leq c$ for any $t \in [0, T)$.*

Here we omit the proof of Lemma 3.8 since it is similar to Lemma 3.9 presented in [17].

Lemma 3.9 (see Li and Olver [13]). *If u and f are functions in $H^{q+1} \cap \{\|u_x\|_{L^\infty} < \infty\}$, then*

$$
\left|\int_R \Lambda^q u \Lambda^q (uf)_x dx\right| \leq
\begin{cases}
c_q\|f\|_{H^{q+1}}\|u\|_{H^q}^2, \quad q \in \left(\dfrac{1}{2}, 1\right], \\
c_q\Big(\|f\|_{H^{q+1}}\|u\|_{H^q}\|u\|_{L_\infty} \\
\qquad +\|f_x\|_{L_\infty}\|u\|_{H^q}^2 + \|f\|_{H^q}\|u\|_{H^q}\|u_x\|_{L_\infty}\Big), \\
\qquad\qquad q \in (0, \infty).
\end{cases}
\tag{3.24}
$$

Lemma 3.10 (see Lai and Wu [16]). *For $u, v \in H^s(R)$ with $s > 3/2$, $w = u - v$, $q > 1/2$, and a natural number n, it holds that*

$$
\left|\int_R \Lambda^s w \Lambda^s \left(u^{n+1} - v^{n+1}\right)_x dx\right| \leq c\left(\|w\|_{H^s}\|w\|_{H^q}\|v\|_{H^{s+1}} + \|w\|_{H^s}^2\right).
\tag{3.25}
$$

Lemma 3.11 (see Lai and Wu [16]). *If $1/2 < q < \min\{1, s-1\}$ and $s > 3/2$, then for any functions w, f defined on R, it holds that*

$$
\left|\int_R \Lambda^q w \Lambda^{q-2}(wf)_x dx\right| \leq c\|w\|_{H^q}^2\|f\|_{H^q},
\tag{3.26}
$$

$$
\left|\int_R \Lambda^q w \Lambda^{q-2}(w_x f_x)_x dx\right| \leq c\|w\|_{H^q}^2\|f\|_{H^s}.
\tag{3.27}
$$

Lemma 3.12. *For problem* (3.17), $s > 3/2$, *and* $u_0 \in H^s(R)$, *there exist two positive constants* c *and* M, *which are independent of* ε, *such that the following inequalities hold for any sufficiently small* ε *and* $t \in [0, T)$:

$$\begin{aligned} \|u_\varepsilon\|_{H^s} &\le Me^{ct}, \\ \|u_\varepsilon\|_{H^{s+k_1}} &\le \varepsilon^{-k_1/4} Me^{ct}, \quad k_1 > 0, \\ \|u_{\varepsilon t}\|_{H^{s+k_1}} &\le \varepsilon^{-(k_1+1)/4} Me^{ct}, \quad k_1 > -1. \end{aligned} \tag{3.28}$$

Slightly modifying the methods presented in [16] can complete the proof of Lemma 3.12.

Our next step is to demonstrate that u_ε is a Cauchy sequence. Let u_ε and u_δ be solutions of problem (3.17), corresponding to the parameters ε and δ, respectively, with $0 < \varepsilon < \delta < 1/4$, and let $w = u_\varepsilon - u_\delta$. Then, w satisfies the problem

$$\begin{aligned} &(1-\varepsilon)w_t - \varepsilon w_{xxt} + (\delta - \varepsilon)(u_{\delta t} + u_{\delta xxt}) \\ &\quad = \left(1 - \partial_x^2\right)^{-1} \Bigg[-\varepsilon w_t + (\delta - \varepsilon) u_{\delta t} - \frac{k}{m+1} \partial_x \left(u_\varepsilon^{m+1} - u_\delta^{m+1}\right) - \partial_x \left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right) \\ &\qquad - \partial_x \left[\partial_x \left(u_\varepsilon^{m+1}\right) \partial_x w + \partial_x \left(u_\varepsilon^{m+1} - u_\varepsilon^{m+1}\right) \partial_x u_\delta\right] \\ &\qquad + \left[u_\varepsilon^m u_{\varepsilon x} u_{\varepsilon xx} - u_\delta^m u_{\delta x} u_{\delta xx}\right] \Bigg] - \frac{1}{m+2} \partial_x \left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right) + \lambda w, \end{aligned} \tag{3.29}$$

$$w(x, 0) = w_0(x) = u_{\varepsilon 0}(x) - u_{\delta 0}(x). \tag{3.30}$$

Lemma 3.13. *For* $s > 3/2$, $u_0 \in H^s(R)$, *there exists* $T > 0$ *such that the solution* u_ε *of* (3.17) *is a Cauchy sequence in* $C([0, T]; H^s(R)) \bigcap C^1([0, T]; H^{s-1}(R))$.

