

## Research Article

# Nontrivial Solutions for a Class of Fractional Differential Equations with Integral Boundary Conditions and a Parameter in a Banach Space with Lattice

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Existence of nontrivial solutions for the following fractional differential equation with integral boundary conditions  $D_{0+}^{\alpha}u(t) + h(t)f(t, u(t)) = 0$ ,  $0 < t < 1$ ,  $u(0) = u'(0) = u''(0) = 0$ ,  $u(1) = \lambda \int_0^{\eta} u(s)ds$  is investigated by using results for the computation of topological degree under the lattice structure, where  $3 < \alpha \leq 4$ ,  $0 < \eta \leq 1$ ,  $0 \leq \lambda\eta^{\alpha}/\alpha < 1$ ,  $D_{0+}^{\alpha}$  is the standard Riemann-Liouville derivative.  $h(t)$  is allowed to be singular at  $t = 0$  and  $t = 1$ .

## 1. Introduction

Fractional differential equations have been of great interest for many researchers recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various fields of sciences and engineering such as control, porous media, electromagnetic, and other fields. For an extensive collection of such results, we refer the readers to the monographs by Samko et al. [1], Podlubny [2], and Kilbas et al. [3]. Recently, there are some papers dealing with the existence of solutions (or positive solutions) for nonlinear fractional differential equation by means of techniques of nonlinear analysis (fixed point theorems, Leray-Schauder theory, adomian decomposition method, lower and upper solution method, etc.); see [4–16].

As is well known, the first eigenvalue is a character of great significance for the linear operator. For some integer order differential equations, many authors have investigated the existence of positive and nontrivial solutions concerning the first eigenvalue corresponding to the relevant linear operators when the nonlinearities are sublinear, see [17–22] for reference.

On the other hand, papers [23–26] obtained similar results to the sublinear case. The main discussion is based on the concepts of dual space, dual cone, and a constructed cone on them.

Recently, Xu et al. [27] and Bai [28] obtained the existence results of positive solutions for some fractional differential equations under the conditions with respect to the first eigenvalue corresponding to the relevant linear operators.

In two recent papers [29, 30], Sun and Liu established some results about the computation of the topological degree for nonlinear operators which are not cone mappings using the lattice structure.

Motivated by the above papers, by using results for the computation of topological degree under the lattice structure, we investigate the existence of nontrivial solutions for the following nonlinear fractional differential equations with integral boundary conditions:

$$\begin{aligned} D_{0+}^{\alpha} u(t) + h(t)f(t, u(t)) &= 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) &= 0, \\ u(1) &= \lambda \int_0^{\eta} u(s)ds, \end{aligned} \tag{1.1}$$

where  $3 < \alpha \leq 4$ ,  $0 < \eta \leq 1$ ,  $0 \leq \lambda\eta^{\alpha}/\alpha < 1$ ,  $D_{0+}^{\alpha}$  is the standard Riemann-Liouville derivative. In this paper, it is not required that nonlinearity  $f(t, u) \geq 0$ , for all  $u \geq 0$ . To the author's knowledge, few papers are available in the literature to study the existence of solutions for fractional differential equations with integral boundary conditions under the lattice structure. The method used in this paper is different from those in previous works.

This paper is organized as follows. In Section 2 corresponding Green's function for BVP (1.1) is derived and its properties are also discussed. The main results and their proof are presented in Section 3.

## 2. Background Materials and Green's Function

Let  $E$  be a Banach space with a cone  $P$ . Then  $E$  becomes an ordered Banach space under the partial ordering  $\leq$  which is induced by  $P$ .  $P$  is said to be normal if there exists a positive constant  $N$  such that  $\theta \leq x \leq y$  implies  $\|x\| \leq N\|y\|$ .  $P$  is called solid if it contains interior points, that is,  $\text{int } P \neq \emptyset$ .  $P$  is called total if  $E = \overline{P - P}$ . If  $P$  is solid, then  $P$  is total. For the concepts and the properties about the cone we refer to [31, 32].

We call  $E$  a lattice under the partial ordering  $\leq$  if  $\sup\{x, y\}$  and  $\inf\{x, y\}$  exist for arbitrary  $x, y \in E$ . For  $x \in E$ , let  $x^+ = \sup\{x, \theta\}$ ,  $x^- = \sup\{-x, \theta\}$ ,  $x^+$  and  $x^-$  are called the positive part and the negative part of  $x$ , respectively, and obviously  $x = x^+ - x^-$ . Take  $|x| = x^+ + x^-$ , then  $|x| \in P$ . One can refer to [33] for the definition and the properties of the lattice. Let  $x_+ = x^+$ ,  $x_- = -x^-$  as denoted in [29, 30]. Then  $x_+ \in P$ ,  $x_- \in -P$  and  $x = x_+ + x_-$ .

Let  $B : E \rightarrow E$  be a bounded linear operator.  $B$  is said to be positive if  $B(P) \rightarrow P$ . In this case,  $B$  is an increase operator, namely, for  $x, y \in E$ ,  $x \leq y$  implies  $Bx \leq By$ . Let  $B : E \rightarrow E$  be a positively completely continuous operator,  $r(B)$  a spectral radius of  $B$ ,  $B^*$  the conjugated operator of  $B$ ,  $P^*$  the conjugated cone of  $P$ . Since  $P \subset E$  is a total cone, according to the famous

Krein-Rutman theorem (see [34]), we infer that if  $r(B) \neq 0$ , then there exist  $\bar{\varphi} \in P \setminus \{\theta\}$  and  $g^* \in P^* \setminus \{\theta\}$  such that

$$B\varphi = r(B)\varphi, \quad B^*g^* = r(B)g^*. \quad (2.1)$$

For a given constant  $\delta > 0$ , set

$$P(g^*, \delta) = \{x \in P, g^*(x) \geq \delta\|x\|\}, \quad (2.2)$$

then  $P(g^*, \delta)$  is also a cone in  $E$ .

