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Research Article

Bounded Oscillation of a Forced Nonlinear Neutral Differential Equation

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This paper is concerned with the nth-order forced nonlinear neutral differential equation $[x(t)-p(t)x(\tau(t))]^{(n)}+\sum_{i=1}^{m}q_i(t)f_i(x(\sigma_{i1}(t)),x(\sigma_{i2}(t)),\dots,x(\sigma_{ik_i}(t)))=g(t)$, $t\geq t_0$. Some necessary and sufficient conditions for the oscillation of bounded solutions and several sufficient conditions for the existence of uncountably many bounded positive and negative solutions of the above equation are established. The results obtained in this paper improve and extend essentially some known results in the literature. Five interesting examples that point out the importance of our results are also included.

1. Introduction

Consider the following nth-order forced nonlinear neutral differential equation:

$$\left[x(t)-p(t)x(\tau(t))\right]^{(n)}+\sum_{i=1}^{m}q_i(t)f_i(x(\sigma_{i1}(t)),x(\sigma_{i2}(t)),\dots,x(\sigma_{ik_i}(t)))=g(t),\quad t\geq t_0,\tag{1.1}$$

where $t_0\in\mathbb{R}$ and $n,m,k_i\in\mathbb{N}$ are constants for $1\leq i\leq m$. In what follows, we assume that

$(A1)$ $p,g,\tau,\sigma_{ij}\in C([t_0,+\infty),\mathbb{R})$ and $q_i\in C([t_0,+\infty),\mathbb{R}^+)$ satisfy that

$$\lim_{t\to+\infty}\tau(t)=\lim_{t\to+\infty}\sigma_{ij}(t)=+\infty,\quad 1\leq j\leq k_i,\ 1\leq i\leq m,\tag{1.2}$$

and there exists $1 \le i_0 \le m$ such that q_{i_0} is positive eventually:

$(A2)$ τ is strictly increasing and $\tau(t) < t$ in $[t_0, +\infty)$;

$(A3)$ $f_i \in C(\mathbb{R}^{k_i}, \mathbb{R})$ satisfies that

$$\begin{aligned} &f_i(u_1, u_2, \ldots, u_{k_i}) > 0, \quad \forall (u_1, u_2, \ldots, u_{k_i}) \in (\mathbb{R}^+ \setminus \{0\})^{k_i}, \\ &f_i(u_1, u_2, \ldots, u_{k_i}) < 0, \quad \forall (u_1, u_2, \ldots, u_{k_i}) \in (\mathbb{R}_- \setminus \{0\})^{k_i} \end{aligned} \tag{1.3}$$

for $1 \le i \le m$.

During the last decades, the oscillation criteria and the existence results of nonoscillatory solutions for various linear and nonlinear differential equations have been studied extensively, for example, see [1–28] and the references cited therein. In particular, Zhang and Yan [25] obtained some sufficient conditions for the oscillation of the first-order linear neutral delay differential equation with positive and negative coefficients:

$$\left[x(t) - p(t)x(t-\tau)\right]' + q(t)x(t-\sigma) - r(t)x(t-\delta) = 0, \quad t \ge t_0, \tag{1.4}$$

where $p, q, r \in C([t_0, +\infty), \mathbb{R}^+)$, $\tau > 0$, and $\sigma \ge \delta \ge 0$. Das and Misra [7] studied the nonhomogeneous neutral delay differential equation:

$$[x(t) - cx(t-\tau)]' + q(t)f(x(t-\sigma)) = g(t), \quad t \ge t_0, \tag{1.5}$$

where $q, g \in C([T, +\infty), \mathbb{R}^+ \setminus \{0\})$, $\sigma > 0$, $\tau > 0$, $c \in [0,1)$, $f : \mathbb{R} \to \mathbb{R}$, $tf(t) > 0$ for $t \neq 0$, f is nondecreasing, Lipschitzian, and satisfies $\int_0^k (1/f(t))dt < +\infty$ for every $k > 0$, and they obtained a necessary and sufficient condition for the solutions of (1.5) to be oscillatory or tend to zero asymptotically. Parhi and Rath [18] extended Das and Misra's result to the following forced first-order neutral differential equation with variable coefficients:

$$\left[x(t) - p(t)x(t-\tau)\right]' + q(t)f(x(t-\sigma)) = g(t), \quad t \ge 0, \tag{1.6}$$

where $p \in C(\mathbb{R}^+, \mathbb{R})$, and they got necessary and sufficient conditions which ensures every solution of (1.6) is oscillatory or tends to zero or to $\pm\infty$ as $t \to +\infty$. By using Banach's fixed point theorem, Zhang et al. [24] proved the existence of a nonoscillatory solution for the first-order linear neutral delay differential equation:

$$\left[x(t) + p(t)x(t-\tau)\right]' + \sum_{i=1}^{n} f_i(t)x(t-\sigma_i) = 0, \quad t \ge t_0, \tag{1.7}$$

where $p \in C([t_0, +\infty), \mathbb{R})$, $\tau > 0$, $\sigma_i \in \mathbb{R}^+$, and $f_i \in C([t_0, +\infty), \mathbb{R})$ for $1 \le i \le m$. Çakmak and Tiryaki [6] showed several sufficient conditions for the oscillation of the forced second-order nonlinear differential equations with delayed argument in the form:

$$x''(t) + p(t)f(x(\alpha(t))) = g(t), \quad t \ge t_0 \ge 0, \tag{1.8}$$

where $p, \alpha, g \in C([t_0, +\infty), \mathbb{R})$, $\alpha(t) \le t$, $\lim_{t\to+\infty}\alpha(t) = +\infty$, and $f \in C(\mathbb{R}, \mathbb{R})$. Travis [20] investigated the oscillatory behavior of the second-order differential equation with functional argument:

$$x''(t) + p(t)f(x(t), x(\alpha(t))) = 0, \quad t \ge t_0, \tag{1.9}$$

where $p, \alpha \in C([t_0, +\infty), \mathbb{R})$ and $f \in C(\mathbb{R}^2, \mathbb{R})$ satisfies that $f(s,t)$ has the same sign of s and t when they have the same sign. Lin [12] got some sufficient conditions for oscillation and nonoscillation of the second order nonlinear neutral differential equation:

$$\left[x(t) - p(t)x(t-\tau)\right]'' + q(t)f(x(t-\sigma)) = 0, \quad t \ge 0, \tag{1.10}$$

where $p, q \in C(\mathbb{R}^+, \mathbb{R})$, $\overline{p} \in [0,1)$ with $0 \le p(t) \le \overline{p}$ eventually, $f \in C(\mathbb{R}, \mathbb{R})$, f is nondecreasing and $tf(t) > 0$ for $t \neq 0$. Kulenović and Hadžiomerspahić [9] deduced the existence of a nonoscillatory solution for the neutral delay differential equation of second order with positive and negative coefficients:

$$[x(t) + cx(t-\tau)]'' + q_1(t)x(t-\sigma_1) - q_2(t)x(t-\sigma_2) = 0, \quad t \ge t_0, \tag{1.11}$$

where $c \neq \pm 1$, $\tau > 0$, $\sigma_i \in \mathbb{R}^+$, $q_i \in C([t_0, +\infty), \mathbb{R}^+)$, and $\int_{t_0}^{+\infty} q_i(t)dt < +\infty$ for $i \in \{1,2\}$. Utilizing the fixed point theorems due to Banach, Schauder and Krasnoselskii, and Zhou and Zhang [27], and Zhou et al. [28] established some sufficient conditions for the existence of a nonoscillatory solution of the following higher-order neutral functional differential equations:

$$\begin{gathered}
[x(t) + cx(t-\tau)]^{(n)} + (-1)^{n+1}[P(t)x(t-\sigma) - Q(t)x(t-\delta)] = 0, \quad t \ge t_0, \\
\left[x(t) + p(t)x(t-\tau)\right]^{(n)} + \sum_{i=1}^{m} q_i(t)f_i(x(t-\sigma_i)) = g(t), \quad t \ge t_0,
\end{gathered} \tag{1.12}$$

where $c \in \mathbb{R} \setminus \{\pm 1\}$, $\tau, \sigma, \delta, \sigma_i \in \mathbb{R}^+$, $P, Q \in C([t_0, +\infty), \mathbb{R}^+)$, and $p, g, f_i \in C([t_0, +\infty), \mathbb{R})$ for $1 \le i \le m$. Li et al. [11] investigated the existence of an unbounded positive solution, bounded oscillation, and nonoscillation criteria for the following even-order neutral delay differential equation with unstable type:

$$\left[x(t) - p(t)x(t-\tau)\right]^{(n)} - q(t)|x(t-\sigma)|^{\alpha-1}x(t-\sigma) = 0, \quad t \ge t_0, \tag{1.13}$$

where $\tau > 0$, $\sigma > 0$, $\alpha \ge 1$, and $p, q \in C([t_0, +\infty), \mathbb{R}^+)$. Zhang and Yan [22] obtained some sufficient conditions for oscillation of all solutions of the even-order neutral differential equation with variable coefficients and delays:

$$\left[x(t) + p(t)x(\tau(t))\right]^{(n)} + q(t)x(\sigma(t)) = 0, \quad t \ge t_0, \tag{1.14}$$

where n is even, $p, q, \tau, \sigma \in C([t_0, +\infty), \mathbb{R}^+)$, $p(t) < 1$, $\tau(t) \le t$ and $\sigma(t) \le t$ for $t \in [t_0, +\infty)$, and $\lim_{t\to+\infty}\tau(t) = \lim_{t\to+\infty}\sigma(t) = +\infty$. Yilmaz and Zafer [21] discussed sufficient conditions for the existence of positive solutions and the oscillation of bounded solutions of the nth-order neutral type differential equations:

$$\begin{gathered}[x(t)+cx(\tau(t))]^{(n)}+q(t)f(x(\sigma(t)))=0, \quad t\geq t_0,\\ \left[x(t)+p(t)x(\tau(t))\right]^{(n)}+q(t)f(x(\sigma(t)))=g(t), \quad t\geq t_0,\end{gathered} \tag{1.15}$$

where $c\in\mathbb{R}\setminus\{\pm1\}$, $\tau,\sigma\in C([t_0,+\infty),\mathbb{R}^+)$, $p,q,g\in C([t_0,+\infty),\mathbb{R})$, and $f\in C(\mathbb{R},\mathbb{R})$. Bolat and Akin [4, 5] got sufficient criteria for oscillatory behaviour of solutions for the higher-order neutral type nonlinear forced differential equations with oscillating coefficients:

$$\begin{gathered}\left[x(t)+p(t)x(\tau(t))\right]^{(n)}+\sum_{i=1}^{m}q_i(t)f_i(x(\sigma_i(t)))=0, \quad t\geq t_0,\\ \left[x(t)+p(t)x(\tau(t))\right]^{(n)}+\sum_{i=1}^{m}q_i(t)f_i(x(\sigma_i(t)))=g(t), \quad t\geq t_0,\end{gathered} \tag{1.16}$$

where $n\in\mathbb{N}\setminus\{1\}$, $m\in\mathbb{N}$, $p,f_i,g,\tau,\sigma_i\in C([t_0,+\infty),\mathbb{R})$, f_i is nondecreasing and $uf_i(u)>0$ for $u\neq0$, $\sigma_i\in C^1([t_0,+\infty),\mathbb{R})$, $\sigma_i'(t)>0$, $\sigma_i(t)\leq t$ for $t\in[t_0,+\infty)$, $\lim_{t\to+\infty}\tau(t)=\lim_{t\to+\infty}\sigma_i(t)=+\infty$ for $1\leq i\leq m$, and p and g are oscillating functions. Zhou and Yu [26] attempted to extend the result of Bolat and Akin [4] and established a necessary and sufficient condition for the oscillation of bounded solutions of the higher-order nonlinear neutral forced differential equation of the form:

$$\left[x(t)-p(t)x(\tau(t))\right]^{(n)}+\sum_{i=1}^{m}q_i(t)f_i(x(\sigma_i(t)))=g(t), \quad t\geq t_0, \tag{1.17}$$

where $n\in\mathbb{N}\setminus\{1\}$, $m\in\mathbb{N}$, and

(C_1) $p,q_i,\tau,g\in C([t_0,+\infty),\mathbb{R})$ for $i=1,2,\ldots,m$ and $\lim_{t\to+\infty}\tau(t)=+\infty$;

(C_2) p and g are oscillating functions;

(C_3) $\sigma_i\in C([t_0,+\infty),\mathbb{R})$, $\sigma_i'(t)>0$, $\sigma_i(t)\leq t$ and $\lim_{t\to+\infty}\sigma_i(t)=+\infty$ for $i=1,2,\ldots,m$;

(C_4) $f_i\in C(\mathbb{R},\mathbb{R})$ is nondecreasing function, $uf_i(u)>0$ for $u\neq0$ and $i=1,2,\ldots,m$.

