Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 302635, 9 pages
doi:10.1155/2012/302635

Research Article

Optimal Bounds for Neuman-Sándor Mean in Terms of the Convex Combinations of Harmonic, Geometric, Quadratic, and Contraharmonic Means

Tie-Hong Zhao,[1] Yu-Ming Chu,[2] and Bao-Yu Liu[3]

[1] *Department of Mathematics, Hangzhou Normal University, Hangzhou 313036, China*
[2] *School of Mathematics and Computation Sciences, Hunan City University, Yiyang 413000, China*
[3] *School of Science, Hangzhou Dianzi University, Hangzhou 310018, China*

Correspondence should be addressed to Yu-Ming Chu, chuyuming2005@yahoo.com.cn

Received 15 October 2012; Accepted 5 December 2012

Academic Editor: Julian López-Gómez

We present the best possible lower and upper bounds for the Neuman-Sándor mean in terms of the convex combinations of either the harmonic and quadratic means or the geometric and quadratic means or the harmonic and contraharmonic means.

1. Introduction

For $a, b > 0$ with $a \neq b$, the Neuman-Sándor mean $M(a,b)$ [1] is defined by

$$M(a,b) = \frac{a-b}{2\sinh^{-1}[(a-b)/(a+b)]}, \tag{1.1}$$

where $\sinh^{-1}(x) = \log(x + \sqrt{1+x^2})$ is the inverse hyperbolic sine function.

Recently, the theory of bivariate means have been the subject of intensive research [2–17]. In particular, many remarkable inequalities for the Neuman-Sándor mean $M(a,b)$ can be found in the literature [1, 18–20].

Let $H(a,b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (b-a)/(\log b - \log a)$, $P(a,b) = (a-b)/(4\arctan\sqrt{a/b} - \pi)$, $A(a,b) = (a+b)/2$, $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$ and $C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic, geometric, logarithmic, first Seiffert, arithmetic, second Seiffert, quadratic, and contraharmonic means of a and b, respectively. Then it is well known that the inequalities

$$\begin{aligned} H(a,b) &< G(a,b) < L(a,b) < P(a,b) < A(a,b) \\ &< M(a,b) < T(a,b) < Q(a,b) < C(a,b) \end{aligned} \tag{1.2}$$

hold for all $a, b > 0$ with $a \neq b$.

In [1, 18], Neuman and Sándor proved that the double inequalities

$$\begin{gathered} A(a,b) < M(a,b) < T(a,b), \\ P(a,b)M(a,b) < A^2(a,b), \\ A(a,b)T(a,b) < M^2(a,b) < \frac{A^2(a,b) + T^2(a,b)}{2} \end{gathered} \tag{1.3}$$

hold for all $a, b > 0$ with $a \neq b$.

Let $0 < a, b < 1/2$ with $a \neq b$, $a' = 1 - a$ and $b' = 1 - b$. Then the following Ky Fan inequalities

$$\frac{G(a,b)}{G(a',b')} < \frac{L(a,b)}{L(a',b')} < \frac{P(a,b)}{P(a',b')} < \frac{A(a,b)}{A(a',b')} < \frac{M(a,b)}{M(a',b')} < \frac{T(a,b)}{T(a',b')} \tag{1.4}$$

were presented in [1].

The double inequality $L_{p_0}(a,b) < M(a,b) < L_2(a,b)$ for all $a, b > 0$ with $a \neq b$ was established by Li et al. in [19], where $L_p(a,b) = [(b^{p+1} - a^{p+1})/((p+1)(b-a))]^{1/p}$ $(p \neq -1, 0)$, $L_0(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$ and $L_{-1}(a,b) = (b-a)/(\log b - \log a)$ is the pth generalized logarithmic mean of a and b, and $p_0 = 1.843\ldots$ is the unique solution of the equation $(p+1)^{1/p} = 2\log(1+\sqrt{2})$.