*Definition 2.1* (see [30]). Let  $D \subset E$  and  $F : D \rightarrow E$  a nonlinear operator.  $F$  is said to be quasiadditive on lattice, if there exists  $y_0 \in E$  such that

$$Fx = Fx_+ + Fx_- + y_0, \quad \forall x \in D. \quad (2.3)$$

*Definition 2.2* (see [30]). Let  $B$  be a positive linear operator. The operator  $B$  is said to satisfy  $H$  condition, if there exist  $\bar{\varphi} \in P \setminus \{\theta\}$ ,  $g^* \in P^* \setminus \{\theta\}$  such that (2.1) holds, and  $B$  maps  $P$  into  $P(g^*, \delta)$ .

*Definition 2.3* (see [4]). The Riemann-Liouville fractional integral of order  $\alpha > 0$  of a function  $y : (0, \infty) \rightarrow R$  is given by

$$I_{0+}^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds \quad (2.4)$$

provided the right-hand side is pointwise defined on  $(0, \infty)$ .

*Definition 2.4* (see [4]). The Riemann-Liouville fractional derivative of order  $\alpha > 0$  of a continuous function  $y : (0, \infty) \rightarrow R$  is given by

$$D_{0+}^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds, \quad (2.5)$$

where  $n = [\alpha] + 1$ ,  $[\alpha]$  denotes the integer part of the number  $\alpha$ , provided that the right-hand side is pointwise defined on  $(0, \infty)$ .

**Lemma 2.5** (see [29]). Let  $P$  be a normal solid cone in  $E$  and  $A : E \rightarrow E$  completely continuous and quasiadditive on lattice. Suppose that the following conditions are satisfied:

(i) there exist a positive bounded linear operator  $B_1$ ,  $u^* \in P$  and  $u_1 \in P$ , such that

$$-u^* \leq Ax \leq B_1x + u_1, \quad \forall x \in P; \quad (2.6)$$

(ii) there exist a positive bounded linear operator  $B_2$  and  $u_2 \in P$ , such that

$$Ax \geq B_2x - u_2, \quad \forall x \in (-P); \quad (2.7)$$

(iii)  $r(B_1) < 1$ ,  $r(B_2) < 1$ , where  $r(B_i)$  is the spectral radius of  $B_i$  ( $i = 1, 2$ ). Then there exists  $R_0 > 0$  such that for  $R > R_0$ , the topological degree  $\deg(I - A, B_R, \theta) = 1$ .

**Lemma 2.6** (see [29]). Let  $P$  be a normal cone of  $E$ , and  $A : E \rightarrow E$  a completely continuous operator. Suppose that there exist positive bounded linear operator  $B_0$  and  $u_0 \in P$ , such that

$$|Ax| \leq B_0|x| + u_0, \quad \forall x \in E. \quad (2.8)$$

If  $r(B_0) < 1$ , then there exists  $R_0 > 0$  such that for  $R > R_0$  the topological degree  $\deg(I - A, B_R, \theta) = 1$ .

**Lemma 2.7** (see [30]). Let  $P$  be a solid cone in  $E$  and  $A : E \rightarrow E$  a completely continuous operator with  $A = BF$ , where  $F$  is quasiadditive on lattice, and  $B$  is a positive bounded linear operator satisfying  $H$  condition. Suppose that

(i) there exist  $a_1 > r^{-1}(B)$  and  $y_1 \in P$  such that

$$Fx \geq a_1x - y_1, \quad \forall x \in P; \quad (2.9)$$

(ii) there exist  $0 < a_2 < r^{-1}(B)$  and  $y_2 \in P$  such that

$$Fx \geq a_2x - y_2, \quad \forall x \in (-P). \quad (2.10)$$

Then there exists  $R_0 > 0$  such that for  $R > R_0$  the topological degree  $\deg(I - A, B_R, \theta) = 0$ .

**Lemma 2.8** (see [30]). Let  $\Omega \subset E$  be a bounded open set which contains  $\theta$ . Suppose that  $A : \overline{\Omega} \rightarrow E$  is a completely continuous operator which has no fixed point on  $\partial\Omega$ . If

(i) there exists a positive bounded linear operator  $B$  such that

$$|Ax| \leq B_0|x|, \quad \forall x \in \partial\Omega; \quad (2.11)$$

(ii)  $r(B_0) \leq 1$ , then the topological degree  $\deg(I - A, \Omega, \theta) = 1$ .

**Lemma 2.9** (see [4]). Let  $\alpha > 0$ . If one assumes  $u \in C(0, 1) \cap L(0, 1)$ , then the fractional differential equation

$$D_{0+}^\alpha u(t) = 0, \quad (2.12)$$

has  $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_N t^{\alpha-N}$ ,  $C_i \in R$ ,  $i = 1, 2, \dots, N$ , as unique solution, where  $N$  is the smallest integer greater than or equal to  $\alpha$ .

**Lemma 2.10** (see [4]). Assume that  $u \in C(0, 1) \cap L(0, 1)$  with a fractional derivative of order  $\alpha > 0$  that belongs to  $C(0, 1) \cap L(0, 1)$ .

Then

$$I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_N t^{\alpha-N}, \quad (2.13)$$

for some  $C_i \in R$ ,  $i = 1, 2, \dots, N$ , where  $N$  is the smallest integer greater than or equal to  $\alpha$ .

In the following, we present Green's function of the fractional differential equation boundary value problem.