That is, they claimed the following result.

Theorem 1.1 (see [26, Theorem 2.1]). *Assume that*

(C_5) *there is an oscillating function* $r\in C([t_0,+\infty),\mathbb{R})$ *such that* $r^{(n)}(t)=g(t)$ *and* $\lim_{t\to+\infty}r(t)=0$;

(C_6) p *is an oscillating function and* $|p(t)|\leq p_0<1/2$;

(C_7) $q_i(t)\geq0$, $i=1,2,\ldots,m$.

Then, every bounded solution of (1.17) *either oscillates or tends to zero if and only if*

$$\int_{t_0}^{+\infty}s^{n-1}q_i(s)ds=+\infty, \quad i=1,2,\ldots,m. \tag{1.18}$$

We, unfortunately, point out that the necessary part in Theorem 1.1 is false, see Remark 4.2 and Example 4.7 below. It is clear that (1.1) includes (1.4)–(1.17) as special cases. To the best of our knowledge, there is no literature referred to the oscillation and existence of uncountably many bounded nonoscillatory solutions of (1.1). The aim of this paper is to establish the bounded oscillation and the existence of uncountably many bounded positive and negative solutions for (1.1) without the monotonicity of the nonlinear term f_i. Our results extend and improve substantially some known results in [4, 5, 9, 10, 20, 24, 26–28] and correct Theorem 2.1 in [26].

The paper is organized as follows. In Section 2, a few notation and lemmas are introduced and proved, respectively. In Section 3, by employing Krasnoselskii's fixed point theorem and some techniques, the existence of uncountably many bounded positive and negative solutions for (1.1) are given, and some necessary and sufficient conditions for all bounded solutions of (1.1) to be oscillatory or tend to zero as $t \to +\infty$ are provided. In Section 4, a number of examples which clarify advantages of our results are constructed.

2. Preliminaries

It is assumed throughout this paper that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R}_- = (-\infty, 0]$ and

$$\beta = \min\{t_0, \inf\{\tau(t), \sigma_{ij}(t) : t \in [t_0, +\infty), 1 \le j \le i_k, 1 \le i \le m\}\}. \tag{2.1}$$

By a solution of (1.1), we mean a function $x \in C([\beta, +\infty), \mathbb{R})$ for some $T \ge t_0 + \beta$, such that $x(t) - p(t)x(\tau(t))$ is n times continuously differentiable in $[T, +\infty)$ and such that (1.1) is satisfied for $t \ge T$. As is customary, a solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, it is nonoscillatory, that is, if it is eventually positive or eventually negative. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Let $BC([\beta, +\infty), \mathbb{R})$ stand for the Banach space of all bounded continuous functions in $[\beta, +\infty)$ with the norm $\|x\| = \sup_{t \ge \beta}|x(t)|$ for each $x \in BC([\beta, +\infty), \mathbb{R})$ and

$$A(N, M) = \{x \in BC([\beta, +\infty), \mathbb{R}) : N \le x(t) \le M,\ t \ge \beta\} \quad \text{for } M, N \in \mathbb{R} \text{ with } M > N. \tag{2.2}$$

It is easy to see that $A(N, M)$ is a bounded closed and convex subset of the Banach space $BC([\beta, +\infty), \mathbb{R})$.

Lemma 2.1. *Let $n \in \mathbb{N}$ and $x \in C^n([t_0, +\infty), \mathbb{R})$ be bounded. If $x^{(n)}(t) \le 0$ eventually, then*

(a) $\lim_{t \to +\infty} x(t)$ *exists and* $\lim_{t \to +\infty} x^{(i)}(t) = 0$ *for* $1 \le i \le n - 1$*; furthermore, there exists* $\theta = 0$ *for n odd and* $\theta = 1$ *for n even such that*

(b) $(-1)^{\theta+i} x^{(i)}(t) \ge 0$ *eventually for* $1 \le i \le n$*;*

(c) $(-1)^{\theta+i} x^{(i)}$ *is nonincreasing eventually for* $0 \le i \le n - 1$.

Proof. Now, we consider two possible cases below.

Case 1. Assume that $n = 1$. Let $\theta = 0$. Note that $x'(t) \le 0$ eventually. It follows that there exists a constant $t_1 > t_0$ satisfying $x'(t) \le 0$, for all $t \ge t_1$, which yields that x is nonincreasing in $[t_1, +\infty)$. Since x is bounded in $[t_0, +\infty)$, it follows that $\lim_{t \to +\infty} x(t)$ exists.

Case 2. Assume that $n \geq 2$. Notice that $\theta+n$ is odd. It follows that $(-1)^{\theta+n}x^{(n)}(t) \geq 0$ eventually, which implies that there exists a constant $t_1 > t_0$ satisfying

$$(-1)^{\theta+n}x^{(n)}(t) \geq 0, \quad \forall t \geq t_1, \tag{2.3}$$

which means that

$$(-1)^{\theta+n-1}x^{(n-1)}(t) \text{ is nonincreasing in } [t_1, +\infty). \tag{2.4}$$

Suppose that there exists a constant $t_2 \geq t_1$ satisfying $(-1)^{\theta+n-1}x^{(n-1)}(t_2) < 0$, which together with (2.4) gives that

$$(-1)^{\theta+n-1}x^{(n-1)}(t) \leq (-1)^{\theta+n-1}x^{(n-1)}(t_2) < 0, \quad \forall t \geq t_2, \tag{2.5}$$

which guarantees that $(-1)^{\theta+n-2}x^{(n-2)}(t)$ is increasing in $[t_2, +\infty)$ and

$$\begin{aligned}
&(-1)^{\theta+n-1}x^{(n-2)}(t) - (-1)^{\theta+n-1}x^{(n-2)}(t_2) \\
&\quad = \int_{t_2}^{t} (-1)^{\theta+n-1}x^{(n-1)}(s)ds \leq (-1)^{\theta+n-1}x^{(n-1)}(t_2)(t-t_2) \longrightarrow -\infty \quad \text{as } t \longrightarrow +\infty,
\end{aligned} \tag{2.6}$$

that is,

$$\lim_{t \to +\infty} x^{(n-2)}(t) = -\infty, \tag{2.7}$$

which means that

$$\lim_{t \to +\infty} x^{(n-3)}(t) = \lim_{t \to +\infty} x^{(n-4)}(t) = \cdots = \lim_{t \to +\infty} x'(t) = \lim_{t \to +\infty} x(t) = -\infty, \tag{2.8}$$

which contradicts the boundedness of x. Consequently, we have

$$(-1)^{\theta+n-1}x^{(n-1)}(t) \geq 0, \quad \forall t \geq t_1. \tag{2.9}$$

Combining (2.4) and (2.9), we conclude easily that there exists a constant $L \geq 0$ with

$$\lim_{t \to +\infty} (-1)^{\theta+n-1}x^{(n-1)}(t) = L. \tag{2.10}$$

Next, we claim that $L = 0$. Otherwise, there exists a constant $b > t_1$ satisfying

$$(-1)^{\theta+n-1}x^{(n-1)}(t) \geq \frac{L}{2} > 0, \quad \forall t \geq b, \tag{2.11}$$

which yields that

$$(-1)^{\theta+n-1}x^{(n-2)}(t) - (-1)^{\theta+n-1}x^{(n-2)}(b) = \int_b^t (-1)^{\theta+n-1}x^{(n-1)}(s)ds \geq \frac{L(t-b)}{2} \longrightarrow +\infty \quad \text{as } t \longrightarrow +\infty, \tag{2.12}$$

which gives that

$$\lim_{t\to+\infty} x^{(n-2)}(t) = +\infty, \tag{2.13}$$

which means that

$$\lim_{t\to+\infty} x^{(n-3)}(t) = \lim_{t\to+\infty} x^{(n-4)}(t) = \cdots = \lim_{t\to+\infty} x'(t) = \lim_{t\to+\infty} x(t) = +\infty, \tag{2.14}$$

which contradicts the boundedness of x in $[t_0, +\infty)$. Hence, $L = 0$, that is,

$$\lim_{t\to+\infty} x^{(n-1)}(t) = 0. \tag{2.15}$$

Repeating the proof of (2.3)–(2.15), we deduce similarly that

$$\begin{gathered}(-1)^{\theta+j}x^{(j)} \text{ is nonincreasing and nonnegative in } [t_1, +\infty), \\ \lim_{t\to+\infty} x^{(j)}(t) = 0, \quad 1 \leq j \leq n-1,\end{gathered} \tag{2.16}$$

which together with the boundedness of x implies that $(-1)^{\theta}x$ is nonincreasing in $[t_1, +\infty)$ and $\lim_{t\to+\infty}x(t)$ exists.

Thus, (2.3) and (2.16) yield (a)–(c). This completes the proof. □

Lemma 2.2. *Let $x, p, \tau, r, y \in C([t_0, +\infty), \mathbb{R})$ satisfy $(A2)$ and*

$$y(t) = x(t) - p(t)x(\tau(t)) - r(t), \quad \forall t \geq t_0; \tag{2.17}$$

$$x \text{ is bounded and } \lim_{t\to+\infty} \tau(t) = +\infty; \tag{2.18}$$

$$\lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} r(t) = 0, \qquad |p(t)| \geq p_0 > 1 \text{ eventually}, \tag{2.19}$$

where p_0 is a fixed constant. Then, $\lim_{t\to+\infty}x(t) = 0$.

Proof. Since τ is a strictly increasing continuous function, $\tau(t) < t$ in $[t_0, +\infty)$ and $\lim_{t\to+\infty}\tau(t) = +\infty$, it follows that the inverse function τ^{-1} of τ is also strictly increasing continuous, $\tau^{-1}(t) > t$ in $[\tau(t_0), +\infty)$ and $\lim_{j\to\infty}\tau^{-j}(t) = +\infty$, where $\tau^{-j} = \tau^{-(j-1)}(\tau^{-1})$ for all $j \in \mathbb{N}$. Equation (2.18) implies that there exists a constant $B > 0$ with

$$|x(t)| \leq B, \quad \forall t \geq t_0. \tag{2.20}$$

Using (2.18) and (2.19), we deduce that, for any $\varepsilon > 0$, there exist sufficiently large numbers $T > 1 + |t_0|$ and $K \in \mathbb{N}$ satisfying

$$\frac{B}{p_0^K} < \frac{\varepsilon}{4}, \qquad \max\{|y(t)|, |r(t)|\} < \frac{\varepsilon(p_0 - 1)}{4}, \qquad |p(t)| \geq p_0, \quad \forall t \geq T. \tag{2.21}$$

In view of (2.17), (2.20), and (2.21), we infer that for all $t \geq T$

$$\begin{aligned}
|x(t)| &= \frac{|x(\tau^{-1}(t)) - y(\tau^{-1}(t)) - r(\tau^{-1}(t))|}{|p(\tau^{-1}(t))|} \\
&\leq \frac{|x(\tau^{-1}(t))| + |y(\tau^{-1}(t))| + |r(\tau^{-1}(t))|}{|p(\tau^{-1}(t))|} \\
&< \frac{1}{p_0}\left|x\left(\tau^{-1}(t)\right)\right| + \frac{\varepsilon(p_0 - 1)}{2p_0} \\
&\leq \frac{1}{p_0}\left[\frac{1}{p_0}\left|x\left(\tau^{-2}(t)\right)\right| + \frac{\varepsilon(p_0 - 1)}{2p_0}\right] + \frac{\varepsilon(p_0 - 1)}{2p_0} \\
&= \frac{1}{p_0^2}\left|x\left(\tau^{-2}(t)\right)\right| + \frac{\varepsilon(p_0 - 1)}{2p_0}\left(1 + \frac{1}{p_0}\right) \\
&\leq \cdots \\
&\leq \frac{1}{p_0^K}\left|x\left(\tau^{-K}(t)\right)\right| + \frac{\varepsilon(p_0 - 1)}{2p_0}\left(1 + \frac{1}{p_0} + \cdots + \frac{1}{p_0^{K-1}}\right) \\
&\leq \frac{B}{p_0^K} + \frac{\varepsilon(p_0 - 1)}{2p_0} \cdot \frac{1}{1 - 1/p_0} \\
&< \varepsilon,
\end{aligned} \tag{2.22}$$

which gives that $\lim_{t \to +\infty} x(t) = 0$. This completes the proof. □

Lemma 2.3. *Let x, p, τ, r, and y be in $C([t_0, +\infty), \mathbb{R})$ satisfying $(A2)$, (2.17), (2.18), and*

$$\lim_{t \to +\infty} |y(t)| = d > 0, \qquad \lim_{t \to +\infty} r(t) = 0; \tag{2.23}$$

$$p_1 \geq |p(t)| \geq p_0 > 1 \text{ eventually}, \qquad p_0^2 > p_0 + p_1, \tag{2.24}$$

where d, p_0, and p_1 are constants. Then, there exists $L > 0$ such that $|x(t)| \geq L$ eventually.