Neuman [20] proved that the double inequalities

$$\begin{gathered} \alpha Q(a,b) + (1-\alpha)A(a,b) < M(a,b) < \beta Q(a,b) + (1-\beta)A(a,b), \\ \lambda Q(a,b) + (1-\lambda)A(a,b) < M(a,b) < \mu Q(a,b) + (1-\mu)A(a,b) \end{gathered} \tag{1.5}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq 1 - \log(\sqrt{2}+1)/[(\sqrt{2}-1)\log(\sqrt{2}+1)] = 0.3249\ldots$, $\beta \geq 1/3$, $\lambda \leq 1 - \log(\sqrt{2}+1)/\log(\sqrt{2}+1) = 0.1345\ldots$ and $\mu \geq 1/6$.

The main purpose of this paper is to find the least values α_1, α_2, α_3, and the greatest values β_1, β_2, β_3, such that the double inequalities

$$\begin{gathered} \alpha_1 H(a,b) + (1-\alpha_1)Q(a,b) < M(a,b) < \beta_1 H(a,b) + (1-\beta_1)Q(a,b), \\ \alpha_2 G(a,b) + (1-\alpha_2)Q(a,b) < M(a,b) < \beta_2 G(a,b) + (1-\beta_2)Q(a,b), \\ \alpha_3 H(a,b) + (1-\alpha_3)C(a,b) < M(a,b) < \beta_3 H(a,b) + (1-\beta_3)C(a,b) \end{gathered} \tag{1.6}$$

hold true for all $a, b > 0$ with $a \neq b$.

Our main results are presented in Theorems 1.1–1.3.

Theorem 1.1. *The double inequality*

$$\alpha_1 H(a,b) + (1-\alpha_1)Q(a,b) < M(a,b) < \beta_1 H(a,b) + (1-\beta_1)Q(a,b) \tag{1.7}$$

holds for all $a,b>0$ *with* $a \neq b$ *if and only if* $\alpha_1 \geq 2/9 = 0.2222\ldots$ *and* $\beta_1 \leq 1 - 1/[\sqrt{2}\log(1+\sqrt{2})] = 0.1977\ldots$.

Theorem 1.2. *The double inequality*

$$\alpha_2 G(a,b) + (1-\alpha_2)Q(a,b) < M(a,b) < \beta_2 G(a,b) + (1-\beta_2)Q(a,b) \tag{1.8}$$

holds for all $a,b>0$ *with* $a \neq b$ *if and only if* $\alpha_2 \geq 1/3 = 0.3333\ldots$ *and* $\beta_2 \leq 1 - 1/[\sqrt{2}\log(1+\sqrt{2})] = 0.1977\ldots$.

Theorem 1.3. *The double inequality*

$$\alpha_3 H(a,b) + (1-\alpha_3)C(a,b) < M(a,b) < \beta_3 H(a,b) + (1-\beta_3)C(a,b) \tag{1.9}$$

holds for all $a,b>0$ *with* $a \neq b$ *if and only if* $\alpha_3 \geq 1 - 1/[2\log(1+\sqrt{2})] = 0.4327\ldots$ *and* $\beta_3 \leq 5/12 = 0.4166\ldots$.

2. Lemmas

In order to prove our main results we need two Lemmas, which we present in this section.

Lemma 2.1 (see [21, Lemma 1.1]). *Suppose that the power series* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ *have the radius of convergence* $r > 0$ *and* $b_n > 0$ *for all* $n \in \{0,1,2,\ldots\}$. *Let* $h(x) = f(x)/g(x)$, *then the following statements are true.*

(1) *If the sequence* $\{a_n/b_n\}_{n=0}^{\infty}$ *is (strictly) increasing (decreasing), then* $h(x)$ *is also (strictly) increasing (decreasing) on* $(0,r)$.

(2) *If the sequence* $\{a_n/b_n\}$ *is (strictly) increasing (decreasing) for* $0 < n \leq n_0$ *and (strictly) decreasing (increasing) for* $n > n_0$, *then there exists* $x_0 \in (0,r)$ *such that* $h(x)$ *is (strictly) increasing (decreasing) on* $(0,x_0)$ *and (strictly) decreasing (increasing) on* (x_0,r).