**Lemma 2.11.** *Given  $y \in C[0, 1]$ , the problem*

$$\begin{aligned} D_{0+}^{\alpha} u(t) + y(t) &= 0, \\ u(0) = u'(0) = u''(0) &= 0, \\ u(1) &= \lambda \int_0^{\eta} u(s) ds, \end{aligned} \quad (2.14)$$

where  $0 < t < 1$ ,  $3 < \alpha \leq 4$ ,  $0 < \eta \leq 1$ ,  $0 \leq \lambda \eta^{\alpha} / \alpha < 1$  is equivalent to

$$u(t) = \int_0^1 G(t, s) y(s) ds, \quad (2.15)$$

where

$$G(t, s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1} - (\lambda/\alpha)(\eta-s)^{\alpha} t^{\alpha-1} - (1 - (\lambda/\alpha)\eta^{\alpha})(t-s)^{\alpha-1}}{p(0)\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, s \leq \eta; \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1} - (1 - (\lambda/\alpha)\eta^{\alpha})(t-s)^{\alpha-1}}{p(0)\Gamma(\alpha)}, & 0 < \eta \leq s \leq t \leq 1; \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1} - (\lambda/\alpha)(\eta-s)^{\alpha} t^{\alpha-1}}{p(0)\Gamma(\alpha)}, & 0 \leq t \leq s \leq \eta \leq 1; \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{p(0)\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \eta \leq s. \end{cases} \quad (2.16)$$

Here,  $p(s) := 1 - (\lambda \eta^{\alpha} / \alpha)(1 - s)$ ,  $G(t, s)$  is called the Green function of BVP (2.14). Obviously,  $G(t, s)$  is continuous on  $[0, 1] \times [0, 1]$ .

*Proof.* We may apply Lemma 2.10 to reduce (2.14) to an equivalent integral equation

$$u(t) = -I_{0+}^{\alpha} y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3} + C_4 t^{\alpha-4}, \quad (2.17)$$

for some  $C_1, C_2, C_3, C_4 \in \mathbb{R}$ . Consequently, the general solution of (2.14) is

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3} + C_4 t^{\alpha-4}. \quad (2.18)$$

By  $u(0) = u'(0) = u''(0) = 0$ , we get that  $C_2 = C_3 = C_4 = 0$ . On the other hand, boundary condition  $u(1) = \lambda \int_0^\eta u(s)ds$  combining with

$$\begin{aligned}
 u(1) &= - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + C_1 \\
 \int_0^\eta u(t)dt &= - \frac{1}{\Gamma(\alpha)} \int_0^\eta \int_0^t (t-s)^{\alpha-1} y(s)ds dt + C_1 \int_0^\eta s^{\alpha-1} ds \\
 &= - \frac{1}{\Gamma(\alpha)} \int_0^\eta \int_s^\eta (t-s)^{\alpha-1} y(s)dt ds + C_1 \int_0^\eta s^{\alpha-1} ds \\
 &= - \frac{1}{\Gamma(\alpha)} \int_0^\eta \frac{(\eta-s)^\alpha}{\alpha} y(s)ds + \frac{C_1 \eta^\alpha}{\alpha}
 \end{aligned} \tag{2.19}$$

yields

$$C_1 = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-(\lambda\eta^\alpha/\alpha))} y(s)ds - \lambda \int_0^\eta \frac{(\eta-s)^\alpha}{\alpha\Gamma(\alpha)(1-(\lambda\eta^\alpha/\alpha))} y(s)ds. \tag{2.20}$$

Therefore, the unique solution of the problem (2.14) is

$$\begin{aligned}
 u(t) &= - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \frac{1}{(1-(\lambda\eta^\alpha/\alpha))} \int_0^1 \frac{(1-s)^{\alpha-1} t^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \\
 &\quad - \frac{1}{(1-(\lambda\eta^\alpha/\alpha))} \int_0^\eta \frac{(\lambda/\alpha)(\eta-s)^\alpha t^{\alpha-1}}{\Gamma(\alpha)} y(s)ds.
 \end{aligned} \tag{2.21}$$

For  $t \leq \eta$ , one has

$$\begin{aligned}
 u(t) &= - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \frac{1}{(1-(\lambda/\alpha)\eta^\alpha)} \left[ \left( \int_0^t + \int_t^\eta + \int_\eta^1 \right) \frac{(1-s)^{\alpha-1} t^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \right] \\
 &\quad - \frac{\lambda}{(1-(\lambda/\alpha)\eta^\alpha)} \left[ \left( \int_0^t + \int_t^\eta \right) \frac{(1/\alpha)(\eta-s)^\alpha t^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \right] \\
 &= \int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-1} - (\lambda/\alpha)(\eta-s)^\alpha t^{\alpha-1} - (1-(\lambda/\alpha)\eta^\alpha)(t-s)^{\alpha-1}}{(1-(\lambda/\alpha)\eta^\alpha)\Gamma(\alpha)} y(s)ds \\
 &\quad + \int_t^\eta \frac{t^{\alpha-1}(1-s)^{\alpha-1} - (\lambda/\alpha)(\eta-s)^\alpha t^{\alpha-1}}{(1-(\lambda/\alpha)\eta^\alpha)\Gamma(\alpha)} y(s)ds + \int_\eta^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-(\lambda/\alpha)\eta^\alpha)\Gamma(\alpha)} y(s)ds \\
 &= \int_0^1 G(t,s)y(s)ds.
 \end{aligned} \tag{2.22}$$