Proof. For q with $1/2 < q < \min\{1, s-1\}$, multiplying both sides of (3.29) by $\Lambda^q w \Lambda^q$ and then integrating with respect to x give rise to

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_R \left[(1-\varepsilon)(\Lambda^q w)^2 + \varepsilon(\Lambda^q w_x)^2\right] dx \\ &\quad = (\varepsilon - \delta) \int_R (\Lambda^q w)[(\Lambda^q u_{\delta t}) + (\Lambda^q u_{\delta xxt})] dx - \varepsilon \int_R \Lambda^q w \Lambda^{q-2} w_t dx \\ &\qquad + (\delta - \varepsilon) \int_R \Lambda^q w \Lambda^{q-2} u_{\delta t} dx - \frac{1}{m+2} \int_R (\Lambda^q w) \Lambda^q \left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right)_x dx \end{aligned}$$

$$
\begin{aligned}
&-\frac{k}{m+1}\int_R \Lambda^q w \Lambda^{q-2}\left(u_\varepsilon^{m+1}-u_\delta^{m+1}\right)_x dx-\int_R \Lambda^q w \Lambda^{q-2}\left(u_\varepsilon^{m+2}-u_\delta^{m+2}\right)_x dx\\
&-\int_R \Lambda^q w \Lambda^{q-2}\left[\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w\right]_x dx-\int_R \Lambda^q w \Lambda^{q-2}\left[\partial_x\left(u_\varepsilon^{m+1}-u_\delta^{m+1}\right)\partial_x u_\delta\right]_x dx\\
&+\int_R \Lambda^q w \Lambda^{q-2}\left[u_\varepsilon^m u_{\varepsilon x}u_{\varepsilon xx}-u_\delta^m u_{\delta x}u_{\delta xx}\right]dx+\lambda\int_R \Lambda^q w \Lambda^q w dx.
\end{aligned}
\tag{3.31}
$$

It follows from the Schwarz inequality that

$$
\begin{aligned}
&\frac{d}{dt}\int\left[(1-\varepsilon)(\Lambda^q w)^2+\varepsilon(\Lambda^q w_x)^2\right]dx\\
&\quad\le c\Big\{\|\Lambda^q w\|_{L^2}\left[(\delta-\varepsilon)(\|\Lambda^q u_{\delta t}\|_{L^2}+\|\Lambda^q u_{\delta xxt}\|_{L^2})+\varepsilon\left\|\Lambda^{q-2}w_t\right\|_{L^2}+(\delta-\varepsilon)\left\|\Lambda^{q-2}u_{\delta t}\right\|_{L^2}\right]\\
&\qquad+|\lambda|\int_R(\Lambda^q w)^2\,dx+\left|\int_R \Lambda^q w\Lambda^q\left(u_\varepsilon^{m+2}-u_\delta^{m+2}\right)_x dx\right|\\
&\qquad\left|\int \Lambda^q w\Lambda^{q-2}\left(u_\varepsilon^{m+1}-u_\delta^{m+1}\right)_x dx\right|+\left|\int \Lambda^q w\Lambda^{q-2}\left(u_\varepsilon^{m+2}-u_\delta^{m+2}\right)_x dx\right|\\
&\qquad+\left|\int_R \Lambda^q w\Lambda^{q-2}\left[\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w\right]_x dx\right|+\left|\int_R \Lambda^q w\Lambda^{q-2}\left[\partial_x\left(u_\varepsilon^{m+1}-u_\delta^{m+1}\right)\partial_x u_\delta\right]_x dx\right|\\
&\qquad+\left|\int_R \Lambda^q w\Lambda^{q-2}\left[u_\varepsilon^m u_{\varepsilon x}u_{\varepsilon xx}-u_\delta^m u_{\delta x}u_{\delta xx}\right]dx\right|\Big\}.
\end{aligned}
\tag{3.32}
$$