Lemma 2.2. *Let* $p \in (0,1)$, $\lambda_0 = 1 - 1/[\sqrt{2}\log(1+\sqrt{2})] = 0.1977\ldots$ *and*

$$f_p(x) = \sinh^{-1}(x) - \frac{x}{\sqrt{1+x^2} - p\left(\sqrt{1+x^2} - \sqrt{1-x^2}\right)}. \tag{2.1}$$

Then $f_{1/3}(x) < 0$ *and* $f_{\lambda_0}(x) > 0$ *for all* $x \in (0,1)$.

Proof. From (2.1), one has

$$f_p(0) = 0, \tag{2.2}$$

$$f_p(1) = \log\left(1+\sqrt{2}\right) - \frac{1}{\sqrt{2}(1-p)}, \tag{2.3}$$

$$f_p'(x) = \frac{g_p(x)}{\sqrt{1-x^4}\left(\sqrt{1+x^2} + p\left(\sqrt{1-x^2} - \sqrt{1+x^2}\right)\right)^2}, \tag{2.4}$$

where

$$g_p(x) = \sqrt{1-x^2}\left(\sqrt{1+x^2} + p\left(\sqrt{1-x^2} - \sqrt{1+x^2}\right)\right)^2 - \sqrt{1-x^2} - p\left(\sqrt{1+x^2} - \sqrt{1-x^2}\right). \tag{2.5}$$

We divide the proof into two cases.

Case 1. ($p = 1/3$). Then (2.5) leads to

$$g_{1/3}(0) = 0, \qquad g_{1/3}(1) = -\frac{\sqrt{2}}{3} < 0, \tag{2.6}$$

$$g'_{1/3}(x) = \frac{x^3}{\sqrt{1-x^4}} h_{1/3}(x), \tag{2.7}$$

where

$$h_{1/3}(x) = \frac{14}{9\left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)} - \left(\sqrt{1+x^2} + \sqrt{1-x^2}\right) - \frac{\sqrt{1-x^2}}{3}. \tag{2.8}$$

We clearly see that the function $\sqrt{1+x^2} + \sqrt{1-x^2}$ is strictly decreasing in $(0,1)$. Then from (2.8), we get

$$h_{1/3}(x) < h_{1/3}(1) = -\frac{2\sqrt{2}}{9} < 0 \tag{2.9}$$

for $x \in (0,1)$.

Therefore, $f_{1/3}(x) < 0$ for all $x \in (0,1)$ follows easily from (2.2), (2.4), (2.6), (2.7), and (2.9).

Case 2. ($p = \lambda_0$). Then (2.3) and (2.5) yield

$$f_{\lambda_0}(1) = g_{\lambda_0}(0) = 0, \qquad g_{\lambda_0}(1) = -\sqrt{2}\lambda_0 < 0, \tag{2.10}$$

$$g'_{\lambda_0}(x) = \frac{x}{\sqrt{1-x^4}} h_{\lambda_0}(x), \tag{2.11}$$

where

$$h_{\lambda_0}(x) = \left[\left(2 - 3\lambda_0 - 2\lambda_0^2\right) - (3 - 6\lambda_0)x^2\right]\sqrt{1+x^2} - \left[\left(3\lambda_0 - 2\lambda_0^2\right) + \left(6\lambda_0 - 6\lambda_0^2\right)x^2\right]\sqrt{1-x^2}. \tag{2.12}$$

We divide the discussion of this case into two subcases, and all computations are carried out using MATHEMATICA software.

Subcase A. $x \in (0.9, 1)$. Then from (2.12) and the fact that

$$\left(2 - 3\lambda_0 - 2\lambda_0^2\right) - (3 - 6\lambda_0)x^2 < \left(2 - 3\lambda_0 - 2\lambda_0^2\right) - (3 - 6\lambda_0) \times (0.9)^2 = -0.1404\cdots < 0, \tag{2.13}$$

we know that

$$h_{\lambda_0}(x) < 0 \tag{2.14}$$

for $x \in (0.9, 1)$.