For  $t \geq \eta$ , one has

$$\begin{aligned}
 u(t) &= - \left( \int_0^\eta + \int_\eta^t \right) \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \frac{1}{(1-(\lambda/\alpha)\eta^\alpha)} \left[ \left( \int_0^\eta + \int_\eta^t + \int_t^1 \right) \frac{(1-s)^{\alpha-1} t^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \right] \\
 &\quad - \frac{\lambda}{(1-(\lambda/\alpha)\eta^\alpha)} \int_0^\eta \frac{(1/\alpha)(\eta-s)^\alpha t^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\
 &= \int_0^\eta \frac{t^{\alpha-1}(1-s)^{\alpha-1} - (\lambda/\alpha)(\eta-s)^\alpha t^{\alpha-1} - (1-(\lambda/\alpha)\eta^\alpha)(t-s)^{\alpha-1}}{(1-(\lambda/\alpha)\eta^\alpha)\Gamma(\alpha)} y(s) ds \\
 &\quad + \int_\eta^t \frac{t^{\alpha-1}(1-s)^{\alpha-1} - (1-(\lambda/\alpha)\eta^\alpha)(t-s)^{\alpha-1}}{(1-(\lambda/\alpha)\eta^\alpha)\Gamma(\alpha)} y(s) ds + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-(\lambda/\alpha)\eta^\alpha)\Gamma(\alpha)} y(s) ds \\
 &= \int_0^1 G(t,s) y(s) ds.
 \end{aligned} \tag{2.23}$$

The proof is complete.  $\square$

**Lemma 2.12.** *The function  $G(t,s)$  defined by (2.16) satisfies*

- (a1)  $G(t,s) \geq m_1 t^{\alpha-1} s(1-s)^{\alpha-1}$ , for all  $t, s \in [0,1]$ ;
- (a2)  $G(t,s) \leq M_1 t^{\alpha-1} (1-s)^{\alpha-1}$ , for all  $t, s \in [0,1]$ ;
- (a3)  $G(t,s) \leq M_1 s(1-s)^{\alpha-1}$ , for all  $t, s \in [0,1]$ ;
- (a4)  $p(s) > 0$ , and  $p(s)$  is not decreasing on  $[0,1]$ ;
- (a5)  $G(t,s) > 0$ , for all  $t, s \in (0,1)$ ,

where  $m_1 = (1-p(0))/\Gamma(\alpha)p(0)$ ,  $M_1 = (\alpha-1)/\Gamma(\alpha) + 4(\lambda/\alpha)\eta^{\alpha-1}/p(0)\Gamma(\alpha)$ .

*Proof.* For  $s \leq t$ ,  $s \leq \eta$ ,

$$\begin{aligned}
 G(t,s) &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} (\eta-s)^\alpha t^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
 &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} \eta^\alpha \left(1 - \frac{s}{\eta}\right)^\alpha t^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
 &\geq \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} \eta^\alpha (1-s)^\alpha t^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
 &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} \left[ 1 - \frac{\lambda}{\alpha} \eta^\alpha (1-s) \right] - p(0)(t-s)^{\alpha-1} \right\}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} p(s) - p(0)(t-s)^{\alpha-1} \right\} \\
&= \frac{1}{\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1} \right\} + \frac{p(s) - p(0)}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
&= \frac{1}{\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-2} t(1-s) - (t-s)^{\alpha-2} (t-s) \right\} + \frac{p(s) - p(0)}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
&\geq \frac{1}{\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-2} [t(1-s) - (t-s)] \right\} + \frac{p(s) - p(0)}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
&\geq \frac{1}{\Gamma(\alpha)} [t(1-s)]^{\alpha-1} s(1-t) + \frac{1-p(0)}{p(0)\Gamma(\alpha)} s [t(1-s)]^{\alpha-1} \\
&\geq \frac{1-p(0)}{p(0)\Gamma(\alpha)} t^{\alpha-1} s(1-s)^{\alpha-1}, \\
G(t, s) &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} (\eta-s)^\alpha t^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
&= \frac{1}{p(0)\Gamma(\alpha)} \left\{ \left( 1 - \frac{\lambda}{\alpha} \eta^\alpha + \frac{\lambda}{\alpha} \eta^\alpha \right) [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} (\eta-s)^\alpha t^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
&= \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0) [t^{\alpha-1} (1-s)^{\alpha-1} - (t-s)^{\alpha-1}] \right. \\
&\quad \left. + \frac{\lambda}{\alpha} \eta^\alpha \left[ t^{\alpha-1} (1-s)^{\alpha-1} - \left( 1 - \frac{s}{\eta} \right)^\alpha t^{\alpha-1} \right] \right\} \\
&= \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) \int_{t-s}^{t(1-s)} x^{\alpha-2} dx \right. \\
&\quad \left. + \frac{\lambda}{\alpha} \eta^\alpha \left[ t^{\alpha-1} (1-s)^{\alpha-1} - \left( 1 - \frac{s}{\eta} \right)^\alpha t^{\alpha-1} (1-s)^{\alpha-1} \right] \right\} \\
&\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) t^{\alpha-2} (1-s)^{\alpha-2} s(1-t) + \frac{\lambda}{\alpha} \eta^\alpha t^{\alpha-1} (1-s)^{\alpha-1} \left[ 1 - \left( 1 - \frac{s}{\eta} \right)^\alpha \right] \right\} \\
&\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) t^{\alpha-2} (1-s)^{\alpha-2} s(1-t) + \frac{\lambda}{\alpha} \eta^\alpha t^{\alpha-1} (1-s)^{\alpha-1} \left[ 1 - \left( 1 - \frac{s}{\eta} \right)^4 \right] \right\} \\
&\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) t^{\alpha-1} (1-s)^{\alpha-1} + \frac{\lambda}{\alpha} \eta^\alpha t^{\alpha-1} (1-s)^{\alpha-1} \right. \\
&\quad \left. \times \left[ 1 - \left( 1 - \frac{s}{\eta} \right) \right] \left[ 1 + \left( 1 - \frac{s}{\eta} \right) \right] \left[ 1 + \left( 1 - \frac{s}{\eta} \right)^2 \right] \right\} \\
&\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) t^{\alpha-1} (1-s)^{\alpha-1} + 4 \frac{\lambda}{\alpha} \eta^{\alpha-1} t^{\alpha-1} (1-s)^{\alpha-1} \right\} \\
&= \left[ \frac{\alpha-1}{\Gamma(\alpha)} + \frac{4(\lambda/\alpha) \eta^{\alpha-1}}{p(0)\Gamma(\alpha)} \right] t^{\alpha-1} (1-s)^{\alpha-1},
\end{aligned}$$