Using the first inequality in Lemma 3.9, we have

$$
\begin{aligned}
\left|\int_R \Lambda^q w\Lambda^q\left(u_\varepsilon^{m+2}-u_\delta^{m+2}\right)_x dx\right|&=\left|\int_R \Lambda^q w\Lambda^q(w g_{m+1})_x dx\right|\\
&\le c\|w\|_{H^q}^2\left\|g_{m+1}\right\|_{H^{q+1}},
\end{aligned}
\tag{3.33}
$$

where $g_{m+1}=\sum_{j=0}^{m+1}u_\varepsilon^{m+1-j}u_\delta^j$. For the last three terms in (3.32), using Lemmas 3.4 and 3.12, $1/2<q<\min\{1,s-1\}$, $s>3/2$, the algebra property of H^{s_0} with $s_0>1/2$, and (3.23), we have

$$\left|\int_R \Lambda^q w \Lambda^{q-2}\left(\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w\right)_x dx\right| \le c\|w\|_{H^q}^2 \|u_\varepsilon\|_{H^s}^{m+1}, \tag{3.34}$$

$$\begin{aligned}
&\left|\int_R \Lambda^q w \Lambda^{q-2}\left(\partial_x\left(u_\varepsilon^{m+1}-u_\delta^{m+1}\right)\partial_x u_\delta\right)_x dx\right| \\
&\quad\le c\|w\|_{H^q}\|u_\delta\|_{H^s}\left\|u_\varepsilon^{m+1}-u_\delta^{m+1}\right\|_{H^q} \\
&\quad\le c\|w\|_{H^q}^2\|u_\delta\|_{H^s},
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
&\left|\int_R \Lambda^q w \Lambda^{q-2}\left[u_\varepsilon^m u_{\varepsilon x} u_{\varepsilon xx} - u_\delta^m u_{\delta x} u_{\delta xx}\right] dx\right| \\
&\quad\le c\|w\|_{H^q}\left\|\left(u_\varepsilon^m - u_\delta^m\right)\left(u_{\varepsilon x}^2\right)_x + u_\delta^m\left[u_{\varepsilon x}^2 - u_{\delta x}^2\right]_x\right\|_{H^{q-2}} \\
&\quad\le c\|w\|_{H^q}\left(\left\|\left(u_\varepsilon^m - u_\delta^m\right)\left(u_{\varepsilon x}^2\right)_x\right\|_{H^{q-1}} + \left\|u_\delta^m\left[u_{\varepsilon x}^2 - u_{\delta x}^2\right]_x\right\|_{H^{q-2}}\right) \\
&\quad\le c\|w\|_{H^q}\left(\left\|u_\varepsilon^m - u_\delta^m\right\|_{H^q}\left\|\left(u_{\varepsilon x}^2\right)_x\right\|_{H^{q-1}} + \left\|u_\delta^m\right\|_{H^s}\left\|\left[u_{\varepsilon x}^2 - u_{\delta x}^2\right]_x\right\|_{H^{q-2}}\right) \\
&\quad\le c\|w\|_{H^q}\left(\|w\|_{H^q}\left\|g_{m-1}\right\|_{H^q}\|u\|_{H^s}^2 + \left\|u_\delta^m\right\|_{H^s}\|u_{\varepsilon x}+u_{\delta x}\|_{H^q}\|w\|_{H^q}\right) \\
&\quad\le c\|w\|_{H^q}^2\left(\left\|g_{m-1}\right\|_{H^q}\|u\|_{H^s}^2 + \left\|u_\delta^m\right\|_{H^s}\|u_{\varepsilon x}+u_{\delta x}\|_{H^q}\right).
\end{aligned} \tag{3.36}$$

