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Research Article

Homotopy Perturbation Method with an Auxiliary Term

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The two most important steps in application of the homotopy perturbation method are to construct a suitable homotopy equation and to choose a suitable initial guess. The homotopy equation should be such constructed that when the homotopy parameter is zero, it can approximately describe the solution property, and the initial solution can be chosen with an unknown parameter, which is determined after one or two iterations. This paper suggests an alternative approach to construction of the homotopy equation with an auxiliary term; Dufing equation is used as an example to illustrate the solution procedure.

1. Introduction

The homotopy perturbation method [1–7] has been worked out over a number of years by numerous authors, and it has matured into a relatively fledged theory thanks to the efforts of many researchers, see Figure 1. For a relatively comprehensive survey on the concepts, theory, and applications of the homotopy perturbation method, the reader is referred to review articles [8–11].

The homotopy perturbation method has been shown to solve a large class of nonlinear differential problems effectively, easily, and accurately; generally one iteration is enough for engineering applications with acceptable accuracy, making the method accessible to nonmathematicians.

In case of higher-order approximates needed, we can use parameter-expansion technology [12–14]; in this paper, we suggest an alternative approach by adding a suitable term in the homotopy equation.

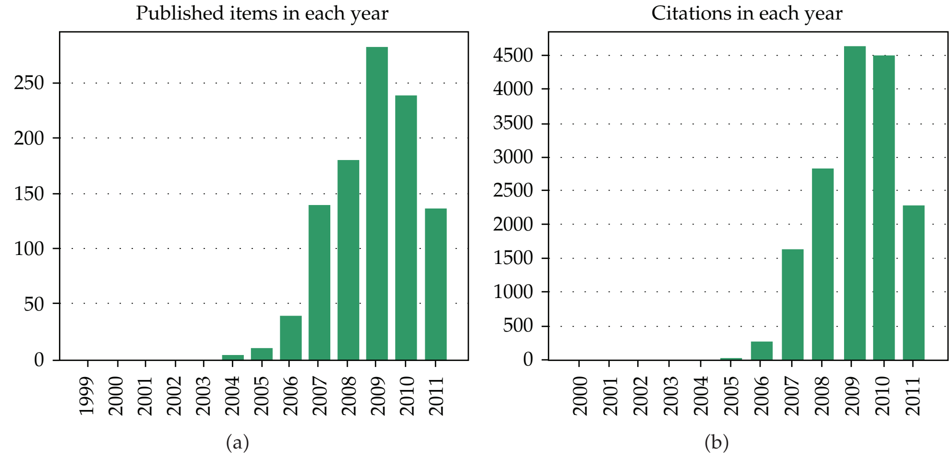

Figure 1: Number of publications on homotopy perturbation according to web of science, August 20, 2011.

2. Homotopy Perturbation Method

Consider a general nonlinear equation

$$Lu + Nu = 0, \tag{2.1}$$

where L and N are, respectively, the linear operator and nonlinear operator.

The first step for the method is to construct a homotopy equation in the form [3–5]

$$\tilde{L}u + p\left(Lu - \tilde{L}u + Nu\right) = 0, \tag{2.2}$$

where $\tilde{L}$ is a linear operator with a possible unknown constant and $\tilde{L}u = 0$ can approximately describe the solution property. The embedding parameter p monotonically increases from zero to unit as the trivial problem ($\tilde{L}u = 0$) is continuously deformed to the original one ($Lu + Nu = 0$).

For example, consider a nonlinear oscillator

$$u'' + \varepsilon u^3 = 0, \qquad u(0) = A, \qquad u'(0) = 0. \tag{2.3}$$

For an oscillator, we can use sine or cosine function. We assume that the approximate solution of (2.3) is

$$u(t) = A\cos\omega t, \tag{2.4}$$

where ω is the frequency to be determined later. We, therefore, can choose

$$\tilde{L}u = \ddot{u} + \omega^2 u. \tag{2.5}$$

Accordingly, we can construct a homotopy equation in the form

$$\ddot{u} + \omega^2 u + p\left(u^3 - \omega^2 u\right) = 0. \tag{2.6}$$

When $p = 0$, we have

$$\ddot{u} + \omega^2 u = 0, \qquad u(0) = A, \qquad u'(0) = 0, \tag{2.7}$$

which describes the basic solution property of the original nonlinear equation, (2.3).