Subcase B. $x \in (0, 0.9]$. Then from (2.12), one has

$$\begin{gathered} h_{\lambda_0}(0) = 0.8137\cdots > 0, \qquad h_{\lambda_0}(0.9) = -0.7494\cdots < 0, \\ h'_{\lambda_0}(x) = \frac{x}{\sqrt{1-x^4}}\mu(x), \end{gathered} \tag{2.15}$$

where

$$\mu(x) = \left[\left(18\lambda_0 - 18\lambda_0^2\right)x^2 - \left(9\lambda_0 - 10\lambda_0^2\right)\right]\sqrt{1+x^2} - \left[(9-18\lambda_0)x^2 + \left(4 - 9\lambda_0 + 2\lambda_0^2\right)\right]\sqrt{1-x^2}. \tag{2.16}$$

We conclude that

$$\mu(x) < 0 \tag{2.17}$$

for all $x \in (0, 0.9]$. Indeed, if $x \in (0, 1/2)$, then (2.17) follows from (2.16) and the inequality

$$\left(18\lambda_0 - 18\lambda_0^2\right)x^2 - \left(9\lambda_0 - 10\lambda_0^2\right) < 5.5\lambda_0^2 - 4.5\lambda_0 = -0.6747\cdots < 0. \tag{2.18}$$

If $x \in [1/2, 0.9]$, then (2.17) follows from (2.16) and the inequalities

$$\begin{aligned} &\left(18\lambda_0 - 18\lambda_0^2\right)x^2 - \left(9\lambda_0 - 10\lambda_0^2\right) \\ &\quad \le \left(18\lambda_0 - 18\lambda_0^2\right) \times (0.9)^2 - \left(9\lambda_0 - 10\lambda_0^2\right) \\ &\quad = 5.58\lambda_0 - 4.58\lambda_0^2 = 0.9242\ldots, \\ &(9-18\lambda_0)x^2 + \left(4 - 9\lambda_0 + 2\lambda_0^2\right) \ge \frac{1}{4}(9 - 18\lambda_0) + \left(4 - 9\lambda_0 + 2\lambda_0^2\right) \\ &\quad = 6.25 - 13.5\lambda_0 + 2\lambda_0^2 = 3.6589\ldots, \\ &\left[\left(18\lambda_0 - 18\lambda_0^2\right)x^2 - \left(9\lambda_0 - 10\lambda_0^2\right)\right]\sqrt{1+x^2} - \left[(9-18\lambda_0)x^2 + \left(4 - 9\lambda_0 + 2\lambda_0^2\right)\right]\sqrt{1-x^2} \\ &\quad \le \left(5.58\lambda_0 - 4.58\lambda_0^2\right)\sqrt{1+(0.9)^2} - \left(6.25 - 13.5\lambda_0 + 2\lambda_0^2\right)\sqrt{1-(0.9)^2} \\ &\quad = -0.3514\cdots < 0. \end{aligned} \tag{2.19}$$

From (2.15) together with (2.17) we clearly see that there exists $x_0 \in (0, 0.9)$ such that $h_{\lambda_0}(x) > 0$ for $x \in [0, x_0)$ and $h_{\lambda_0}(x) < 0$ for $(x_0, 0.9]$.

Subcases A and B lead to the conclusion that $h_{\lambda_0}(x) > 0$ for $x \in [0, x_0)$ and $h_{\lambda_0}(x) < 0$ for $x \in (x_0, 1)$. Thus from (2.11), we know that $g_{\lambda_0}(x)$ is strictly increasing in $(0, x_0]$ and strictly decreasing in $[x_0, 1)$.

It follows from (2.4) and (2.10) together with the piecewise monotonicity of $g_{\lambda_0}(x)$ that there exists $x_1 \in (0,1)$ such that $f_{\lambda_0}(x)$ is strictly increasing in $[0, x_1)$ and strictly decreasing in $[x_1, 1)$.