Proof. Obviously, (2.20) holds. It follows from (2.18), (2.23), and (2.24) that for $\varepsilon = d[p_0(p_0 - 1) - p_1]/(p_0(p_0 - 1) + p_1) > 0$, there exist $K \in \mathbb{N}$ and $T > 1 + |t_0|$ satisfying

$$\frac{B}{p_0^K} < \frac{\varepsilon}{4p_1}, \quad d - \frac{\varepsilon}{4} < |y(t)| < d + \frac{\varepsilon}{4}, \quad |r(t)| < \frac{\varepsilon}{4p_0}, \quad p_1 \geq |p(t)| \geq p_0, \quad \forall t \geq T. \tag{2.25}$$

Put $L = d[p_0(p_0-1)-p_1]/2p_1p_0(p_0-1)$. In light of (2.17), we conclude that for each $t \ge T$

$$\begin{aligned}
x(t) &= \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{y(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} \\
&= \frac{1}{p(\tau^{-1}(t))}\left[\frac{x(\tau^{-2}(t))}{p(\tau^{-2}(t))} - \frac{y(\tau^{-2}(t))}{p(\tau^{-2}(t))} - \frac{r(\tau^{-2}(t))}{p(\tau^{-2}(t))}\right] - \frac{y(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} \\
&= \frac{x(\tau^{-2}(t))}{\prod_{i=1}^{2} p(\tau^{-i}(t))} - \sum_{j=1}^{2} \frac{y(\tau^{-j}(t))}{\prod_{i=1}^{j} p(\tau^{-i}(t))} - \sum_{j=1}^{2} \frac{r(\tau^{-j}(t))}{\prod_{i=1}^{j} p(\tau^{-i}(t))} \\
&= \cdots \\
&= \frac{x(\tau^{-K}(t))}{\prod_{i=1}^{K} p(\tau^{-i}(t))} - \sum_{j=1}^{K} \frac{y(\tau^{-j}(t))}{\prod_{i=1}^{j} p(\tau^{-i}(t))} - \sum_{j=1}^{K} \frac{r(\tau^{-j}(t))}{\prod_{i=1}^{j} p(\tau^{-i}(t))},
\end{aligned} \tag{2.26}$$

which together with (2.20) and (2.25) yields that for any $t \ge T$

$$\begin{aligned}
|x(t)| &\ge \frac{|y(\tau^{-1}(t))|}{|p(\tau^{-1}(t))|} - \frac{|x(\tau^{-K}(t))|}{\prod_{i=1}^{K} |p(\tau^{-i}(t))|} - \sum_{j=2}^{K} \frac{|y(\tau^{-j}(t))|}{\prod_{i=1}^{j} |p(\tau^{-i}(t))|} - \sum_{j=1}^{K} \frac{|r(\tau^{-j}(t))|}{\prod_{i=1}^{j} |p(\tau^{-i}(t))|} \\
&\ge \frac{d-\varepsilon/4}{p_1} - \frac{B}{p_0^K} - \left(d+\frac{\varepsilon}{4}\right)\sum_{j=2}^{K}\frac{1}{p_0^j} - \frac{\varepsilon}{4p_0}\sum_{j=1}^{K}\frac{1}{p_0^j} \\
&\ge \frac{d-\varepsilon/4}{p_1} - \frac{\varepsilon}{4p_1} - \left(d+\frac{\varepsilon}{4}\right)\cdot\frac{1/p_0^2}{1-1/p_0} - \frac{\varepsilon}{4p_0}\cdot\frac{1/p_0}{1-1/p_0} \\
&= \frac{d-\varepsilon/2}{p_1} - \frac{d+\varepsilon/2}{p_0(p_0-1)} = \frac{d[p_0(p_0-1)-p_1] - (\varepsilon/2)[p_0(p_0-1)+p_1]}{p_1p_0(p_0-1)} \\
&= L.
\end{aligned} \tag{2.27}$$

This completes the proof. □

Similar to the proof of Lemma 3.2 in [26], we have the following two lemmas.

Lemma 2.4. *Let x, p, τ, r, and y be in $C([t_0,+\infty),\mathbb{R})$ satisfying $(A2)$, (2.17), (2.18), and*

$$\lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} r(t) = 0; \tag{2.28}$$

$$|p(t)| \le p_0 < \frac{1}{2} \text{ eventually,} \tag{2.29}$$

where p_0 is a constant. Then, $\lim_{t\to+\infty} x(t) = 0$.

Lemma 2.5. *Let x, p, τ, r, and y be in $C([t_0,+\infty),\mathbb{R})$ satisfying $(A2)$,* (2.17), (2.18), (2.23), *and* (2.29). *Then, there exists $L>0$ such that $|x(t)|\geq L$ eventually.*

Lemma 2.6 (Krasnoselskii's fixed point theorem). *Let X be a Banach space, let Y be a nonempty bounded closed convex subset of X, and let f, g be mappings of Y into X such that $fx+gy\in Y$ for every pair $x,y\in Y$. If f is a contraction mapping and g is completely continuous, then the mapping $f+g$ has a fixed point in Y.*

3. Main Results

First, we use the Krasnoselskii's fixed point theorem to show the existence and multiplicity of bounded positive and negative solutions of (1.1).

Theorem 3.1. *Let $(A1)$, $(A2)$, and $(A3)$ hold. Assume that there exist $p_0,p_1\in\mathbb{R}^+\backslash\{0\}$, $r_0,r_1\in\mathbb{R}^+$, and $r\in C^n([t_0,+\infty),\mathbb{R})$ satisfying*

$$p_1\geq p(t)\geq p_0>1 \text{ eventually},\qquad p_0^2>p_0+p_1; \tag{3.1}$$

$$r^{(n)}(t)=g(t),\qquad -r_0\leq r(t)\leq r_1 \text{ eventually}; \tag{3.2}$$

$$\int_{t_0}^{+\infty}s^{n-1}\sum_{i=1}^{m}q_i(s)ds<+\infty. \tag{3.3}$$

Then, the following hold:

(a) *for arbitrarily positive constants M and N with*

$$(p_0-1)M>(p_1-1)N+\frac{p_1r_1}{p_0}+r_0, \tag{3.4}$$

equation (1.1) *has uncountably many bounded positive solutions $x\in A(N,M)$ with*

$$N\leq\liminf_{t\to+\infty}x(t)\leq\limsup_{t\to+\infty}x(t)\leq M; \tag{3.5}$$

(b) *for arbitrarily positive constants M and N with*

$$(p_0-1)N>(p_1-1)M+\frac{p_1r_0}{p_0}+r_1, \tag{3.6}$$

equation (1.1) *has uncountably many bounded negative solutions $x\in A(-N,-M)$ with*

$$-N\leq\liminf_{t\to+\infty}x(t)\leq\limsup_{t\to+\infty}x(t)\leq -M. \tag{3.7}$$

Proof. It follows from (3.1) and (3.2) that there exists an enough large constant T_0 with $\tau^{-1}(T_0) > 1 + |t_0| + |\beta|$ satisfying

$$p_0 \le p(t) \le p_1, \quad r^{(n)}(t) = g(t), \quad -r_0 \le r(t) \le r_1, \quad \forall t \ge T_0. \tag{3.8}$$

(a) Assume that M and N are arbitrary positive constants satisfying (3.4). Let $D \in ((p_1 - 1)N + (p_1 r_1/p_0)\ ,\ (p_0 - 1)M - r_0)$. First of all, we prove that there exist two mappings $F_D, G_D : A(N, M) \to BC([\beta, +\infty), \mathbb{R})$ and a constant $T_D > \tau^{-1}(T_0)$ such that $F_D + G_D$ has a fixed point $x \in A(N, M)$, which is also a bounded positive solution of (1.1) with $N \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le M$. Put

$$B = \max\left\{ \left| f_i(u_1, u_2, \dots, u_{k_i}) \right| : u_j \in [N, M], 1 \le j \le k_i, 1 \le i \le m \right\}. \tag{3.9}$$

In light of (3.3), (3.9), and $(A2)$, we infer that there exists a sufficiently large number $T_D > \tau^{-1}(T_0)$ satisfying

$$\frac{B}{p_0(n-1)!} \int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s) ds < \min\left\{ M - \frac{D + M + r_0}{p_0}, \frac{D + N}{p_1} - \frac{r_1}{p_0} - N \right\}. \tag{3.10}$$

Define two mappings $F_D, G_D : A(N, M) \to C([\beta, +\infty), \mathbb{R})$ by

$$(F_D x)(t) = \begin{cases} \dfrac{D}{p(\tau^{-1}(t))} + \dfrac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \dfrac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))}, & t \ge T_D \\ (F_D x)(T_D), & \beta \le t < T_D, \end{cases} \tag{3.11}$$

$$(G_D x)(t) = \begin{cases} \dfrac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\ \quad \times \displaystyle\int_{\tau^{-1}(t)}^{+\infty} \left(s - \tau^{-1}(t)\right)^{n-1} \\ \quad \times \displaystyle\sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \dots, x(\sigma_{ik_i}(s))) ds, & t \ge T_D, \\ (G_D x)(T_D), & \beta \le t < T_D, \end{cases} \tag{3.12}$$

for each $x \in A(N, M)$. In view of (3.1), (3.8), and (3.10)–(3.12), we conclude that for any $x, u \in A(N, M)$ and $t \geq T_D$