Using (3.26), we derive that the inequality

$$\begin{aligned}
\left|\int_R \Lambda^q w \Lambda^{q-2}\left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right)_x dx\right| &= \left|\int_R \Lambda^q w \Lambda^{q-2}\left(w g_{m+1}\right)_x dx\right| \\
&\le c\left\|g_{m+1}\right\|_{H^q}\|w\|_{H^q}^2
\end{aligned} \tag{3.37}$$

holds for some constant c, where $g_{m+1} = \sum_{j=0}^{m+1} u_\varepsilon^{m+1-j} u_\delta^j$. Using the algebra property of H^q with $q > 1/2$, $q+1 < s$ and Lemma 3.11, we have $\|g_m\|_{H^{q+1}} \le c$ for $t \in (0, \widetilde{T}]$. Then, it follows from (3.28) and (3.33)–(3.37) that there is a constant c depending on $\widetilde{T}$ such that the estimate

$$\frac{d}{dt}\int_R \left[(1-\varepsilon)(\Lambda^q w)^2 + \varepsilon(\Lambda^q w_x)^2\right] dx \le c\left(\delta^\gamma \|w\|_{H^q} + \|w\|_{H^q}^2\right) \tag{3.38}$$

holds for any $t \in [0, \widetilde{T})$, where $\gamma = 1$ if $s \geq 3 + q$ and $\gamma = (1 + s - q)/4$ if $s < 3 + q$. Integrating (3.38) with respect to t, one obtains the estimate

$$\begin{aligned}
\frac{1}{2}\|w\|_{H^q}^2 &= \frac{1}{2}\int_R (\Lambda^q w)^2 dx \\
&\leq \int_R \left[(1-\varepsilon)(\Lambda^q w)^2 + \varepsilon(\Lambda^q w)^2\right] dx \\
&\leq \int_R \left[(\Lambda^q w_0)^2 + \varepsilon(\Lambda^q w_{0x})^2\right] dx + c\int_0^t \left(\delta^\gamma \|w\|_{H^q} + \|w\|_{H^q}^2\right) d\tau.
\end{aligned} \tag{3.39}$$

Applying the Gronwall inequality and using (3.20) and (3.22) yield

$$\|u\|_{H^q} \leq c\delta^{(s-q)/4} e^{ct} + \delta^\gamma \left(e^{ct} - 1\right) \tag{3.40}$$

for any $t \in [0, \widetilde{T})$.

Multiplying both sides of (3.29) by $\Lambda^s w \Lambda^s$ and integrating the resultant equation with respect to x, one obtains

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_R \left[(1-\varepsilon)(\Lambda^s w)^2 + \varepsilon(\Lambda^s w_x)^2\right] dx \\
&\quad = (\varepsilon - \delta)\int_R (\Lambda^s w)[(\Lambda^s u_{\delta t}) + (\Lambda^s u_{\delta xxt})] dx - \varepsilon\int_R \Lambda^s w \Lambda^{s-2} w_t dx \\
&\qquad + (\delta - \varepsilon)\int_R \Lambda^s w \Lambda^{s-2} u_{\delta t} dx - \frac{k}{m+1}\int_R (\Lambda^s w)\Lambda^s \left(u_\varepsilon^{m+1} - u_\delta^{m+1}\right)_x dx \\
&\qquad - \frac{1}{m+2}\int_R (\Lambda^s w)\Lambda^s \left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right)_x dx - \int_R \Lambda^s w \Lambda^{s-2}\left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right)_x dx \\
&\qquad - \int_R \Lambda^s w \Lambda^{s-2}\left[\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w\right]_x dx - \int_R \Lambda^s w \Lambda^{s-2}\left[\partial_x\left(u_\varepsilon^{m+1} - u_\delta^{m+1}\right)\partial_x u_\delta\right]_x dx \\
&\qquad + \int_R \Lambda^s w \Lambda^{s-2}\left[u_\varepsilon^m u_{\varepsilon x} u_{\varepsilon xx} - u_\delta^m u_{\delta x} u_{\delta xx}\right] dx + \lambda\int_R (\Lambda^s w)^2 dx.
\end{aligned} \tag{3.41}$$

From Lemma 3.12, we have

$$\left|\int_R \Lambda^s w \Lambda^{s-2}\left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right)_x dx\right| \leq c_3 \left\|g_{m+1}\right\|_{H^s} \|w\|_{H^s}^2. \tag{3.42}$$