Therefore, $f_{\lambda_0}(x) > 0$ for $x \in (0,1)$ follows from (2.2) and (2.10) together with the piecewise monotonicity of $f_{\lambda_0}(x)$. □

3. Proof of Theorems 1.1–1.3

Proof of Theorem 1.1. Since $H(a,b)$, $M(a,b)$ and $Q(a,b)$ are symmetric and homogeneous of degree 1. Hence, without loss of generality, we assume that $a > b$. Let $x = (a-b)/(a+b)$ and $t = \sinh^{-1}(x)$. Then $x \in (0,1)$, $t \in (0, \log(1+\sqrt{2}))$, $M(a,b)/A(a,b) = x/\sinh^{-1}(x) = \sinh(t)/t$, $H(a,b)/A(a,b) = 1 - x^2 = 1 - \sinh^2(t) = [3 - \cosh(2t)]/2$, $Q(a,b)/A(a,b) = \sqrt{1+x^2} = \cosh(t)$ and

$$\begin{aligned}\frac{Q(a,b) - M(a,b)}{Q(a,b) - H(a,b)} &= \frac{\sqrt{1+x^2}\sinh^{-1}(x) - x}{\left[\sqrt{1+x^2} - (1-x^2)\right]\sinh^{-1}(x)} \\ &= \frac{t\cosh(t) - \sinh(t)}{t[(1/2)\cosh(2t) + \cosh(t) - (3/2)]} := \varphi(t).\end{aligned} \tag{3.1}$$

Making use of power series $\sinh(t) = \sum_{n=0}^{\infty} t^{2n+1}/(2n+1)!$ and $\cosh(t) = \sum_{n=0}^{\infty} t^{2n}/(2n)!$, we can express (3.1) as follows:

$$\varphi(t) = \frac{\sum_{n=1}^{\infty}[2n/((2n+1)(2n)!)]t^{2n+1}}{\sum_{n=1}^{\infty}\left[\left(2^{2n-1}+1\right)/(2n)!\right]t^{2n+1}}. \tag{3.2}$$

Let $a_n = 2n/((2n+1)(2n)!)$ and $b_n = (2^{2n-1}+1)/(2n)!$ Then $a_n/b_n = 2n/[(2n+1)(2^{2n-1}+1)]$. Moreover, by a simple calculation, we see that

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = \frac{2 + \left(2 - 18n - 12n^2\right)2^{2n-1}}{(2n+1)(2n+3)\left(2^{2n-1}+1\right)\left(2^{2n+1}+1\right)} < 0 \tag{3.3}$$

for $n \geq 1$.

Equations (3.1) and (3.2) together with inequality (3.3) and Lemma 2.1 lead to the conclusion that $\varphi(t)$ is strictly decreasing in $(0, \log(1+\sqrt{2}))$. This in turn implies that

$$\lim_{t\to 0^+}\varphi(t) = \frac{2}{9}, \qquad \lim_{t\to\log(1+\sqrt{2})}\varphi(t) = 1 - \frac{1}{\sqrt{2}\log\left(1+\sqrt{2}\right)}. \tag{3.4}$$

Therefore, Theorem 1.1 follows from (3.1) and (3.4) together with the monotonicity of $\varphi(t)$. □

Proof of Theorem 1.2. Since $G(a,b)$, $M(a,b)$ and $Q(a,b)$ are symmetric and homogeneous of degree 1. Hence, without loss of generality, we assume that $a > b$. Let $x = (a-b)/(a+b)$, $p \in (0,1)$ and $\lambda_0 = 1-1/[\sqrt{2}\log(1+\sqrt{2})]$. Then making use of $G(a,b)/A(a,b) = \sqrt{1-x^2}$ gives

$$\frac{Q(a,b)-M(a,b)}{Q(a,b)-G(a,b)} = \frac{\sqrt{1+x^2}\sinh^{-1}(x)-x}{\left(\sqrt{1+x^2}-\sqrt{1-x^2}\right)\sinh^{-1}(x)}. \tag{3.5}$$

Moreover, we obtain

$$\lim_{x\to 0^+}\frac{\sqrt{1+x^2}\sinh^{-1}(x)-x}{\left(\sqrt{1+x^2}-\sqrt{1-x^2}\right)\sinh^{-1}(x)} = \frac{1}{3}, \tag{3.6}$$

$$\lim_{x\to 1^-}\frac{\sqrt{1+x^2}\sinh^{-1}(x)-x}{\left(\sqrt{1+x^2}-\sqrt{1-x^2}\right)\sinh^{-1}(x)} = 1-\frac{1}{\sqrt{2}\log\left(1+\sqrt{2}\right)} = \lambda_0. \tag{3.7}$$