$$\begin{aligned}
G(t, s) &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} (\eta-s)^\alpha t^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
&= \frac{1}{p(0)\Gamma(\alpha)} \left\{ \left( 1 - \frac{\lambda}{\alpha} \eta^\alpha + \frac{\lambda}{\alpha} \eta^\alpha \right) [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} (\eta-s)^\alpha t^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
&= \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0) \left[ t^{\alpha-1} (1-s)^{\alpha-1} - (t-s)^{\alpha-1} \right] + \frac{\lambda}{\alpha} \eta^\alpha \left[ t^{\alpha-1} (1-s)^{\alpha-1} - \left( 1 - \frac{s}{\eta} \right)^\alpha t^{\alpha-1} \right] \right\} \\
&\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) \int_{t-s}^{t(1-s)} x^{\alpha-2} dx + \frac{\lambda}{\alpha} \eta^\alpha \left[ t^{\alpha-1} (1-s)^{\alpha-1} - \left( 1 - \frac{s}{\eta} \right)^\alpha t^{\alpha-1} (1-s)^{\alpha-1} \right] \right\} \\
&\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) t^{\alpha-2} (1-s)^{\alpha-2} s(1-t) + \frac{\lambda}{\alpha} \eta^\alpha t^{\alpha-1} (1-s)^{\alpha-1} \left[ 1 - \left( 1 - \frac{s}{\eta} \right)^\alpha \right] \right\} \\
&\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) t^{\alpha-2} (1-s)^{\alpha-2} s(1-t) + \frac{\lambda}{\alpha} \eta^\alpha t^{\alpha-1} (1-s)^{\alpha-1} \left[ 1 - \left( 1 - \frac{s}{\eta} \right)^4 \right] \right\} \\
&\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) t^{\alpha-2} (1-s)^{\alpha-2} s(1-t) + \frac{\lambda}{\alpha} \eta^\alpha t^{\alpha-1} (1-s)^{\alpha-1} \right. \\
&\quad \left. \times \left[ 1 - \left( 1 - \frac{s}{\eta} \right) \right] \left[ 1 + \left( 1 - \frac{s}{\eta} \right) \right] \left[ 1 + \left( 1 - \frac{s}{\eta} \right)^2 \right] \right\} \\
&\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) (1-s)^{\alpha-1} s + 4 \frac{\lambda}{\alpha} \eta^{\alpha-1} s(1-s)^{\alpha-1} \right\} \\
&= \left[ \frac{\alpha-1}{\Gamma(\alpha)} + \frac{4(\lambda/\alpha) \eta^{\alpha-1}}{p(0)\Gamma(\alpha)} \right] s(1-s)^{\alpha-1}.
\end{aligned}$$

(2.24)

For  $\eta \leq s \leq t$ ,

$$\begin{aligned}
G(t, s) &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
&\geq \frac{p(s)}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} - \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} \\
&= \frac{1}{\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-2} t(1-s) - (t-s)^{\alpha-2} (t-s) \right\} + \frac{p(s) - p(0)}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
&\geq \frac{1}{\Gamma(\alpha)} [t(1-s)]^{\alpha-1} s(1-t) + \frac{1-p(0)}{p(0)\Gamma(\alpha)} s [t(1-s)]^{\alpha-1} \\
&\geq \frac{1-p(0)}{p(0)\Gamma(\alpha)} t^{\alpha-1} s(1-s)^{\alpha-1},
\end{aligned}$$

$$\begin{aligned}
G(t, s) &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
&= \frac{1}{p(0)\Gamma(\alpha)} \left\{ \left( 1 - \frac{\lambda}{\alpha} \eta^\alpha + \frac{\lambda}{\alpha} \eta^\alpha \right) [t(1-s)]^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
&= \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0) \left[ t^{\alpha-1} (1-s)^{\alpha-1} - (t-s)^{\alpha-1} \right] + \frac{\lambda}{\alpha} \eta^\alpha [t(1-s)]^{\alpha-1} \right\} \\
&\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) \int_{t-s}^{t(1-s)} x^{\alpha-2} dx + \frac{\lambda}{\alpha} \eta^{\alpha-1} s t^{\alpha-1} (1-s)^{\alpha-1} \right\} \\
&\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) t^{\alpha-2} (1-s)^{\alpha-2} s (1-t) + \frac{\lambda}{\alpha} \eta^{\alpha-1} s t^{\alpha-1} (1-s)^{\alpha-1} \right\} \\
&\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) t^{\alpha-1} (1-s)^{\alpha-1} + \frac{\lambda}{\alpha} \eta^{\alpha-1} t^{\alpha-1} (1-s)^{\alpha-1} \right\} \\
&\leq \left[ \frac{\alpha-1}{\Gamma(\alpha)} + \frac{4(\lambda/\alpha) \eta^{\alpha-1}}{p(0)\Gamma(\alpha)} \right] t^{\alpha-1} (1-s)^{\alpha-1}, \\
G(t, s) &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
&= \frac{1}{p(0)\Gamma(\alpha)} \left\{ \left( 1 - \frac{\lambda}{\alpha} \eta^\alpha + \frac{\lambda}{\alpha} \eta^\alpha \right) [t(1-s)]^{\alpha-1} - p(0)(t-s)^{\alpha-1} \right\} \\
&= \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0) \left[ t^{\alpha-1} (1-s)^{\alpha-1} - (t-s)^{\alpha-1} \right] + \frac{\lambda}{\alpha} \eta^\alpha [t(1-s)]^{\alpha-1} \right\} \\
&\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) \int_{t-s}^{t(1-s)} x^{\alpha-2} dx + \frac{\lambda}{\alpha} \eta^{\alpha-1} s t^{\alpha-1} (1-s)^{\alpha-1} \right\} \\
&\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) t^{\alpha-2} (1-s)^{\alpha-2} s (1-t) + \frac{\lambda}{\alpha} \eta^{\alpha-1} s t^{\alpha-1} (1-s)^{\alpha-1} \right\} \\
&\leq \frac{1}{p(0)\Gamma(\alpha)} \left\{ p(0)(\alpha-1) (1-s)^{\alpha-1} s + \frac{\lambda}{\alpha} \eta^{\alpha-1} s (1-s)^{\alpha-1} \right\} \\
&\leq \left[ \frac{\alpha-1}{\Gamma(\alpha)} + \frac{4(\lambda/\alpha) \eta^{\alpha-1}}{p(0)\Gamma(\alpha)} \right] s (1-s)^{\alpha-1}.
\end{aligned} \tag{2.25}$$