From Lemma 3.10, it holds that

$$\left|\int_R \Lambda^s w \Lambda^s \left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right)_x dx\right| \leq c\left(\|w\|_{H^s}\|w\|_{H^q}\|u_\delta\|_{H^{s+1}} + \|w\|_{H^s}^2\right). \tag{3.43}$$

Using the Cauchy-Schwartz inequality and the algebra property of H^{s_0} with $s_0 > 1/2$, for $s > 3/2$, we have

$$\begin{aligned}
&\left|\int_R \Lambda^s w \Lambda^{s-2}\left[\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w\right]_x dx\right| \\
&\quad = \left|\int_R \Lambda^q w \Lambda^{s-2}\left[\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w\right]_x dx\right| \\
&\quad \le c\|\Lambda^s w\|_{L^2}\left\|\Lambda^{s-2}\left[\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w\right]_x\right\|_{L^2} \\
&\quad \le c\|w\|_{H^q}\left\|\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w\right\|_{H^{s-1}} \\
&\quad \le c\left\|u_{\varepsilon}{}^{m+1}_{H^s}\right\|\|w\|_{H^s}^2,
\end{aligned} \tag{3.44}$$

$$\begin{aligned}
&\left|\int_R \Lambda^s w \Lambda^{s-2}\left[\partial_x\left(u_\varepsilon^{m+1} - u_\delta^{m+1}\right)\partial_x u_\delta\right]_x dx\right| \\
&\quad \le c\|w\|_{H^s}\left\|\Lambda^{s-2}\left[\partial_x\left(u_\varepsilon^{m+1} - u_\delta^{m+1}\right)\partial_x u_\delta\right]_x\right\|_{L^2} \\
&\quad \le c\|u_\delta\|_{H^s}\|g_m\|_{H^s}\|w\|_{H^s}^2,
\end{aligned} \tag{3.45}$$

$$\begin{aligned}
&\left|\int_R \Lambda^s w \Lambda^{s-2}\left[u_\varepsilon^m u_{\varepsilon x} u_{\varepsilon xx} - u_\delta^m u_{\delta x} u_{\delta xx}\right] dx\right| \\
&\quad \le c\|w\|_{H^s}\left(\left\|\left(u_\varepsilon^m - u_\delta^m\right)\left(u_{\varepsilon x}^2\right)_x\right\|_{H^{s-2}} + \left\|u_\delta^m\left[u_{\varepsilon x}^2 - u_{\delta x}^2\right]_x\right\|_{H^{s-2}}\right) \\
&\quad \le c\|w\|_{H^s}\left(\|u_\varepsilon^m - u_\delta^m\|_{H^s}\left\|\left(u_{\varepsilon x}^2\right)_x\right\|_{H^{s-2}} + \|u_\delta^m\|_{H^s}\left\|\left[u_{\varepsilon x}^2 - u_{\delta x}^2\right]_x\right\|_{H^{s-2}}\right) \\
&\quad \le c\|w\|_{H^s}^2,
\end{aligned} \tag{3.46}$$

in which we have used Lemma 3.4 and the bounded property of $\|u_\varepsilon\|_{H^s}$ and $\|u_\delta\|_{H^s}$ (see Remark 3.7). It follows from (3.41)–(3.46) and the inequalities (3.28) and (3.40) that there exists a constant c depending on m such that

$$\begin{aligned}
&\frac{d}{dt}\int_R\left[(1-\varepsilon)(\Lambda^s w)^2 + \varepsilon(\Lambda^s w_x)^2\right] dx \\
&\quad \le 2\delta\left(\|u_{\delta t}\|_{H^s} + \|u_{\delta xxt}\|_{H^s} + \left\|\Lambda^{s-2} w_t\right\|_{L^2} + \left\|\Lambda^{s-2} u_{\delta t}\right\|\right)\|w\|_{H^s} \\
&\qquad + c\left(\|w\|_{H^s}^2 + \|w\|_{H^q}\|w\|_{H^s}\|u_\delta\|_{H^{s+1}}\right) \\
&\quad \le c\left(\delta^{\gamma_1}\|w\|_{H^s} + \|w\|_{H^s}^2\right),
\end{aligned} \tag{3.47}$$

where $\gamma_1 = \min(1/4, (s-q-1)/4) > 0$. Integrating (3.47) with respect to t leads to the estimate

$$\begin{aligned}\frac{1}{2}\|w\|_{H^s}^2 &\le \int_R \left[(1-\varepsilon)(\Lambda^s w)^2 + \varepsilon(\Lambda^s w_x)^2\right]dx \\ &\le \int_R \left[(\Lambda^s w_0)^2 + \varepsilon(\Lambda^s w_{0x})^2\right]dx + c\int_0^t \left(\delta^{\gamma_1}\|w\|_{H^s} + \|w\|_{H^s}^2\right)d\tau.\end{aligned} \tag{3.48}$$