We take the difference between the additive convex combination of $G(a,b)$, $Q(a,b)$, and $M(a,b)$ as follows:

$$\begin{aligned} &pG(a,b)+(1-p)Q(a,b)-M(a,b) \\ &\quad = A(a,b)\left[p\sqrt{1-x^2}+(1-p)\sqrt{1+x^2}-\frac{x}{\sinh^{-1}(x)}\right] \\ &\quad = \frac{A(a,b)\left[p\sqrt{1-x^2}+(1-p)\sqrt{1+x^2}\right]}{\sinh^{-1}(x)} f_p(x), \end{aligned} \tag{3.8}$$

where $f_p(x)$ is defined as in Lemma 2.2.

Therefore, $(1/3)G(a,b)+(2/3)Q(a,b) < M(a,b) < \lambda_0 G(a,b)+(1-\lambda_0)Q(a,b)$ for all $a,b>0$ with $a \neq b$ follows from (3.8) and Lemma 2.2. This conjunction with the following statement gives the asserted result.

(i) If $p<1/3$, then (3.5) and (3.6) imply that there exists $0<\delta_1<1$ such that $M(a,b) < pG(a,b)+(1-p)Q(a,b)$ for all $a,b>0$ with $(a-b)/(a+b) \in (0,\delta_1)$.

(ii) If $p>\lambda_0$, then (3.5) and (3.7) imply that there exists $0<\delta_2<1$ such that $M(a,b) > pG(a,b)+(1-p)Q(a,b)$ for all $a,b>0$ with $(a-b)/(a+b) \in (1-\delta_2,1)$. □

Proof of Theorem 1.3. We will follow, to some extent, lines in the proof of Theorem 1.1. First we rearrange terms of (1.9) to obtain

$$\beta_3 < \frac{C(a,b)-M(a,b)}{C(a,b)-H(a,b)} < \alpha_3. \tag{3.9}$$

Use of $C(a,b)/A(a,b) = 1+x^2$ followed by a substitution $x = \sinh(t)$ gives

$$\beta_3 < \phi(t) < \alpha_3, \tag{3.10}$$

where

$$\phi(t) = \frac{t[\cosh(2t) + 1] - 2\sinh(t)}{2t[\cosh(2t) - 1]}, \quad |t| < \log\left(1 + \sqrt{2}\right). \tag{3.11}$$

Since the function $\phi(t)$ is an even function, it suffices to investigate its behavior on the interval $(0, \log(1 + \sqrt{2}))$.

Using power series of $\sinh(t)$ and $\cosh(t)$, then (3.11) can be rewritten as

$$\phi(t) = \frac{\sum_{n=1}^{\infty} \left[2^{2n}/(2n)! - 2/(2n+1)!\right] t^{2n+1}}{\sum_{n=1}^{\infty} \left[2^{2n+1}/(2n)!\right] t^{2n+1}}. \tag{3.12}$$

Let $c_n = 2^{2n}/(2n)! - 2/(2n+1)!$ and $d_n = 2^{2n+1}/(2n)!$. Then

$$\frac{c_n}{d_n} = \frac{1}{2} - \frac{1}{(2n+1)2^{2n}}. \tag{3.13}$$

It follows from (3.13) that the sequence $\{c_n/d_n\}$ is strictly increasing for $n \geq 1$.

Equations (3.12) and (3.13) together with Lemma 2.1 and the monotonicity of $\{c_n/d_n\}$ lead to the conclusion that $\phi(t)$ is strictly increasing in $(0, \log(1 + \sqrt{2}))$. Moreover,

$$\lim_{t \to 0^+} \phi(t) = \frac{c_1}{d_1} = \frac{5}{12}, \qquad \lim_{t \to \log(1+\sqrt{2})} \phi(t) = 1 - \frac{1}{2\log\left(1 + \sqrt{2}\right)}. \tag{3.14}$$

Making use of (3.14) and (3.10) together with the monotonicity of $\phi(t)$ gives the asserted result. □

Acknowledgments

This research was supported by the Natural Science Foundation of China under Grants 11071069 and 11271105, and Innovation Team Foundation of the Department of Education of Zhejiang Province under Grant T200924.