For  $t \leq s \leq \eta$ ,

$$\begin{aligned}
G(t, s) &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} (\eta-s)^\alpha t^{\alpha-1} \right\} \\
&= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} \eta^\alpha \left( 1 - \frac{s}{\eta} \right)^\alpha t^{\alpha-1} \right\}
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} \eta^\alpha (1-s)^\alpha t^{\alpha-1} \right\} \\
&= \frac{1}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \left[ 1 - \frac{\lambda}{\alpha} \eta^\alpha (1-s) \right] \\
&= \frac{p(0) + p(s) - p(0)}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
&= \frac{1}{\Gamma(\alpha)} [t(1-s)]^{\alpha-1} + \frac{p(s) - p(0)}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
&\geq \frac{1 - p(0)}{p(0)\Gamma(\alpha)} t^{\alpha-1} s (1-s)^{\alpha-1}, \\
G(t, s) &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} (\eta - s)^\alpha t^{\alpha-1} \right\} \\
&\leq \frac{1}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
&\leq \left[ \frac{\alpha - 1}{\Gamma(\alpha)} + \frac{4(\lambda/\alpha)\eta^{\alpha-1}}{p(0)\Gamma(\alpha)} \right] t^{\alpha-1} (1-s)^{\alpha-1}, \\
G(t, s) &= \frac{1}{p(0)\Gamma(\alpha)} \left\{ [t(1-s)]^{\alpha-1} - \frac{\lambda}{\alpha} (\eta - s)^\alpha t^{\alpha-1} \right\} \\
&\leq \frac{1}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
&\leq \left[ \frac{\alpha - 1}{\Gamma(\alpha)} + \frac{4(\lambda/\alpha)\eta^{\alpha-1}}{p(0)\Gamma(\alpha)} \right] s (1-s)^{\alpha-1}.
\end{aligned} \tag{2.26}$$

For  $\eta \leq s, t \leq s$ ,

$$\begin{aligned}
G(t, s) &= \frac{1}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
&\geq \frac{1 - p(0)}{p(0)\Gamma(\alpha)} t^{\alpha-1} s (1-s)^{\alpha-1}, \\
G(t, s) &= \frac{1}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
&\leq \left[ \frac{\alpha - 1}{\Gamma(\alpha)} + \frac{4(\lambda/\alpha)\eta^{\alpha-1}}{p(0)\Gamma(\alpha)} \right] t^{\alpha-1} (1-s)^{\alpha-1},
\end{aligned}$$

$$\begin{aligned}
G(t, s) &= \frac{1}{p(0)\Gamma(\alpha)} [t(1-s)]^{\alpha-1} \\
&\leq \left[ \frac{\alpha-1}{\Gamma(\alpha)} + \frac{4(\lambda/\alpha)\eta^{\alpha-1}}{p(0)\Gamma(\alpha)} \right] s(1-s)^{\alpha-1}.
\end{aligned} \tag{2.27}$$

From above, (a1), (a2), (a3), (a5) are complete. Clearly, (a4) is true. The proof is complete.  $\square$

### 3. Main Results and Proof

Let  $E = C[0, 1]$ ,  $\|u\| = \max_{t \in [0, 1]} |u(t)|$ ,  $P = \{u \in C[0, 1] \mid u(t) \geq 0, t \in [0, 1]\}$ . Obviously,  $P$  is a normal solid cone with normal constant 1 in Banach space  $E$ , and  $E$  is a lattice under the partial ordering  $\leq$  which is deduced by  $P$ .

Throughout this paper, we always assume that

(H1)  $f : [0, 1] \times R \rightarrow R$  is continuous;

(H2)  $h : (0, 1) \rightarrow [0, +\infty)$  is continuous and not identical zero on any closed subinterval of  $[0, +\infty)$  with  $0 < \int_0^1 h(t)t(1-t)^{\alpha-1} dt < +\infty$ .

*Remark 3.1.* In the assumption (H1), it is not required that  $f(t, u) \geq 0, \forall u \geq 0$ .

Define operators  $A$  and  $B$  as follows:

$$(Au)(t) = \int_0^1 G(t, s)h(s)f(s, u(s))ds, \quad (Bu)(t) = \int_0^1 G(t, s)h(s)u(s)ds, \quad t \in [0, 1]; \tag{3.1}$$

$$(Fu)(t) = f(t, u(t)). \tag{3.2}$$

*Remark 3.2.* By Lemma 2.12, (H1) and (H2), it is easy to see that operators  $A$  and  $B$  defined by (3.1) are well defined.