$$
\begin{aligned}
&|(F_D x)(t) - (F_D u)(t)| = \left| \frac{x(\tau^{-1}(t)) - u(\tau^{-1}(t))}{p(\tau^{-1}(t))} \right| \leq \frac{1}{p_0} \|x - u\|, \\
&(F_D x)(t) + (G_D u)(t) \\
&\quad = \frac{D}{p(\tau^{-1}(t))} + \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\
&\qquad \times \int_{\tau^{-1}(t)}^{+\infty} \left(s - \tau^{-1}(t)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \ldots, u(\sigma_{ik_i}(s))) ds \\
&\quad \leq \frac{D}{p_0} + \frac{M}{p_0} + \frac{r_0}{p_0} + \frac{B}{p_0(n-1)!} \int_{\tau^{-1}(t)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s) ds \\
&\quad < \frac{D + M + r_0}{p_0} + \min\left\{ M - \frac{D + M + r_0}{p_0}, \frac{D + N}{p_1} - \frac{r_1}{p_0} - N \right\} \\
&\quad \leq M, \\
&(F_D x)(t) + (G_D u)(t) \\
&\quad = \frac{D}{p(\tau^{-1}(t))} + \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\
&\qquad \times \int_{\tau^{-1}(t)}^{+\infty} \left(s - \tau^{-1}(t)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \ldots, u(\sigma_{ik_i}(s))) ds \\
&\quad \geq \frac{D}{p_1} + \frac{N}{p_1} - \frac{r_1}{p_0} - \frac{B}{p_0(n-1)!} \int_{\tau^{-1}(t)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s) ds \\
&\quad > \frac{D + N}{p_1} - \frac{r_1}{p_0} - \min\left\{ M - \frac{D + M + r_0}{p_0}, \frac{D + N}{p_1} - \frac{r_1}{p_0} - N \right\} \\
&\quad \geq N, \\
&|(G_D u)(t)| \\
&\quad = \left| \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \right. \\
&\qquad \left. \times \int_{\tau^{-1}(t)}^{+\infty} \left(s - \tau^{-1}(t)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \ldots, u(\sigma_{ik_i}(s))) ds \right| \\
&\quad \leq \frac{B}{p_0(n-1)!} \int_{\tau^{-1}(t)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s) ds \\
&\quad < \min\left\{ M - \frac{D + M + r_0}{p_0}, \frac{D + N}{p_1} - \frac{r_1}{p_0} - N \right\} \\
&\quad < M,
\end{aligned}
\tag{3.13}
$$

which ensures that

$$\|F_D x - F_D u\| = \sup_{t\ge T_D}|(F_D x)(t) - (F_D u)(t)| \le \frac{1}{p_0}\|x-u\|, \quad \forall x,u \in A(N,M), \tag{3.14}$$

$$F_D x + G_D u \in A(N,M), \quad \forall x,u \in A(N,M), \tag{3.15}$$

$$\|G_D u\| \le M, \quad \forall u \in A(N,M). \tag{3.16}$$

It follows from (3.11), (3.12), (3.15), and (3.16) that F_D and G_D map $A(N,M)$ into $BC([\beta,+\infty),\mathbb{R})$, respectively.

Now, we show that G_D is continuous in $A(N,M)$. Let $\{x_l\}_{l\in\mathbb{N}} \subset A(N,M)$ and $x \in A(N,M)$ with $\lim_{l\to\infty} x_l = x$, given $\varepsilon > 0$. It follows from the uniform continuity of f_i in $[N,M]^{k_i}$ for $1 \le i \le m$ and $\lim_{l\to\infty} x_l = x$ that there exist $\delta > 0$ and $K \in \mathbb{N}$ satisfying

$$\begin{gathered}
\left|f_i(u_{i1},u_{i2},\dots,u_{ik_i}) - f_i(v_{i1},v_{i2},\dots,v_{ik_i})\right| \\
< \frac{\varepsilon}{1+(1/p_0(n-1)!)\int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1}\sum_{i=1}^m q_i(s)ds}, \quad \forall u_{ij},v_{ij} \in [N,M], \\
\left|u_{ij}-v_{ij}\right| < \delta,\ 1 \le j \le k_i,\ 1 \le i \le m, \\
\|x_l - x\| < \delta, \quad \forall l \ge K.
\end{gathered} \tag{3.17}$$

In view of (3.8), (3.12), (3.17), we arrive at

$$\begin{aligned}
&\|G_D x_l - G_D x\| \\
&\quad = \sup_{t\ge T_D}\Bigg|\frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\
&\qquad \times \int_{\tau^{-1}(t)}^{+\infty}\left(s-\tau^{-1}(t)\right)^{n-1}\sum_{i=1}^m q_i(s)\left[f_i(x_l(\sigma_{i1}(s)),x_l(\sigma_{i2}(s)),\dots,x_l(\sigma_{ik_i}(s)))\right. \\
&\qquad\qquad \left.-f_i(x(\sigma_{i1}(s)),x(\sigma_{i2}(s)),\dots,x(\sigma_{ik_i}(s)))\right]ds\Bigg| \\
&\quad \le \sup_{t\ge T_D}\frac{1}{p_0(n-1)!} \\
&\qquad \times \int_{\tau^{-1}(t)}^{+\infty} s^{n-1}\sum_{i=1}^m q_i(s)\left|f_i(x_l(\sigma_{i1}(s)),x_l(\sigma_{i2}(s)),\dots,x_l(\sigma_{ik_i}(s)))\right. \\
&\qquad\qquad \left.-f_i(x(\sigma_{i1}(s)),x(\sigma_{i2}(s)),\dots,x(\sigma_{ik_i}(s)))\right|ds \\
&\quad \le \frac{1}{p_0(n-1)!}\int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1}\sum_{i=1}^m q_i(s)ds \cdot \frac{\varepsilon}{1+1/p_0(n-1)!\int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1}\sum_{i=1}^m q_i(s)ds} \\
&\quad < \varepsilon, \quad \forall l \ge K,
\end{aligned} \tag{3.18}$$

which means that G_D is continuous in $A(N,M)$.

Next, we show that $G_D(A(N,M))$ is equicontinuous in $[\beta,+\infty)$. Let $\varepsilon>0$. Taking into account (3.3) and $(A2)$, we know that there exists $T^*>T_D$ satisfying

$$\frac{1}{p_0(n-1)!}\int_{\tau^{-1}(T^*)}^{+\infty}s^{n-1}\sum_{i=1}^{m}q_i(s)ds<\frac{\varepsilon}{4}. \tag{3.19}$$

Put

$$B_1=\max\left\{s^{n-1}\sum_{i=1}^{m}q_i(s):\tau^{-1}(T_D)\le s\le\tau^{-1}(T^*)\right\}. \tag{3.20}$$

It follows from the uniform continuity of $p\tau^{-1}$ and τ^{-1} in $[T_D,T^*]$ that there exists $\delta>0$ satisfying

$$\begin{aligned}&\left|p\left(\tau^{-1}(t_1)\right)-p\left(\tau^{-1}(t_2)\right)\right|<\frac{\varepsilon p_0^2(n-1)!}{4\left[1+B\int_{\tau^{-1}(T_D)}^{+\infty}s^{n-1}\sum_{i=1}^{m}q_i(s)ds\right]},\\&\qquad\forall t_1,t_2\in[T_D,T^*]\text{ with }|t_1-t_2|<\delta;\\&\left|\tau^{-1}(t_1)-\tau^{-1}(t_2)\right|<\frac{\varepsilon p_0(n-1)!}{4B\left[1+B_1+(n-1)\int_{\tau^{-1}(T_D)}^{+\infty}u^{n-1}\sum_{i=1}^{m}q_i(s)ds\right]},\\&\qquad\forall t_1,t_2\in[T_D,T^*]\text{ with }|t_1-t_2|<\delta.\end{aligned} \tag{3.21}$$

Let $x\in A(N,M)$ and $t_1,t_2\in[\beta,+\infty)$ with $|t_1-t_2|<\delta$. We consider three possible cases.

Case 1. Let $t_1,t_2\in[T^*,+\infty)$. In view of (3.8), (3.9), (3.12), and (3.19), we conclude that

$$\begin{aligned}&|(G_Dx)(t_1)-(G_Dx)(t_2)|\\&\quad=\frac{1}{(n-1)!}\left|\frac{1}{p(\tau^{-1}(t_1))}\right.\\&\qquad\times\int_{\tau^{-1}(t_1)}^{+\infty}\left(s-\tau^{-1}(t_1)\right)^{n-1}\sum_{i=1}^{m}q_i(s)f_i(x(\sigma_{i1}(s)),x(\sigma_{i2}(s)),\dots,x(\sigma_{ik_i}(s)))ds\\&\qquad-\frac{1}{p(\tau^{-1}(t_2))}\\&\qquad\left.\times\int_{\tau^{-1}(t_2)}^{+\infty}\left(s-\tau^{-1}(t_2)\right)^{n-1}\sum_{i=1}^{m}q_i(s)f_i(x(\sigma_{i1}(s)),x(\sigma_{i2}(s)),\dots,x(\sigma_{ik_i}(s)))ds\right|\end{aligned}$$

$$\begin{aligned}
&\le \frac{B}{p_0(n-1)!}\left[\int_{\tau^{-1}(t_1)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds + \int_{\tau^{-1}(t_2)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds\right]\\
&< \frac{\varepsilon}{2}.
\end{aligned} \tag{3.22}$$

Case 2. Let $t_1, t_2 \in [T_D, T^*]$. In terms of (3.8), (3.9), (3.12), (3.21), we arrive at

$$\begin{aligned}
&|(G_D x)(t_1) - (G_D x)(t_2)|\\
&\quad = \frac{1}{(n-1)!}\Bigg|\frac{1}{p(\tau^{-1}(t_1))}\\
&\qquad \times \int_{\tau^{-1}(t_1)}^{+\infty}\left(s-\tau^{-1}(t_1)\right)^{n-1}\sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \dots, x(\sigma_{ik_i}(s)))ds\\
&\qquad - \frac{1}{p(\tau^{-1}(t_2))}\\
&\qquad \times \int_{\tau^{-1}(t_2)}^{+\infty}\left(s-\tau^{-1}(t_2)\right)^{n-1}\sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \dots, x(\sigma_{ik_i}(s)))ds\Bigg|\\
&\quad \le \frac{1}{(n-1)!}\Bigg\{\left|\frac{1}{p(\tau^{-1}(t_1))} - \frac{1}{p(\tau^{-1}(t_2))}\right|\\
&\qquad \times \int_{\tau^{-1}(t_1)}^{+\infty}\left(s-\tau^{-1}(t_1)\right)^{n-1}\sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \dots, x(\sigma_{ik_i}(s)))ds\\
&\qquad + \frac{1}{p(\tau^{-1}(t_2))}\\
&\qquad \times \Bigg[\left|\int_{\tau^{-1}(t_1)}^{\tau^{-1}(t_2)}\left(s-\tau^{-1}(t_1)\right)^{n-1}\sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \dots, x(\sigma_{ik_i}(s)))ds\right|\\
&\qquad\quad + \int_{\tau^{-1}(t_2)}^{+\infty}\left|\left(s-\tau^{-1}(t_1)\right)^{n-1} - \left(s-\tau^{-1}(t_2)\right)^{n-1}\right|\\
&\qquad\quad \times \sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \dots, x(\sigma_{ik_i}(s)))ds\Bigg]\Bigg\}\\
&\quad \le \frac{B}{(n-1)!}\Bigg\{\frac{|p(\tau^{-1}(t_1)) - p(\tau^{-1}(t_2))|}{p(\tau^{-1}(t_1))p(\tau^{-1}(t_2))}\int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds + \frac{1}{p_0}\\
&\qquad \times \Bigg[\left|\int_{\tau^{-1}(t_1)}^{\tau^{-1}(t_2)} s^{n-1}\sum_{i=1}^{m} q_i(s)ds\right|\\
&\qquad\quad + \int_{\tau^{-1}(t_2)}^{+\infty}(n-1)s^{\max\{n-2,0\}}\left|\tau^{-1}(t_1) - \tau^{-1}(t_2)\right|\sum_{i=1}^{m} q_i(s)ds\Bigg]\Bigg\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{B}{p_0^2(n-1)!}\left|p\left(\tau^{-1}(t_1)\right)-p\left(\tau^{-1}(t_2)\right)\right|\int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds \\
&\quad+\frac{B}{p_0(n-1)!}\left[B_1+(n-1)\int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds\right]\left|\tau^{-1}(t_1)-\tau^{-1}(t_2)\right| \\
&<\frac{\varepsilon}{2}.
\end{aligned} \tag{3.23}$$

Case 3. Let $t_1, t_2 \in [\beta, T_D]$. By (3.12), we have

$$|(G_Dx)(t_1)-(G_Dx)(t_2)| = |(G_Dx)(T_D)-(G_Dx)(T_D)| = 0 < \varepsilon. \tag{3.24}$$