It follows from the Gronwall inequality and (3.48) that

$$\begin{aligned}\|w\|_{H^s} &\le \left(2\int_R \left[(\Lambda^s w_0)^2 + \varepsilon(\Lambda^s w_{0x})^2\right]dx\right)^{1/2} e^{ct} + \delta^{\gamma_1}\left(e^{ct}-1\right) \\ &\le c_1\left(\|w_0\|_{H^s} + \delta^{3/4}\right)e^{ct} + \delta^{\gamma_1}\left(e^{ct}-1\right),\end{aligned} \tag{3.49}$$

where c_1 is independent of ε and δ.

Then, (3.22) and the above inequality show that

$$\|w\|_{H^s} \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0, \quad \delta \longrightarrow 0. \tag{3.50}$$

Next, we consider the convergence of the sequence $\{u_{\varepsilon t}\}$. Multiplying both sides of (3.29) by $\Lambda^{s-1}w_t\Lambda^{s-1}$ and integrating the resultant equation with respect to x, we obtain

$$\begin{aligned}&(1-\varepsilon)\|w_t\|_{H^{s-1}}^2 + \frac{1}{m+2}\int_R \left(\Lambda^{s-1}w_t\right)\Lambda^{s-1}\left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right)_x dx \\ &\quad + \int_R \left[-\varepsilon\left(\Lambda^{s-1}w_t\right)\left(\Lambda^{s-1}w_{xxt}\right) + (\delta-\varepsilon)\left(\Lambda^{s-1}w_t\right)\Lambda^{s-1}(u_{\delta t} + u_{\delta xxt})\right]dx \\ &= \int_R \left(\Lambda^{s-1}w_t\right)\Lambda^{s-3}\Bigg[-\varepsilon w_t + (\delta-\varepsilon)u_{\delta t} - \frac{k}{m+1}\partial_x\left(u_\varepsilon^{m+1} - u_\delta^{m+1}\right) - \partial_x\left(u_\varepsilon^{m+2} - u_\delta^{m+2}\right) \\ &\qquad - \partial_x\left[\partial_x\left(u_\varepsilon^{m+1}\right)\partial_x w + \partial_x\left(u_\varepsilon^{m+1} - u_\varepsilon^{m+1}\right)\partial_x u_\delta\right] \\ &\qquad + \left[u_\varepsilon^m u_{\varepsilon x}u_{\varepsilon xx} - u_\delta^m u_{\delta x}u_{\delta xx}\right]\Bigg]dx + \lambda\int_R \Lambda^{s-1}w_t\Lambda^{s-1}w\,dx.\end{aligned} \tag{3.51}$$

It follows from inequalities (3.28) and the Schwartz inequality that there is a constant c depending on $\widetilde{T}$ and m such that

$$(1-\varepsilon)\|w_t\|_{H^{s-1}}^2 \le c\left(\delta^{1/2} + \|w\|_{H^s} + \|w\|_{s-1}\right)\|w_t\|_{H^{s-1}} + \varepsilon\|w_t\|_{H^{s-1}}^2 \tag{3.52}$$

Hence

$$\begin{aligned}\frac{1}{2}\|w_t\|_{H^{s-1}}^2 &\le (1-2\varepsilon)\|w_t\|_{H^{s-1}}^2 \\ &\le c\left(\delta^{1/2}+\|w\|_{H^s}+\|w\|_{H^{s-1}}\right)\|w_t\|_{H^{s-1}},\end{aligned} \tag{3.53}$$

which results in

$$\frac{1}{2}\|w_t\|_{H^{s-1}} \le c\left(\delta^{1/2}+\|w\|_{H^s}+\|w\|_{H^{s-1}}\right). \tag{3.54}$$