When $p = 1$, (2.6) becomes the original one. So the solution procedure is to deform from the initial solution, (2.4), to the real one. Due to one unknown parameter in the initial solution, only one iteration is enough. For detailed solution procedure, refer to [5].

If a higher-order approximate solution is searched for, we can construct a homotopy equation in the form

$$\ddot{u} + 0 \cdot u + pu^3 = 0. \tag{2.8}$$

We expand the solution and the coefficient, zero, of the linear term into a series in p:

$$u = u_0 + pu_1 + p^2 u_2 + \cdots, \tag{2.9}$$

$$0 = \omega^2 + pa_1 + p^2 a_2 + \cdots, \tag{2.10}$$

where the unknown constant, a_i, is determined in the $(i + 1)$th iteration. The solution procedure is given in [5].

3. Homotopy Equation with an Auxiliary Term

In this paper, we suggest an alternative approach to construction of homotopy equation, which is

$$\tilde{L}u + p\left(Lu - \tilde{L}u + Nu\right) + \alpha p(1 - p)u = 0, \tag{3.1}$$

where α is an auxiliary parameter. When $\alpha = 0$, (3.1) turns out to be that of the classical one, expressed in (2.2). The auxiliary term, $\alpha p(1 - p)u$, vanishes completely when $p = 0$ or $p = 1$; so the auxiliary term will affect neither the initial solution ($p = 0$) nor the real solution ($p = 1$). The homotopy perturbation method with an auxiliary term was first considered by Noor [15].

To illustrate the solution procedure, we consider a nonlinear oscillator in the form

$$\frac{d^2u}{dt^2} + bu + cu^3 = 0, \qquad u(0) = A, \qquad u'(0) = 0, \tag{3.2}$$

where b and c are positive constants.

Equation (3.2) admits a periodic solution, and the linearized equation of (3.2) is

$$u'' + \omega^2 u = 0, \qquad u(0) = A, \qquad u'(0) = 0, \tag{3.3}$$

where ω is the frequency of (3.2).

We construct the following homotopy equation with an auxiliary term:

$$u'' + \omega^2 u + p\left[\left(b - \omega^2\right)u + cu^3\right] + \alpha p(1-p)u = 0. \tag{3.4}$$

Assume that the solution can be expressed in a power series in p as shown in (2.9).

Substituting (2.9) into (3.4), and processing as the standard perturbation method, we have

$$u_0'' + \omega^2 u_0 = 0, \qquad u_0(0) = A, \qquad u_0'(0) = 0, \tag{3.5}$$

$$u_1'' + \omega^2 u_1 + \left(b - \omega^2\right)u_0 + cu_0^3 + \alpha u_0 = 0, \qquad u_1(0) = 0, \qquad u_1'(0) = 0, \tag{3.6}$$

$$u_2'' + \omega^2 u_2 + \left(b - \omega^2\right)u_1 + 3cu_0^2 u_1 + \alpha(u_1 - u_0) = 0, \tag{3.7}$$

with initial conditions

$$\sum_{i=0} u_i(0) = A, \qquad \sum_{i=0} u_i'(0) = 0. \tag{3.8}$$

Solving (3.5), we have

$$u_0 = A\cos\omega t. \tag{3.9}$$

Substituting u_0 into (3.6) results into

$$u_1'' + \omega^2 u_1 + A\left(\alpha + b - \omega^2 + \frac{3}{4}cA^2\right)\cos\omega t + \frac{1}{4}cA^3\cos 3\omega t = 0. \tag{3.10}$$

Eliminating the secular term needs

$$\alpha + b - \omega^2 + \frac{3}{4}cA^2 = 0. \tag{3.11}$$

A special solution of (3.10) is

$$u_1 = -\frac{cA^3}{32\omega^2}\cos 3\omega t. \tag{3.12}$$

If only a first-order approximate solution is enough, we just set $\alpha = 0$; this results in

$$\omega = \sqrt{b + \frac{3}{4}cA^2}. \tag{3.13}$$

The accuracy reaches 7.6% even for the case $cA^2 \to \infty$.