Thus, $G_D(A(N,M))$ is equicontinuous in $[\beta,+\infty)$. Consequently, $G_D(A(N,M))$ is relatively compact by (3.16) and the continuity of G_D. By means of (3.14), (3.15), and Lemma 2.6, we infer that $F_D + G_D$ possesses a fixed point $x \in A(N,M)$, that is,

$$\begin{aligned}
x(t) &= \frac{D}{p(\tau^{-1}(t))}+\frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))}-\frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))}+\frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\
&\quad\times\int_{\tau^{-1}(t)}^{+\infty}\left(s-\tau^{-1}(t)\right)^{n-1}\sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)),x(\sigma_{i2}(s)),\dots,x(\sigma_{ik_i}(s)))ds, \quad \forall t \geq T_D,
\end{aligned} \tag{3.25}$$

which gives that

$$\begin{aligned}
x(t)-p(t)x(\tau(t)) &= -D+r(t)+\frac{(-1)^{n-1}}{(n-1)!} \\
&\quad\times\int_{t}^{+\infty}(s-t)^{n-1}\sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)),x(\sigma_{i2}(s)),\dots,x(\sigma_{ik_i}(s)))ds, \\
&\qquad \forall t \geq \tau^{-1}(T_D), \\
[x(t)-p(t)x(\tau(t))]^{(n)} &= g(t)-\sum_{i=1}^{m} q_i(t) f_i(x(\sigma_{i1}(t)),x(\sigma_{i2}(t)),\dots,x(\sigma_{ik_i}(t))), \quad \forall t \geq \tau^{-1}(T_D),
\end{aligned} \tag{3.26}$$

which mean that $x \in A(N,M)$ is a bounded positive solution of (1.1) with

$$N \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq M. \tag{3.27}$$

Let D_1 and D_2 be two arbitrarily different numbers in $((p_1-1)N+(p_1r_1/p_0),\ (p_0-1)M-r_0)$. Similarly, we conclude that for each $l \in \{1,2\}$ there exist two mappings F_{D_j}, G_{D_j} :

$A(N,M) \to BC([\beta,+\infty),\mathbb{R})$ and a sufficiently large number $T_{D_l} > \tau^{-1}(T_0)$ satisfying (3.8)–(3.12), where D, T_D, F_D, and G_D are replaced by D_l, T_{D_l}, F_{D_l}, and G_{D_l}, respectively, and $F_{D_l} + G_{D_l}$ has a fixed point $x_l \in A(N,M)$, which is also a bounded positive solution with $N \le \liminf_{t\to+\infty} x_l(t) \le \limsup_{t\to+\infty} x_l(t) \le M$, that is,

$$\begin{aligned} x_l(t) &= \frac{D_l}{p(\tau^{-1}(t))} + \frac{x_l(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\ &\quad \times \int_{\tau^{-1}(t)}^{+\infty} \left(s-\tau^{-1}(t)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(x_l(\sigma_{i1}(s)), x_l(\sigma_{i2}(s)), \ldots, x_l(\sigma_{ik_i}(s)))ds, \quad \forall t \ge T_{D_l}. \end{aligned} \tag{3.28}$$

It follows from (3.3) that there exists $T_3 > \max\{T_{D_1}, T_{D_2}\}$ satisfying

$$\frac{B}{p_0(n-1)!} \int_{\tau^{-1}(T_3)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s)ds < \frac{|D_1 - D_2|}{4p_1}. \tag{3.29}$$

Combining (3.8), (3.28), and (3.29), we conclude easily that

$$\begin{aligned} &|x_1(t) - x_2(t)| \\ &\quad = \Bigg| \frac{D_1 - D_2}{p(\tau^{-1}(t))} + \frac{x_1(\tau^{-1}(t)) - x_2(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\ &\qquad\quad \times \int_{\tau^{-1}(t)}^{+\infty} \left(s-\tau^{-1}(t)\right)^{n-1} \\ &\qquad\qquad \times \sum_{i=1}^{m} q_i(s)\left[f_i(x_1(\sigma_{i1}(s)), x_1(\sigma_{i2}(s)), \ldots, x_1(\sigma_{ik_i}(s)))\right. \\ &\qquad\qquad\qquad \left. - f_i(x_2(\sigma_{i1}(s)), x_2(\sigma_{i2}(s)), \ldots, x_2(\sigma_{ik_i}(s)))\right] ds \Bigg| \\ &\quad \ge \frac{|D_1 - D_2|}{p(\tau^{-1}(t))} - \frac{\left|x_1(\tau^{-1}(t)) - x_2(\tau^{-1}(t))\right|}{p(\tau^{-1}(t))} - \frac{2B}{p(\tau^{-1}(t))(n-1)!} \int_{\tau^{-1}(t)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s)ds \\ &\quad \ge \frac{|D_1 - D_2|}{p_1} - \frac{\|x_1 - x_2\|}{p_0} - \frac{2B}{p_0(n-1)!} \int_{\tau^{-1}(T_3)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s)ds \\ &\quad > \frac{|D_1 - D_2|}{p_1} - \frac{\|x_1 - x_2\|}{p_0} - \frac{|D_1 - D_2|}{2p_1} \\ &\quad = \frac{|D_1 - D_2\,|}{2p_1} - \frac{\|x_1 - x_2\|}{p_0}, \quad \forall t \ge T_3, \end{aligned} \tag{3.30}$$

which guarantees that

$$\|x_1 - x_2\| \geq \frac{p_0|D_1 - D_2|}{2p_1(1+p_0)} > 0, \tag{3.31}$$

that is, $x_1 \neq x_2$. Hence, (1.1) has uncountably many bounded positive solutions $x \in A(N, M)$ with $N \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq M$.

(b) Assume that M and N are arbitrary positive constants satisfying (3.6) and put

$$B_2 = \max\left\{\left|f_i(u_1, u_2, \dots, u_{k_i})\right| : u_j \in [-N, -M], 1 \leq j \leq k_i, 1 \leq i \leq m\right\}. \tag{3.32}$$

Let $D \in ((1-p_0)N + r_1, (1-p_1)M - (p_1 r_0/p_0))$. It follows from (3.3), (3.8), (3.32), and $(A2)$ that there exists $T_D > \tau^{-1}(T_0)$ satisfying

$$\frac{B_2}{p_0(n-1)!}\int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds < \min\left\{-M + \frac{M-D}{p_1} - \frac{r_0}{p_0}, N + \frac{D-N-r_1}{p_0}\right\}. \tag{3.33}$$

Let the mappings $F_D, G_D : A(-N, -M) \to C([\beta, +\infty), \mathbb{R})$ be defined by (3.11) and (3.12), respectively.

Using (3.1), (3.8), (3.11), (3.12), and (3.33), we deduce that for any $x, u \in A(-N, -M)$ and $t \geq T_D$

$$\begin{aligned}
&(F_D x)(t) + (G_D u)(t) \\
&\quad = \frac{D}{p(\tau^{-1}(t))} + \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\
&\qquad \times \int_{\tau^{-1}(t)}^{+\infty} \left(s - \tau^{-1}(t)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \dots, u(\sigma_{ik_i}(s)))ds \\
&\quad \leq \frac{D}{p_1} - \frac{M}{p_1} + \frac{r_0}{p_0} + \frac{B_2}{p_0(n-1)!}\int_{\tau^{-1}(t)}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds \\
&\quad < \frac{D-M}{p_1} + \frac{r_0}{p_0} + \min\left\{-M + \frac{M-D}{p_1} - \frac{r_0}{p_0},\ N + \frac{D-N-r_1}{p_0}\right\} \\
&\quad \leq -M,
\end{aligned}$$

$$
\begin{aligned}
&(F_D x)(t) + (G_D u)(t) \\
&\quad = \frac{D}{p(\tau^{-1}(t))} + \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\
&\quad\quad \times \int_{\tau^{-1}(t)}^{+\infty} \left(s - \tau^{-1}(t)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \dots, u(\sigma_{ik_i}(s))) ds \\
&\quad \geq \frac{D}{p_0} - \frac{N}{p_0} - \frac{r_1}{p_0} - \frac{B_2}{p_0(n-1)!} \int_{\tau^{-1}(t)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s) ds \\
&\quad > \frac{D - N - r_1}{p_0} - \min\left\{ -M + \frac{M - D}{p_1} - \frac{r_0}{p_0},\ N + \frac{D - N - r_1}{p_0} \right\} \\
&\quad \geq -N,
\end{aligned}
\tag{3.34}
$$

which give that

$$
F_D x + G_D u \in A(-N, -M), \quad \forall x, u \in A(-N, -M). \tag{3.35}
$$

The rest of the proof is similar to the proof of (a) and is omitted. This completes the proof. □

Theorem 3.2. *Let $(A1)$, $(A2)$, and $(A3)$, hold. Assume that there exist $p_0, p_1 \in \mathbb{R}^+ \setminus \{0\}$, $r_0, r_1 \in \mathbb{R}^+$, and $r \in C^n([t_0, +\infty), \mathbb{R})$ satisfying* (3.2), (3.3), *and*

$$
p_1 \geq -p(t) \geq p_0 > 1 \text{ eventually.} \tag{3.36}
$$

Then, the following hold:

(a) *for arbitrarily positive constants M and N with*

$$
N < M, \qquad \left(p_0^2 - p_1\right) M > \left(p_1 - \frac{p_0}{p_1}\right) p_0 N + p_0 r_1 + p_1 r_0, \tag{3.37}
$$

equation (1.1) *has uncountably many bounded positive solutions $x \in A(N, M)$ with*

$$
N \leq \liminf_{t \to +\infty} x(t) \leq \limsup_{t \to +\infty} x(t) \leq M; \tag{3.38}
$$

(b) *for arbitrarily positive constants M and N with*

$$
M < N, \qquad \left(p_0^2 - p_1\right) N > \left(p_1 - \frac{p_0}{p_1}\right) p_0 M + p_1 r_1 + p_0 r_0, \tag{3.39}
$$

equation (1.1) *has uncountably many bounded negative solutions* $x \in A(-N, -M)$ *with*

$$-N \leq \liminf_{t \to +\infty} x(t) \leq \limsup_{t \to +\infty} x(t) \leq -M. \tag{3.40}$$

Proof. It follows from (3.2) and (3.36) that there exists a constant T_0 with $\tau(T_0) > 1 + |t_0| + |\beta|$ satisfying

$$p_0 \leq -p(t) \leq p_1, \quad r^{(n)}(t) = g(t), \quad -r_0 \leq r(t) \leq r_1, \quad \forall t \geq T_0. \tag{3.41}$$

(a) Assume that M and N are arbitrary positive constants satisfying (3.37). Let $D \in (p_1((M + r_0)/p_0 + N),\ p_0(N/p_1 + M) - r_1)$ and B be defined by (3.9). In light of (3.3), (3.9), and $(A2)$, there exists a sufficiently large number $T_D > \tau^{-1}(T_0)$ satisfying

$$\frac{B}{p_0(n-1)!} \int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s) ds < \min\left\{ M - \frac{D + r_1}{p_0} + \frac{N}{p_1}, \frac{D}{p_1} - \frac{M + r_0}{p_0} - N \right\}. \tag{3.42}$$

Define two mappings $F_D, G_D : A(N, M) \to C([\beta, +\infty), \mathbb{R})$ by (3.12) and

$$(F_D x)(t) = \begin{cases} -\dfrac{D}{p(\tau^{-1}(t))} + \dfrac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \dfrac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))}, & t \geq T_D \\ (F_D x)(T_D), & \beta \leq t < T_D \end{cases} \tag{3.43}$$

for each $x \in A(N, M)$. In view of (3.12), (3.36), and (3.41)–(3.43), we conclude that for any $x, u \in A(N, M)$ and $t \geq T_D$

$$\begin{aligned}
&(F_D x)(t) + (G_D u)(t) \\
&\quad = -\frac{D}{p(\tau^{-1}(t))} + \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\
&\qquad \times \int_{\tau^{-1}(t)}^{+\infty} \left(s - \tau^{-1}(t)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \ldots, u(\sigma_{ik_i}(s))) ds \\
&\quad \leq \frac{D}{p_0} - \frac{N}{p_1} + \frac{r_1}{p_0} + \frac{B}{p_0(n-1)!} \int_{\tau^{-1}(t)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s) ds \\
&\quad < \frac{D}{p_0} - \frac{N}{p_1} + \frac{r_1}{p_0} + \min\left\{ M - \frac{D + r_1}{p_0} + \frac{N}{p_1}, \frac{D}{p_1} - \frac{M + r_0}{p_0} - N \right\} \\
&\quad \leq M,
\end{aligned}$$

$$
\begin{aligned}
(F_D x)(t) &+ (G_D u)(t) \\
&= -\frac{D}{p(\tau^{-1}(t))} + \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\
&\quad \times \int_{\tau^{-1}(t)}^{+\infty} \left(s-\tau^{-1}(t)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \dots, u(\sigma_{ik_i}(s)))ds \\
&\geq \frac{D}{p_1} - \frac{M}{p_0} - \frac{r_0}{p_0} - \frac{B}{p_0(n-1)!} \int_{\tau^{-1}(t)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s)ds \\
&> \frac{D}{p_1} - \frac{M+r_0}{p_0} - \min\left\{ M - \frac{D+r_1}{p_0} + \frac{N}{p_1}, \frac{D}{p_1} - \frac{M+r_0}{p_0} - N \right\} \\
&\geq N,
\end{aligned}
\tag{3.44}
$$

which imply (3.15). The rest of the proof is similar to that of Theorem 3.1 and is omitted.