References

[1] E. Neuman and J. Sándor, "On the Schwab-Borchardt mean," *Mathematica Pannonica*, vol. 14, no. 2, pp. 253–266, 2003.

[2] Y.-M. Chu and M.-K. Wang, "Inequalities between arithmetic-geometric, Gini, and Toader means," *Abstract and Applied Analysis*, vol. 2012, Article ID 830585, 11 pages, 2012.

[3] E. Neuman, "On one-parameter family of bivariate means," *Aequationes Mathematicae*, vol. 83, no. 1-2, pp. 191–197, 2012.

[4] M.-K. Wang, Z.-K. Wang, and Y.-M. Chu, "An optimal double inequality between geometric and identric means," *Applied Mathematics Letters*, vol. 25, no. 3, pp. 471–475, 2012.

[5] W.-D. Jiang, "Some sharp inequalities involving reciprocals of the Seiffert and other means," *Journal of Mathematical Inequalities*, vol. 6, no. 4, pp. 593–599, 2012.

[6] E. Neuman, "Inequalities for the Schwab-Borchardt mean and their applications," *Journal of Mathematical Inequalities*, vol. 5, no. 4, pp. 601–609, 2011.

[7] E. Neuman and J. Sándor, "Companion inequalities for certain bivariate means," *Applicable Analysis and Discrete Mathematics*, vol. 3, no. 1, pp. 46–51, 2009.

[8] L. Zhu, "Some new inequalities for means in two variables," *Mathematical Inequalities & Applications*, vol. 11, no. 3, pp. 443–448, 2008.
[9] L. Zhu, "New inequalities for means in two variables," *Mathematical Inequalities & Applications*, vol. 11, no. 2, pp. 229–235, 2008.
[10] E. Neuman and J. Sándor, "Inequalities for the ratios of certain bivariate means," *Journal of Mathematical Inequalities*, vol. 2, no. 3, pp. 383–396, 2008.
[11] H. Alzer and S.-L. Qiu, "Inequalities for means in two variables," *Archiv der Mathematik*, vol. 80, no. 2, pp. 201–215, 2003.
[12] J. Sándor, "On certain inequalities for means III," *Archiv der Mathematik*, vol. 76, no. 1, pp. 34–40, 2001.
[13] J. Sándor, "On certain inequalities for means II," *Journal of Mathematical Analysis and Applications*, vol. 199, no. 2, pp. 629–635, 1996.
[14] J. Sándor, "On certain inequalities for means," *Journal of Mathematical Analysis and Applications*, vol. 189, no. 2, pp. 602–606, 1995.
[15] M. K. Vamanamurthy and M. Vuorinen, "Inequalities for means," *Journal of Mathematical Analysis and Applications*, vol. 183, no. 1, pp. 155–166, 1994.
[16] J. Sándor, "A note on some inequalities for means," *Archiv der Mathematik*, vol. 56, no. 5, pp. 471–473, 1991.
[17] P. S. Bullen, D. S. Mitrinović, and P. M. Vasić, *Means and Their Inequalities*, D. Reidel, Dordrecht, The Netherlands, 1988.
[18] E. Neuman and J. Sándor, "On the Schwab-Borchardt mean. II," *Mathematica Pannonica*, vol. 17, no. 1, pp. 49–59, 2006.
[19] Y.-M. Li, B.-Y. Long, and Y.-M. Chu, "Sharp bounds for the Neuman-Sándor mean in terms of generalized logarithmic mean," *Journal of Mathematical Inequalities*, vol. 6, no. 4, pp. 567–577, 2012.
[20] E. Neuman, "A note on certain bivariate mean," *Journal of Mathematical Inequalities*, vol. 6, no. 4, pp. 637–643, 2012.
[21] S. Simić and M. Vuorinen, "Landen inequalities for zero-balanced hypergeometric functions," *Abstract and Applied Analysis*, vol. 2012, Article ID 932061, 11 pages, 2012.