The solution procedure continues by submitting u_1 into (3.7); after some simple calculation, we obtain

$$\begin{aligned} &u_2'' + \omega^2 u_2 - \left(\alpha A + \frac{3c^2A^5}{128\omega^2}\right)\cos\omega t - \left(\frac{cA^3(b-\omega^2)}{32\omega^2} + \frac{3c^2A^5}{64\omega^2} + \frac{\alpha c A^3}{32\omega^2}\right)\cos 3\omega t \\ &\quad - \frac{3c^2A^5}{128\omega^2}\cos 5\omega = 0. \end{aligned} \tag{3.14}$$

No secular term in u_2 requires

$$\alpha A + \frac{3c^2A^5}{128\omega^2} = 0. \tag{3.15}$$

Solving (3.11) and (3.15) simultaneously, we obtain

$$\omega = \sqrt{\frac{b + (3/4)cA^2 + \sqrt{(b + (3/4)cA^2)^2 + (3/32)c^2A^4}}{2}}, \tag{3.16}$$

and the approximate solution is $u(t) = A\cos\omega t$, where ω is given in (3.16).

In order to compare with the perturbation solution and the exact solution, we set $b = 1$. In case $c \ll 1$, (3.16) agrees with that obtained by classical perturbation method; when $c \to \infty$, we have

$$\lim_{c\to\infty}\omega = \sqrt{\frac{(3/4) + \sqrt{(3/4)^2 + 3/32}}{2}}\sqrt{cA^2} = 0.8832\sqrt{cA^2}. \tag{3.17}$$

The exact period reads

$$T_{ex} = \frac{4}{\sqrt{1+cA^2}}\int_0^{\pi/2}\frac{\mathrm{d}x}{\sqrt{1-k\sin^2 x}}, \tag{3.18}$$

where $k = cA^2/2(1+cA^2)$.

In case $c \to \infty$, we have

$$\lim_{c\to\infty} T_{ex} = \frac{6.743}{\sqrt{cA^2}}, \tag{3.19}$$

$$\omega_{ex} \approx \frac{2\pi}{6.743}\sqrt{cA^2}. \tag{3.20}$$

Comparing between (3.17) and (3.20), we find that the accuracy reaches 5.5%, while accuracy of the first-order approximate frequency is 7.6%.

If a higher-order approximate solution is needed, we rewrite the homotopy equation in the form

$$u'' + \omega^2 u + p\left[\left(b - \omega^2\right)u + cu^3\right] + 1 \cdot p(1-p)u = 0. \tag{3.21}$$

The coefficient, 1, in the auxiliary term, is also expanded in a series in p in the form

$$1 = \alpha_0 + p\alpha_1 + p^2\alpha_2 + \cdots, \tag{3.22}$$

where α_i is identified in the $(i+2)$th iteration. The solution procedure is similar to that illustrated above.

4. Discussions and Conclusions

Generally the homotopy equation can be constructed in the form

$$\tilde{L}u + p\left(Lu - \tilde{L}u + Nu\right) + 1 \cdot f(p)g(p)h(u, u', u'', \ldots) = 0, \tag{4.1}$$

where f and g are functions of p, satisfying $f(0) = 0$ and $g(1) = 0$, and h can be generally expressed in the form

$$h = u + \beta_1 u' + \beta_2 u'' + \cdots. \tag{4.2}$$

For example, for the Blasius equation

$$u'''(\eta) + \frac{1}{2}u(\eta)u''(\eta) = 0, \qquad u(0) = u'(0) = 0, \qquad u'(+\infty) = 1, \tag{4.3}$$

where the superscript denotes derivative with respect to η, we can construct a homotopy equation in the form

$$u''' + au'' + p\left(\frac{1}{2}uu'' - au''\right) + bp(1-p)u'' = 0, \tag{4.4}$$

where a and b are unknown constants to be determined.

The operator $\widetilde{L}()$ can be also a nonlinear one, for example, if we want to search for a solitonary solution, we can choose $\widetilde{L}u = u_t + 6uu_x + u_{xxx}$.

The homotopy equation can be easily constructed, and the solution procedure is simple. This paper can be considered a standard homotopy perturbation algorithm and can be used as a paradigm for many other applications.

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