(b) Assume that M and N are arbitrary positive constants satisfying (3.39). Let $D \in (-p_0(N+(M/p_1))M+r_0,\ -Mp_1-(p_1/p_0)(N+r_1))$ and B_2 be defined by (3.32). Note that (3.3), (3.32), and $(A2)$ yield that there exists a sufficiently large number $T_D > \tau^{-1}(T_0)$ satisfying

$$
\frac{B_2}{p_0(n-1)!} \int_{\tau^{-1}(T_D)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s)ds < \min\left\{ -M - \frac{D}{p_1} - \frac{N+r_1}{p_0}, N + \frac{D-r_0}{p_0} + \frac{M}{p_1} \right\}. \tag{3.45}
$$

Let the mappings $F_D, G_D : A(-N,-M) \to C([\beta,+\infty),\mathbb{R})$ be defined by (3.12) and (3.43), respectively.

Using (3.12), (3.36), (3.41), and (3.45), we infer that for any $x, u \in A(N,M)$ and $t \geq T_D$

$$
\begin{aligned}
(F_D x)(t) &+ (G_D u)(t) \\
&= -\frac{D}{p(\tau^{-1}(t))} + \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\
&\quad \times \int_{\tau^{-1}(t)}^{+\infty} \left(s-\tau^{-1}(t)\right)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \dots, u(\sigma_{ik_i}(s)))ds \\
&\leq \frac{D}{p_1} + \frac{N}{p_0} + \frac{r_1}{p_0} + \frac{B_2}{p_0(n-1)!} \int_{\tau^{-1}(t)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s)ds \\
&< \frac{D}{p_1} + \frac{N}{p_0} + \frac{r_1}{p_0} + \min\left\{ -M - \frac{D}{p_1} - \frac{N+r_1}{p_0}, N + \frac{D-r_0}{p_0} + \frac{M}{p_1} \right\} \\
&\leq -M,
\end{aligned}
$$

$$
\begin{aligned}
&(F_D x)(t) + (G_D u)(t) \\
&\quad = -\frac{D}{p(\tau^{-1}(t))} + \frac{x(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{r(\tau^{-1}(t))}{p(\tau^{-1}(t))} + \frac{(-1)^n}{p(\tau^{-1}(t))(n-1)!} \\
&\qquad \times \int_{\tau^{-1}(t)}^{+\infty} \left(s - \tau^{-1}(t)\right)^{n-1} \sum_{i=1}^{m} q_i(s)\ f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \dots, u(\sigma_{ik_i}(s)))ds \\
&\quad \geq \frac{D}{p_0} + \frac{M}{p_1} - \frac{r_0}{p_0} - \frac{B_2}{p_0(n-1)!} \int_{\tau^{-1}(t)}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s) ds \\
&\quad > \frac{D}{p_0} + \frac{M}{p_1} - \frac{r_0}{p_0} - \min\left\{ -M - \frac{D}{p_1} - \frac{N + r_1}{p_0}, N + \frac{D - r_0}{p_0} + \frac{M}{p_1} \right\} \\
&\quad \geq -N,
\end{aligned}
\tag{3.46}
$$

which give (3.15). The rest of the proof is similar to the proof of Theorem 3.1 and is omitted. This completes the proof. □

Theorem 3.3. *Let* $(A1)$ *and* $(A3)$ *hold. Assume that there exist* $p_0, p_1 \in \mathbb{R}^+ \setminus \{0\}$, $r_0, r_1 \in \mathbb{R}^+$, *and* $r \in C^n([t_0, +\infty), \mathbb{R})$ *satisfying* (3.2), (3.3), *and*

$$-p_0 \leq p(t) \leq p_1 \text{ eventually}, \qquad p_0 + p_1 < 1. \tag{3.47}$$

Then, the following hold:

(a) for arbitrarily positive constants M *and* N *with*

$$r_0 + r_1 + N < (1 - p_0 - p_1)M, \tag{3.48}$$

equation (1.1) *has uncountably many bounded positive solutions* $x \in A(N, M)$ *with*

$$N \leq \liminf_{t \to +\infty} x(t) \leq \limsup_{t \to +\infty} x(t) \leq M; \tag{3.49}$$

for arbitrarily positive constants M *and* N *with*

$$r_0 + r_1 + M < (1 - p_0 - p_1)N, \tag{3.50}$$

equation (1.1) *has uncountably many bounded negative solutions* $x \in A(-N, -M)$ *with*

$$-N \leq \liminf_{t \to +\infty} x(t) \leq \limsup_{t \to +\infty} x(t) \leq -M. \tag{3.51}$$

Proof. It follows from (3.2) and (3.47) that there exists a constant $T_0 > 1 + |t_0| + |\beta|$ satisfying

$$-p_0 \leq p(t) \leq p_1, \quad r^{(n)}(t) = g(t), \quad -r_0 \leq r(t) \leq r_1, \quad \forall t \geq T_0. \tag{3.52}$$

(a) Assume that M and N are arbitrary positive constants satisfying (3.48). Let $D \in (p_0 M + r_0 + N,\ (1 - p_1)M_1 - r_1)$ and B be defined by (3.9). In light of (3.3), (3.9), and $(A2)$, we infer that there exists a sufficiently large number $T_D > \max\{T_0, \tau(T_0)\}$ satisfying

$$\frac{B}{p_0(n-1)!}\int_{T_D}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds < \min\{M - D - p_1 M - r_1, D - p_0 M - r_0 - N\}. \tag{3.53}$$

Define two mappings $F_D, G_D : A(N,M) \to C([\beta,+\infty),\mathbb{R})$ by

$$(F_D x)(t) = \begin{cases} D + p(t)x(\tau(t)) + r(t), & t \ge T_D, \\ (F_D x)(T_D), & \beta \le t < T_D, \end{cases} \tag{3.54}$$

$$(G_D x)(t)\begin{cases} \dfrac{(-1)^{n-1}}{(n-1)!} & \\ \quad\times\displaystyle\int_t^{+\infty} (s-t)^{n-1}\sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \dots, x(\sigma_{ik_i}(s)))ds, & t \ge T_D \\ (G_D x)(T_D), & \beta \le t < T_D, \end{cases} \tag{3.55}$$

for each $x \in A(N,M)$. In view of (3.47) and (3.52)–(3.55), we conclude that for any $x, u \in A(N,M)$ and $t \ge T_D$

$$\begin{aligned}
|(F_D x)(t) - (F_D u)(t)| &\le \left|p(t)(x(\tau(t)) - u(\tau(t)))\right| \le (p_0 + p_1)\|x - u\|, \\
(F_D x)(t) + (G_D u)(t) &= D + p(t)x(\tau(t)) + r(t) + \frac{(-1)^{n-1}}{(n-1)!} \\
&\quad\times \int_t^{+\infty} (s-t)^{n-1}\sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \dots, u(\sigma_{ik_i}(s)))ds \\
&\le D + p_1 M + r_1 + \frac{B}{(n-1)\,!}\int_t^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds \\
&< D + p_1 M + r_1 + \min\{M - D - p_1 M - r_1, D - p_0 M - r_0 - N\} \le M, \\
(F_D x)(t) + (G_D u)(t) &= D + p(t)x(\tau(t)) + r(t) + \frac{(-1)^{n-1}}{(n-1)!} \\
&\quad\times \int_t^{+\infty} (s-t)^{n-1}\sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \dots, u(\sigma_{ik_i}(s)))ds
\end{aligned}$$

$$\begin{aligned}
&\geq D - p_0 M - r_0 - \frac{B}{(n-1)!}\int_t^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds \\
&> D - p_0 M - r_0 - \min\{M - D - p_1 M - r_1, D - p_0 M - r_0 - N\} \\
&\geq N,
\end{aligned} \tag{3.56}$$

which yield (3.15). The rest of the proof is similar to that of Theorem 3.1 and is omitted.

(b) Assume that M and N are arbitrary positive constants satisfying (3.50). Let $D \in (r_0 - (1-p_1)N - M - N,\ p_0 - r_1)$ and B_2 be defined by (3.32). In light of (3.3), (3.32), and $(A2)$, we infer that there exists a sufficiently large number $T_D > \max\{T_0, \tau(T_0)\}$ satisfying

$$\frac{B_2}{p_0(n-1)!}\int_{T_D}^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds < \min\{-M - D - p_0 N - r_1, D + N(1-p_1) - r_0\}. \tag{3.57}$$

Define two mappings $F_D, G_D : A(-N,-M) \to C([\beta,+\infty),\mathbb{R})$ by (3.54) and (3.55). In view of (3.47), (3.52), (3.54), (3.55), and (3.57), we conclude that (3.56) holds and

$$\begin{aligned}
(F_D x)(t) + (G_D u)(t) &= D + p(t)x(\tau(t)) + r(t) + \frac{(-1)^{n-1}}{(n-1)!} \\
&\quad \times \int_t^{+\infty} (s-t)^{n-1}\sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \dots, u(\sigma_{ik_i}(s)))ds \\
&\leq D + p_0 N + r_1 + \frac{B}{(n-1)\,!}\int_t^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds \\
&< D + p_0 N + r_1 + \min\{-M - D - p_0 N - r_1, D + N(1-p_1) - r_0\} \\
&\leq -M, \quad \forall x, u \in A(N,M),\ t \geq T_D, \\
(F_D x)(t) + (G_D u)(t) &= D + p(t)x(\tau(t)) + r(t) + \frac{(-1)^{n-1}}{(n-1)!} \\
&\quad \times \int_t^{+\infty} (s-t)^{n-1}\sum_{i=1}^{m} q_i(s) f_i(u(\sigma_{i1}(s)), u(\sigma_{i2}(s)), \dots, u(\sigma_{ik_i}(s)))ds \\
&\geq D - p_1 N - r_0 - \frac{B}{(n-1)!}\int_t^{+\infty} s^{n-1}\sum_{i=1}^{m} q_i(s)ds \\
&> D - p_1 N - r_0 - \min\{-M - D - p_0 N - r_1, D + N(1-p_1) - r_0\} \\
&\geq -N, \quad \forall x, u \in A(N,M),\ t \geq T_D.
\end{aligned} \tag{3.58}$$

Thus, (3.15) follows from (3.58). The rest of the proof is similar to that of Theorem 3.1 and is omitted. This completes the proof. □

Second, we provide necessary and sufficient conditions for the oscillation of bounded solutions of (1.1).