**Lemma 3.3.** Suppose that (H2) holds, then the spectral radius  $r(B) \neq 0$  and  $B$  has a positive eigenfunction corresponding to the first eigenvalue  $\lambda_1 = (r(B))^{-1}$ .

*Proof.* By Lemma 2.11, (H2), similar to the proof of Lemma 4.4 in [28], the proof can be easily given. We omit the details.  $\square$

By standard argument, we have the following.

**Lemma 3.4.** Suppose that (H1) and (H2) hold, then  $A : E \rightarrow E$  is completely continuous.

**Theorem 3.5.** Suppose that conditions (H1) and (H2) are satisfied. If there exists a constant  $b$  such that

$$f(t, u) \geq -b, \quad \forall t \in [0, 1], \quad u \geq 0; \quad (3.3)$$

$$\lim_{|u| \rightarrow +\infty} \sup \frac{f(t, u)}{u} < \lambda_1 \quad \text{uniformly for } t \in [0, 1], \quad (3.4)$$

where  $\lambda_1$  is the first eigenvalue of  $B$  defined by (3.1), then BVP (1.1) has at least one solution.

*Proof.* From Lemma 3.4, we know that  $A : E \rightarrow E$  is completely continuous. By (3.4), there exist  $R_0 > 0$ ,  $\lambda_1 > \varepsilon > 0$  such that

$$\begin{aligned} f(t, u) &\leq (\lambda_1 - \varepsilon)u, \quad t \in [0, 1], \quad u \geq R_0, \\ f(t, u) &\geq (\lambda_1 - \varepsilon)u, \quad t \in [0, 1], \quad u \leq -R_0. \end{aligned} \quad (3.5)$$

This implies

$$\begin{aligned} f(t, u) &\leq (\lambda_1 - \varepsilon)u + M_2, \quad t \in [0, 1], \quad u \geq 0, \\ f(t, u) &\geq (\lambda_1 - \varepsilon)u - M_2, \quad t \in [0, 1], \quad u \leq 0, \end{aligned} \quad (3.6)$$

where  $M_2 = \max_{0 \leq t \leq 1, |u| \leq R_0} |f(t, u)|$ . Set

$$u^*(t) = b \int_0^1 G(t, s)h(s)ds, \quad u_1(t) = M_2 \int_0^1 G(t, s)h(s)ds. \quad (3.7)$$

Obviously,  $u^*, u_1 \in P$ . Let  $B_0 = (\lambda_1 - \varepsilon)B$ , where  $B$  is defined as (3.1). It is clear that  $B_0 : P \rightarrow P$  is a positive bounded linear operator and

$$r(B_0) = r((\lambda_1 - \varepsilon)B) < \lambda_1 r(B) = 1. \quad (3.8)$$

It follows from (3.3), (3.6) that

$$\begin{aligned} -u^* &\leq Au \leq B_0u + u_1, \quad \forall u \in P, \\ Au &\geq B_0u - u_1, \quad \forall u \in -P. \end{aligned} \quad (3.9)$$

It follows from Lemma 2.5 that there exists  $R > 0$  big enough such that

$$\deg(I - A, B_R, \theta) = 1, \quad (3.10)$$

which means that  $A$  has at least one solution.  $\square$

**Theorem 3.6.** Suppose that (H1) and (H2) hold. In addition,

$$\lim_{|u| \rightarrow +\infty} \sup \frac{|f(t, u)|}{|u|} < \lambda_1 \quad \text{uniformly for } t \in [0, 1]. \quad (3.11)$$

Then BVP (1.1) has at least one solution.

*Proof.* Similar to the proof of (3.9), we arrive at

$$|Au| \leq B|u| + u_1, \quad \forall u \in C[0, 1]. \quad (3.12)$$

By Lemma 2.6, there exists  $R > 0$  big enough such that

$$\deg(I - A, B_R, \theta) = 1, \quad (3.13)$$

which shows that  $A$  has at least one solution.  $\square$

**Theorem 3.7.** Suppose that conditions (H1) and (H2) are satisfied. If

$$\liminf_{u \rightarrow +\infty} \frac{f(t, u)}{u} > \lambda_1, \quad \text{uniformly on } t \in [0, 1]; \quad (3.14)$$

$$\limsup_{u \rightarrow -\infty} \frac{f(t, u)}{u} < \lambda_1, \quad \text{uniformly on } t \in [0, 1]; \quad (3.15)$$

$$\limsup_{u \rightarrow 0} \left| \frac{f(t, u)}{u} \right| < \lambda_1, \quad \text{uniformly on } t \in [0, 1], \quad (3.16)$$

where  $\lambda_1$  is the first eigenvalue of  $B$  defined by (3.1), then the singular BVP (1.1) has at least one nontrivial solution.

*Proof.* Let  $E = C[0, 1]$ , and let  $A$ ,  $B$  and  $F$  be defined by (3.1) and (3.2), respectively. Obviously, by remark 3.1 in [27, 28],  $F : E \rightarrow E$  is continuous and quasiadditive on lattice, and  $A = BF$ . By Lemma 3.4, we know that  $A : E \rightarrow E$  is completely continuous.