It follows from (3.40) and (3.50) that $w_t \to 0$ as ε, $\delta \to 0$ in the H^{s-1} norm. This implies that u_ε is a Cauchy sequence in the spaces $C([0,T);H^s(R))$ and $C([0,T);H^{s-1}(R))$, respectively. The proof is completed. □

4. Proof of the Main Result

We consider the problem

$$\begin{gathered}(1-\varepsilon)u_t-\varepsilon u_{txx}=\left(1-\partial_x^2\right)^{-1}\Bigg[-\frac{k}{m+1}\left(u^{m+1}\right)_x-\frac{m+3}{m+2}\left(u^{m+2}\right)_x+\frac{1}{m+2}\partial_x^3\left(u^{m+2}\right) \\ -(m+1)\partial_x\left(u^m u_x^2\right)+u^m u_x u_{xx}\Bigg]+\lambda u, \\ u(0,x)=u_{\varepsilon 0}(x).\end{gathered} \tag{4.1}$$

Letting $u(t,x)$ be the limit of the sequence u_ε and taking the limit in problem (4.1) as $\varepsilon \to 0$, from Lemma 3.13, we know that u is a solution of the problem

$$\begin{gathered}u_t=\left(1-\partial_x^2\right)^{-1}\Bigg[-\frac{k}{m+1}\left(u^{m+1}\right)_x-\frac{m+3}{m+2}\left(u^{m+2}\right)_x+\frac{1}{m+2}\partial_x^3\left(u^{m+2}\right) \\ -(m+1)\partial_x\left(u^m u_x^2\right)+u^m u_x u_{xx}\Bigg]+\lambda u, \\ u(0,x)=u_0(x),\end{gathered} \tag{4.2}$$

and hence u is a solution of problem (4.2) in the sense of distribution. In particular, if $s \ge 4$, u is also a classical solution. Let u and v be two solutions of (4.2) corresponding to the same initial value u_0 such that $u, v \in C([0,T); H^s(R))$. Then, $w = u - v$ satisfies the Cauchy problem

$$\begin{aligned} w_t &= \left(1-\partial_x^2\right)^{-1}\left\{\partial_x\left[-\frac{k}{m+1}wg_m - \frac{m+3}{m+2}wg_{m+1} + \frac{1}{m+2}\partial_x^2(wg_{m+1})\right.\right. \\ &\quad \left.\left. -\partial_x\left(u^{m+1}\right)\partial_x w - \partial_x\left(u^{m+1}-v^{m+1}\right)\partial_x v\right] + u^m u_x u_{xx} - v^m v_x v_{xx}\right\} + \lambda w, \\ w(0,x) &= 0. \end{aligned} \tag{4.3}$$

For any $1/2 < q < \min\{1, s-1\}$, applying the operator $\Lambda^q w \Lambda^q$ to both sides of (4.3) and integrating the resultant equation with respect to x, we obtain the equality

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|_{H^q}^2 &= \int_R (\Lambda^q w)\Lambda^{q-2}\left\{\partial_x\left[-\frac{k}{m+1}wg_m - \frac{m+3}{m+2}wg_{m+1} + \frac{1}{m+2}\partial_x^2(wg_{m+1})\right.\right. \\ &\quad \left. -\partial_x\left(u^{m+1}\right)\partial_x w - \partial_x\left(u^{m+1}-v^{m+1}\right)\partial_x v\right] + u^m u_x u_{xx} \\ &\quad \left. -v^m v_x v_{xx}\right\} dx + |\lambda|\|w\|_{H^q}^2. \end{aligned} \tag{4.4}$$

By the similar estimates presented in Lemma 3.13, we have

$$\frac{d}{dt}\|w\|_{H^q}^2 \le \tilde{c}\|w\|_{H^q}^2. \tag{4.5}$$

Using the Gronwall inequality leads to the conclusion that

$$\|w\|_{H^q} \le 0 \times e^{\tilde{c}t} = 0 \tag{4.6}$$

for $t \in [0, \tilde{T})$. This completes the proof.

Acknowledgments

Thanks are given to the referees whose suggestions were very helpful in improving the paper. This work is supported by the Fundamental Research Funds for the Central Universities (JBK120504).

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