Theorem 3.4. *Let* $(A1)$, $(A2)$, *and* $(A3)$ *hold. Assume that there exist* $p_0, p_1 \in \mathbb{R}^+ \setminus \{0\}$ *and* $r \in C^n([t_0, +\infty), \mathbb{R})$ *satisfying* (2.24) *and*

$$\lim_{t \to +\infty} r(t) = 0, \qquad r^{(n)}(t) = g(t) \text{ eventually.} \tag{3.59}$$

Then, each bounded solution of (1.1) *either oscillates or tends to 0 as* $t \to +\infty$ *if and only if*

$$\int_{t_0}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s) ds = +\infty. \tag{3.60}$$

Proof.

Sufficiency. Suppose, without loss of generality, that (1.1) possesses a bounded eventually positive solution x with $\limsup_{t \to +\infty} x(t) > 0$, which together with $(A1)$, $(A3)$, (2.17), (2.24), and (3.60), yields that there exist constants $M > 0$ and $T > 1 + |t_0| + |\beta|$ satisfying

$$0 < x(t) \le M, \quad \forall t \ge T; \tag{3.61}$$

$$y^{(n)}(t) = -\sum_{i=1}^{m} q_i(t) f_i(x(\sigma_{i1}(t)), x(\sigma_{i2}(t)), \dots, x(\sigma_{ik_i}(t))) < 0, \quad \forall t \ge T. \tag{3.62}$$

Obviously (2.17), (2.24), (3.59), and the boundedness of x imply that y is bounded. It follows from (2.17), (3.62), Lemmas 2.1 and 2.2 that there exists a constant L satisfying

$$\lim_{t \to +\infty} y(t) = L \neq 0, \quad \lim_{t \to +\infty} y^{(i)}(t) = 0, \quad 1 \le i \le n-1. \tag{3.63}$$

Thus, $(A1)$, (3.61), (3.63), and Lemma 2.3 imply that there exist constants N and $T_1 \ge T_0 \ge T$ satisfying

$$\begin{gathered} \inf\{\sigma_{ij}(t) : t \ge T_1, 1 \le j \le k_i, 1 \le i \le m\} \ge T_0, \\ [7pt]0 < N \le x(t), \quad |y(t) - L| < 1, \quad \forall t \ge T_1. \end{gathered} \tag{3.64}$$

Put

$$B_3 = \min\{f_i(u_1, u_2, \dots, u_{k_i}) : u_j \in [N, M],\ 1 \le j \le k_i,\ 1 \le i \le m\}. \tag{3.65}$$

Clearly, $(A3)$ guarantees that $B_3 > 0$. Integrating (3.62) from t to $+\infty$, by (3.63) and (3.64), we have

$$y^{(n-1)}(t) = (-1)^2 \int_t^{+\infty} \sum_{i=1}^{m} q_i(u_1) f_i(x(\sigma_{i1}(u_1)), x(\sigma_{i2}(u_1)), \dots, x(\sigma_{ik_i}(u_1))) du_1, \quad \forall t \geq T_1, \tag{3.66}$$

repeating this procedure, we obtain that

$$\begin{aligned}
y^{(n-2)}(t) &= (-1)^3 \int_t^{+\infty} du_2 \int_{u_2}^{+\infty} \sum_{i=1}^{m} q_i(u_1) f_i(x(\sigma_{i1}(u_1)), x(\sigma_{i2}(u_1)), \dots, x(\sigma_{ik_i}(u_1))) du_1, \quad \forall t \geq T_1, \\
&\cdots \\
y'(t) &= (-1)^n \int_t^{+\infty} du_{n-1} \int_{u_{n-1}}^{+\infty} du_{n-2} \cdots \\
&\quad \times \int_{u_2}^{+\infty} \sum_{i=1}^{m} q_i(u_1) f_i(x(\sigma_{i1}(u_1)), x(\sigma_{i2}(u_1)), \dots, x(\sigma_{ik_i}(u_1))) du_1, \quad \forall t \geq T_1, \\
L - y(t) &= \lim_{u \to +\infty} y(u) - y(t) \\
&= (-1)^n \int_t^{+\infty} du_n \int_{u_n}^{+\infty} du_{n-1} \cdots \\
&\quad \times \int_{u_2}^{+\infty} \sum_{i=1}^{m} q_i(u_1) f_i(x(\sigma_{i1}(u_1)), x(\sigma_{i2}(u_1)), \dots, x(\sigma_{ik_i}(u_1))) du_1 \\
&= \frac{(-1)^n}{(n-1)!} \int_t^{+\infty} (s-t)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \dots, x(\sigma_{ik_i}(s))) ds, \quad \forall t \geq T_1,
\end{aligned} \tag{3.67}$$

which together with (3.64) and $(A3)$ means that

$$\begin{aligned}
1 > \left| L - y(t) \right| &= \left| \frac{(-1)^n}{(n-1)!} \int_t^{+\infty} (s-t)^{n-1} \sum_{i=1}^{m} q_i(s) f_i(x(\sigma_{i1}(s)), x(\sigma_{i2}(s)), \dots, x(\sigma_{ik_i}(s))) ds \right| \\
&\geq \frac{B_3}{(n-1)!} \int_t^{+\infty} (s-t)^{n-1} \sum_{i=1}^{m} q_i(s) ds, \quad \forall t \geq T_1,
\end{aligned} \tag{3.68}$$

which gives that

$$\int_{T_1}^{+\infty} s^{n-1} \sum_{i=1}^{m} q_i(s) ds < +\infty, \tag{3.69}$$

which contradicts (3.60).

Necessity. Suppose that (3.60) does not hold. Observe that $\lim_{t\to+\infty} r(t) = 0$ implies that there exist two positive constants r_0 and r_1 satisfying

$$-r_0 \le r(t) \le r_1 \text{ eventually.} \tag{3.70}$$

It follows from Theorem 3.1 or Theorem 3.2 that, for any positive constants M and N satisfying (3.4) or (3.37), (1.1) possesses uncountably many bounded positive solutions $x \in A(N, M)$ with $M \ge \limsup_{t\to+\infty} x(t) \ge \liminf_{t\to+\infty} x(t) \ge N$. This is a contradiction. This completes the proof. □

As in the proof of Theorem 3.4, by means of Lemmas 2.1, 2.4, and 2.5, we have

Theorem 3.5. *Let* $(A1)$ *and* $(A3)$ *hold. Assume that there exist* $p_0 \in \mathbb{R}^+ \setminus \{0\}$ *and* $r \in C^n([t_0, +\infty), \mathbb{R})$ *satisfying* (2.29) *and* (3.59)*. Then, each bounded solution of* (1.1) *either oscillates or tends to 0 as* $t \to +\infty$ *if and only if* (3.60) *holds.*

4. Remarks and Examples

Now, we compare the results in Section 3 with some known results in the literature. In order to illustrate the advantage and applications of our results, five nontrivial examples are constructed.

Remark 4.1. Theorems 3.1–3.3 extend and improve the Theorem in [9], Theorem 8.4.2 in [10], Theorem 1 in [21], Theorems 1–3 in [24], Theorem 2.2 in [26], and Theorems 1–4 in [27, 28].

Remark 4.2. The sufficient part of Theorem 3.5 is a generalization of Theorem 3.1 in [4, 5]. Theorem 3.5 corrects and perfects Theorem 2.1 in [26].

The examples below show that our results extend indeed the corresponding results in [4, 5, 9, 10, 21, 24, 26–28]. Notice that none of the known results can be applied to these examples.

Example 4.3. Consider the nth-order forced nonlinear neutral differential equation:

$$\begin{aligned}
&\left[x(t) - \frac{3+4t^n}{1+t^n} x\left(\sqrt{t}\right)\right]^{(n)} + \frac{\left(1+\sqrt{3+2t}\right)\left[x^5(3t+\sin t) + x^3(t-1/t)\right]}{(1+t^{n+3})\left[1+\left|x^8(3t^2) - 2x^{21}\left(t-\sqrt{t-1}\right)\right|\right]} \\
&\quad + \frac{tx(3t-\ln t)x^4(t^2-t)x^6(t-2) + 5tx\left(t(1+1/t)^t\right)}{(1+3t^{3n+1})\left[1+|x^3(4t-\cos^3 t) - 4x^4(t-1)|\right]} = \frac{1}{2}\sin\left(t+\frac{n\pi}{2}\right), \quad t \ge 2,
\end{aligned} \tag{4.1}$$

where $t_0 = 2$, $m = 2$, and $n \in \mathbb{N}$. Put $k_1 = 4$, $k_2 = 6$, $\beta = 0$, $r_0 = r_1 = 1/2$, $p_0 = 3$, $p_1 = 4$,

$$\begin{gathered}
p(t) = \frac{3+4t^n}{1+t^n}, \qquad q_1(t) = \frac{1+\sqrt{3+2t}}{1+t^{n+3}}, \qquad q_2(t) = \frac{t}{1+3t^{3n+1}}, \\
g(t) = \frac{1}{2}\sin\left(t+\frac{n\pi}{2}\right), \qquad r(t) = \frac{1}{2}\sin t, \qquad \tau(t) = \sqrt{t}, \qquad \sigma_{11}(t) = 3t + \sin t,
\end{gathered}$$

$$\sigma_{12}(t)=t-\frac{1}{t},\qquad \sigma_{13}(t)=3t^2,\qquad \sigma_{14}(t)=t-\sqrt{t-1},\qquad \sigma_{21}(t)=3t-\ln t,$$
$$\sigma_{22}(t)=t^2-t,\qquad \sigma_{23}(t)=t-2,\qquad \sigma_{24}(t)=t\left(1+\frac{1}{t}\right)^t,\qquad \sigma_{25}(t)=4t-\cos^3 t,$$
$$\sigma_{26}(t)=t-1,\qquad f_1(u,v,w,z)=\frac{u^5+v^3}{1+|w^8-2z^{21}|},$$
$$f_2(u,v,w,z,y,s)=\frac{uv^4w^6+5z}{1+|y^3-4s^4|},\quad \forall(t,u,v,w,z,y,s)\in[t_0,+\infty)\times\mathbb{R}^6. \tag{4.2}$$

Clearly $(A1)$, $(A2)$, $(A3)$, and (3.1)–(3.3) hold.

Let M and N be arbitrarily positive constants satisfying $M>(3/2)N+7/12$. It is easy to verify that (3.4) holds. It follows from Theorem 3.1 that (4.1) has uncountably many bounded positive solutions $x\in A(N,M)$ with $N\le\liminf_{t\to+\infty}x(t)\le\limsup_{t\to+\infty}x(t)\le M$.

Let M and N be arbitrarily positive constants satisfying $N>3M/2+7/12$. It is easy to verify that (3.6) holds. It follows from Theorem 3.1 that (4.1) has uncountably many bounded negative solutions $x\in A(-N,-M)$ with $-N\le\liminf_{t\to+\infty}x(t)\le\limsup_{t\to+\infty}x(t)\le -M$.