It follows from (3.14) and (3.15) that there exist constants  $\varepsilon > 0$  and  $R_0 > 0$  such that

$$\begin{aligned} f(t, u) &\geq (\lambda_1 + \varepsilon)u, \quad t \in [0, 1], \quad u \geq R_0; \\ f(t, u) &\geq (\lambda_1 - \varepsilon)u, \quad t \in [0, 1], \quad u \leq -R_0. \end{aligned} \quad (3.17)$$

Therefore, there exists a constant  $M_3 > 0$  such that

$$\begin{aligned} f(t, u) &\geq (\lambda_1 + \varepsilon)u - M_3, \quad t \in [0, 1], \quad u \geq 0; \\ f(t, u) &\geq (\lambda_1 - \varepsilon)u - M_3, \quad t \in [0, 1], \quad u \leq 0. \end{aligned} \quad (3.18)$$

From (3.18), one can see that (2.9) and (2.10) hold for  $a_1 = \lambda_1 + \varepsilon > \lambda_1 = r^{-1}(B)$  and  $a_2 = \lambda_1 - \varepsilon < \lambda_1 = r^{-1}(B)$ .

Next, we are in position to show that  $B$  satisfies  $H$  condition. Let

$$(B^*u)(t) = \int_0^1 G(s, t)h(t)u(s)ds. \quad (3.19)$$

By Lemma 3.3, we know that  $r(B^*) = r(B) \neq 0$ , and there exists  $g^* \in P^* \setminus \{\theta\}$ , such that

$$g^* = r^{-1}(B)B^*g^*. \quad (3.20)$$

By Lemma 2.12, we have

$$\begin{aligned} g^*(s) &= r^{-1}(B)B^*g^* \\ &= r^{-1}(B) \int_0^1 G(t, s)h(s)g^*(t)dt \\ &\geq m_1 r^{-1}(B) \int_0^1 t^{\alpha-1} s(1-s)^{\alpha-1} g^*(t)dt \\ &= \left[ m_1 r^{-1}(B) \int_0^1 t^{\alpha-1} g^*(t)dt \right] s(1-s)^{\alpha-1}, \quad \forall s \in [0, 1]. \end{aligned} \quad (3.21)$$

Therefore, for  $u \in P$ , we get by Lemma 2.12, (3.20) and (3.21) that

$$\begin{aligned} \int_0^1 g^*(t)(Bu)(t)dt &= \iint_0^1 g^*(t)G(t, s)h(s)u(s)ds dt \\ &= \int_0^1 \left[ \int_0^1 g^*(t)G(t, s)h(s)dt \right] u(s)ds \\ &= r(B) \int_0^1 g^*(s)u(s)ds \\ &\geq r(B)m_1 r^{-1}(B) \int_0^1 t^{\alpha-1} g^*(t)dt \cdot \int_0^1 s(1-s)^{\alpha-1} u(s)ds \\ &\geq \frac{m_1}{M_1} \int_0^1 t^{\alpha-1} g^*(t)dt \cdot M_1 \int_0^1 s(1-s)^{\alpha-1} u(s)ds \\ &\geq \frac{m_1}{M_1} \int_0^1 t^{\alpha-1} g^*(t)dt \cdot \|Bu\|. \end{aligned} \quad (3.22)$$

This means that  $g^*(Bu) \geq \delta \|Bu\|$ , where  $\delta = (m_1/M_1) \int_0^1 t^{\alpha-1} g^*(t)dt$ . So,  $B(P) \subset P(g^*, \delta)$ . Thus, we have shown that  $B$  satisfies  $H$  condition. By Lemma 2.7, we know that there exists  $R > R_0$  such that

$$\deg(I - A, T_R, \theta) = 0. \quad (3.23)$$

On the other hand, by (3.16), we know that there exists  $0 < r < R_0$  such that

$$|f(t, u)| \leq (\lambda_1 - \varepsilon)|u|, \quad \forall t \in [0, 1], \quad |u| \leq r, \quad (3.24)$$

which implies that

$$|Au| \leq B_0|u|, \quad \forall u \in \partial T_r, \quad (3.25)$$

where  $B_0 = (\lambda_1 - \varepsilon)B$ , and  $r(B_0) = (\lambda_1 - \varepsilon)r(B) < 1$ . By virtue of Lemma 2.8, we get that

$$\deg(I - A, T_r, \theta) = 1. \quad (3.26)$$

It follows from the additivity of the topology degree and Lemma 2.11 that  $A$  has at least one nontrivial fixed point in  $\bar{T}_R \setminus T_r$ . That is, BVP (1.1) has at least one nontrivial solution.  $\square$

#### 4. An Example

Consider the following fractional differential equations with integral boundary conditions:

$$\begin{aligned} D_{0+}^{7/2} u(t) + \lambda_0 t^{p-1} (1-t)^{q-1} \left[ \left( \sqrt{u^2 + 1} - 1 \right) - \sin\left(u + \frac{\pi}{2}\right) \right] &= 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) &= 0, \\ u(1) &= \int_0^{1/2} u(s) ds, \end{aligned} \quad (4.1)$$

where  $0 < p, q < 1$ ,  $\lambda_0 < \lambda_1$ ,  $\lambda_1$  is the first eigenvalue of operator  $B$ . It is easy to see that (H1) and (H2) hold for  $h(t) = t^{p-1}(1-t)^{q-1}$ ,  $f(t, u) = \lambda_0(\sqrt{u^2 + 1} - 1) - \sin(u + \pi/2)$ . By Lemma 3.3, we get that  $\lambda_1 > 0$ . Obviously,  $f(u) \geq -1$  is bounded below and sign-changing for  $u \geq 0$ . By direct computation, we have  $\lim_{u \rightarrow +\infty} (f(u)/u) = \lambda_0 < \lambda_1$ ,  $\lim_{u \rightarrow -\infty} (f(u)/u) = -\lambda_0 < \lambda_1$ . Thus (3.3) and (3.4) in Theorem 3.5 hold. It follows from Theorem 3.5 that BVP (4.1) has one solution.

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