Example 4.4. Consider the nth-order forced nonlinear neutral differential equation:

$$\left[x(t)+\frac{8+10t^5}{2+t^5}x\left(\sqrt{t^2+1}-1\right)\right]^{(n)}+\frac{(t^2+3t^3)x^7(3t^2)x(t-1)x^3(t\ln t)}{\left(2+\sin^3(t^2)+t^{n+5}\right)\left[1+x^2(t-1)x^4(t\ln t)\right]}$$
$$+\frac{(3+t^2)\left[x^5(t^2+1)+7x^3(t^4-2)+x^9\left(t+\sqrt{t}\right)x^8(t-4)\right]}{\left(\sqrt{t+1}+t^{n+3}\right)\left[1+\left(x^5\left(t+\sqrt{t}\right)-4x^4(t-4)-3\right)^6\right]}=\frac{(-1)^n n!}{t^{n+1}}+\frac{n!}{(1-t)^{n+1}},\quad t\ge 3, \tag{4.3}$$

where $t_0=3$, $m=2$, and $n\in\mathbb{N}$. Put $k_1=3$, $k_2=4$, $\beta=-1$, $r_0=1/2$, $r_1=0$, $p_0=4$, $p_1=10$,

$$p(t)=-\frac{8+10t^5}{2+t^5},\qquad q_1(t)=\frac{t^2+3t^3}{2+\sin^3(t^2)+t^{n+5}},\qquad q_2(t)=\frac{3+t^2}{\sqrt{t+1}+t^{n+3}},$$
$$g(t)=\frac{(-1)^n n!}{t^{n+1}}+\frac{n!}{(1-t)^{n+1}},\qquad r(t)=\frac{1}{t(1-t)},\qquad \tau(t)=\sqrt{t^2+1}-1,$$
$$\sigma_{11}(t)=3t^2,\qquad \sigma_{12}(t)=t-1,\qquad \sigma_{13}(t)=t\ln t,\qquad \sigma_{21}(t)=t^2+1, \tag{4.4}$$
$$\sigma_{22}(t)=t^4-2,\qquad \sigma_{23}(t)=t+\sqrt{t},\qquad \sigma_{24}(t)=t-4,\qquad f_1(u,v,w)=\frac{u^7vw^3}{1+v^2w^4},$$
$$f_2(u,v,w,z)=\frac{u^5+7v^3+w^9z^8}{1+\left(w^5-4z^4-3\right)^6},\quad \forall(t,u,v,w,z)\in[t_0,+\infty)\times\mathbb{R}^4.$$

Clearly $(A1)$, $(A2)$, $(A3)$, (3.2), (3.3), and (3.36) hold.

Let M and N be arbitrarily positive constants satisfying $M > (32/5)N + 5/6$. It is easy to verify that (3.37) holds. It follows from Theorem 3.2 that (4.3) has uncountably many bounded positive solutions $x \in A(N, M)$ with $N \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le M$.

Let M and N be arbitrarily positive constants satisfying $N > (32/5)M + 1/3$. It is easy to verify that (3.39) holds. It follows from Theorem 3.2 that (4.3) has uncountably many bounded negative solutions $x \in A(-N, -M)$ with $-N \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le -M$.

Example 4.5. Consider the nth-order forced nonlinear neutral differential equation:

$$\left[x(t) - \frac{2\sin\left(t^2 - \sqrt{t}\right)}{5 + \sin\left(t^2 - \sqrt{t}\right)} x\left((t-5)^2\right)\right]^{(n)} + \frac{\left(3\sqrt{t-1} + t^5\right) x^3(t-4)}{\left(\sqrt{t^2+1} + t^{n+6}\right)\left[2 + \cos^5 x\left(\sqrt{t+1} - 3\right)\right]}$$
$$+ \frac{\left(1 - \sqrt{t+1}\ln^2 t + t^4\right) x^9\left(2t + \sin\left(t^2+1\right)\right)}{\left(1 - 2t^3 + 3t^4 + t^{n+5}\right)\ln\left[2 + x^2\left(t^2\sqrt{1+2t}\right)\right]} = (-1)^n \cos\left(t + \frac{n\pi}{2}\right), \quad t \ge 1, \tag{4.5}$$

where $t_0 = 1$, $m = 2$, and $n \in \mathbb{N}$. Put $k_1 = k_2 = 2$, $\beta = -4$, $r_0 = r_1 = 1$, $p_0 = 1/2$, $p_1 = 1/3$,

$$\begin{gathered}
p(t) = \frac{2\sin\left(t^2 - \sqrt{t}\right)}{5 + \sin\left(t^2 - \sqrt{t}\right)}, \qquad q_1(t) = \frac{3\sqrt{t-1} + t^5}{\sqrt{t^2+1} + t^{n+6}}, \qquad q_2(t) = \frac{1 - \sqrt{t+1}\ln^2 t + t^4}{1 - 2t^3 + 3t^4 + t^{n+5}}, \\
g(t) = (-1)^n \cos\left(t + \frac{n\pi}{2}\right), \qquad r(t) = (-1)^n \cos t, \qquad \tau(t) = (t-4)^2 \qquad \sigma_{11}(t) = t - 5, \\
\sigma_{12}(t) = \sqrt{t+1} - 3, \qquad \sigma_{21}(t) = 2t + \sin\left(t^2 + 1\right), \qquad \sigma_{22}(t) = t^2\sqrt{1+2t}, \\
f_1(u, v) = \frac{u^3}{2 + \cos^5 v}, \qquad f_2(u, v) = \frac{u^9}{\ln(2 + v^2)}, \quad \forall (t, u, v) \in [t_0, +\infty) \times \mathbb{R}^2.
\end{gathered} \tag{4.6}$$

Clearly $(A1)$, $(A3)$, (3.2), (3.3), and (3.47) hold.

Let M and N be arbitrarily positive constants satisfying $M > 6N + 12$. It is easy to verify that (3.48) holds. It follows from Theorem 3.3 that (4.5) has uncountably many bounded positive solutions $x \in A(N, M)$ with $N \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le M$.

Let M and N be arbitrarily positive constants satisfying $N > 6M + 12$. It is easy to verify that (3.50) holds. It follows from Theorem 3.3 that (4.5) has uncountably many bounded negative solutions $x \in A(-N, -M)$ with $-N \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le -M$.

Example 4.6. Consider the nth-order forced nonlinear neutral differential equation:

$$\left[x(t) - \frac{(-1)^n\left(5 + 9\ln^2 t\right)}{1 + \ln^2 t} x\left(\sqrt{t} - 1\right)\right]^{(n)} + \left(t^8 + 9t^5 + 3\right)\left[2x^3\left(t\ln^2 t\right) + 5x^7(t - 16)\right]$$

$$
\begin{aligned}
&+ t^2\left(2+\sin\left(t^3-5t\right)\right)x^9\left(t-\frac{\sin t}{t}\right)\left[x^4(t-\cos t)+4x^6\left(\frac{1+t+t^2+t^3}{1+t+t^2}\right)\right]^2 \\
&+ \frac{\left[(t+1)^2-\sqrt{t}\right]x^5\left(t\arctan(t^3+1)/\left(1+\sqrt{t+1}\right)\right)\ln\left(1+x^6(t+1)/(1+x^2(t-2))\right)}{\left[2t^{n+3}+\sqrt{t}\sin(3t^5-1)\right]\left[1+x^2(t^2-t)x^4(t^2+t)\right]} \\
&= \frac{(-1)^n n!\left(\ln t-\sum_{i=1}^{n}(1/i)\right)}{t^{n+1}}, \quad t \geq 4,
\end{aligned}
\tag{4.7}
$$

where $t_0 = 4$, $m = 3$, and $n \in \mathbb{N}$. Put $k_1 = 2$, $k_2 = 3$, $k_3 = 5$, $\beta = -12$, $p_0 = 5$, $p_1 = 9$,

$$
\begin{gathered}
p(t) = \frac{(-1)^n\left(5+9\ln^2 t\right)}{1+\ln^2 t}, \qquad q_1(t) = t^8+9t^5+3, \qquad q_2(t) = t^2\left(2+\sin\left(t^3-5t\right)\right), \\
q_3 = \frac{(t+1)^2-\sqrt{t}}{2t^{n+3}+\sqrt{t}\sin(3t^5-1)}, \qquad g(t) = \frac{(-1)^n n!\left(\ln t-\sum_{i=1}^{n}1/i\right)}{t^{n+1}}, \qquad r(t) = \frac{\ln t}{t}, \\
\tau(t) = \sqrt{t}-1, \qquad \sigma_{11}(t) = t\ln^2 t, \qquad \sigma_{12}(t) = t-16, \qquad \sigma_{21}(t) = t-\frac{\sin t}{t}, \\
\sigma_{22}(t) = t-\cos t, \qquad \sigma_{23}(t) = \frac{1+t+t^2+t^3}{1+t+t^2}, \qquad \sigma_{31}(t) = \frac{t\arctan(t^3+1)}{1+\sqrt{t+1}}, \\
\sigma_{32}(t) = t+1, \qquad \sigma_{33}(t) = t-2, \qquad \sigma_{34}(t) = t^2-t, \qquad \sigma_{35}(t) = t^2+t, \\
f_1(u,v) = 2u^3+5v^7, \qquad f_2(u,v,w) = u^9\left(v^4+4w^6\right)^2, \\
f_3(u,v,w,y,z) = \frac{u^5\ln\left(1+v^6/(1+w^2)\right)}{1+y^2z^4}, \quad \forall (t,u,v,w,y,z) \in [t_0,+\infty)\times\mathbb{R}^5.
\end{gathered}
\tag{4.8}
$$

Clearly $(A1)$, $(A2)$, $(A3)$, (2.24), (3.59), and (3.60) hold. It follows from Theorem 3.4 that each bounded solution of (4.7) either oscillates or tends to 0 as $t \to +\infty$.

Example 4.7. Consider the nth-order forced nonlinear neutral differential equation:

$$
\begin{aligned}
&\left[x(t)-\frac{(-1)^n\cos^3(3t-1)}{4+\cos^3(3t-1)}x(t-\sin t)\right]^{(n)} + \frac{\left(t^3+2t^2-\sqrt{t}+1\right)x^5\left(\sqrt{t-2}-1\right)}{1+x^2\left(\sqrt{t-2}-1\right)} \\
&+ \frac{\sqrt{t^2-1}\left[x^3(t-1/t)+5x^7(t-1/t)\right]\ln\left(2+x^6(t-1/t)\right)}{t^{2n+1}+2t^n\ln^3(1+t^2)+1} = \frac{2^n\sin\left(\sqrt{2}t+n\pi/4\right)}{e^{\sqrt{2}t}}, \quad t \geq 6,
\end{aligned}
\tag{4.9}
$$

where $t_0 = 6,\ m = 2$, and $n \in \mathbb{N}$. Put $k_1 = k_2 = 1,\ \beta = 1,\ p_0 = 1/3$,

$$p(t) = \frac{(-1)^n \cos^3(3t-1)}{4+\cos^3(3t-1)}, \qquad q_1(t) = t^3 + 2t^2 - \sqrt{t} + 1,$$
$$q_2(t) = \frac{\sqrt{t^2-1}}{t^{2n+1} + 2t^n \ln^3(1+t^2) + 1}, \qquad g(t) = \frac{2^n \sin\left(\sqrt{2}t + n\pi/4\right)}{e^{\sqrt{2}t}}, \qquad r(t) = \frac{\sin\left(\sqrt{2}t\right)}{e^{\sqrt{2}t}},$$
$$\tau(t) = t - \sin t, \qquad \sigma_1(t) = \sqrt{t-2} - 1 \qquad \sigma_2(t) = t - \frac{1}{t}, \qquad f_1(u) = \frac{u^5}{1+u^2},$$
$$f_2(u) = u^3 + 5u^7 \ln\left(2 + u^6\right), \quad \forall (t,u) \in [t_0, +\infty) \times \mathbb{R}. \tag{4.10}$$

Clearly $(A1)$, $(A3)$, (2.29), (3.59), and (3.60) hold. It follows from Theorem 3.5 that each bounded solution of (4.9) either oscillates or tends to 0 as $t \to +\infty$.

Next, we prove that the necessary part of Theorem 2.1 in [26] does not hold by means of (4.9). It is easy to verify that the conditions of Theorem 2.1 in [26] are fulfilled. Suppose that the necessary part of Theorem 2.1 in [26] is true. Because each bounded solution of (4.9) either oscillates or tends to 0 as $t \to +\infty$, it follows that the necessary part of Theorem 2.1 in [26] gives that

$$\int_{t_0}^{+\infty} s^{n-1} q_i(s)\, ds = +\infty, \quad i \in \{1,2\}, \tag{4.11}$$

which yields that

$$+\infty = \int_{t_0}^{+\infty} s^{n-1} q_2(s)\, ds = \int_{t_0}^{+\infty} \frac{s^{n-1}\sqrt{s^2-1}}{s^{2n+1} + 2s^n \ln^3(1+s^2) + 1} ds \le \int_{t_0}^{+\infty} \frac{1}{s^{n+1}} ds < +\infty, \tag{4.12}$$

which is a contradiction.

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