

Research Article

Extension on Bifurcations of Traveling Wave Solutions for a Two-Component Fornberg-Whitham Equation

Zhenshu Wen

School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, China

Correspondence should be addressed to Zhenshu Wen, zhenshu.wen@mail.scut.edu.cn

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Fan et al. studied the bifurcations of traveling wave solutions for a two-component Fornberg-Whitham equation. They gave a part of possible phase portraits and obtained some uncertain parametric conditions for solitons and kink (antikink) solutions. However, the exact explicit parametric conditions have not been given for the existence of solitons and kink (antikink) solutions. In this paper, we study the bifurcations for the two-component Fornberg-Whitham equation in details, present all possible phase portraits, and give the exact explicit parametric conditions for various solutions. In addition, not only solitons and kink (antikink) solutions, but also peakons and periodic cusp waves are obtained. Our results extend the previous study.

1. Introduction

In 2011, Fan et al. [1] introduced the following two-component Fornberg-Whitham equation

$$\begin{aligned}u_t &= u_{xxt} - u_x - uu_x + 3u_x u_{xx} + uu_{xxx} + \rho_x, \\ \rho_t &= -(\rho u)_x,\end{aligned}\tag{1.1}$$

where $u = u(x, t)$ denotes the height of the water surface above a horizontal bottom, and $\rho = \rho(x, t)$ indicates the horizontal velocity field. They studied the bifurcations of traveling wave solutions for (1.1) through obtaining some uncertain parametric conditions for solitons, kink (antikink) solutions, and further gave some expressions of those solutions. However, they did not give the explicit parametric conditions for the existence of solitons and kink (antikink) solutions. In this paper, we further analyze the bifurcations for (1.1) systematically

by exploiting the bifurcation method and qualitative theory of dynamical systems [2–7]. We present all possible phase portraits determinately and give all the exact explicit parametric conditions for various solutions. Additionally, we obtain explicit peakons and periodic cusp waves for (1.1), which were not included in [1].

2. Bifurcations of Phase Portraits

In this section, we will present the process of obtaining the bifurcations of phase portraits for (1.1).

For given constant c , substituting $u(x, t) = \varphi(\xi)$, $\rho = \psi(\xi)$ with $\xi = x - ct$ into (1.1), it follows,

$$\begin{aligned} -c\varphi' + c\varphi''' + \varphi' + \varphi\varphi' - 3\varphi'\varphi'' - \varphi\varphi''' - \psi' &= 0, \\ -c\psi' + \varphi'\psi + \varphi\psi' &= 0, \end{aligned} \quad (2.1)$$

where the prime stands for derivative with respect to the variable ξ .

Integrating (2.1) once leads to

$$\begin{aligned} (1-c)\varphi + \frac{1}{2}\varphi^2 - (\varphi')^2 + (c-\varphi)\varphi'' - \psi &= g, \\ (\varphi-c)\psi &= G, \end{aligned} \quad (2.2)$$

where both g and G are integral constants.

From the second equation of system (2.2), we obtain

$$\psi = \frac{G}{\varphi - c}. \quad (2.3)$$

Substituting (2.3) into the first equation of system (2.2), it follows:

$$(1-c)\varphi + \frac{1}{2}\varphi^2 - (\varphi')^2 + (c-\varphi)\varphi'' - \frac{G}{\varphi - c} = g. \quad (2.4)$$

By setting $\varphi = \phi + c - 2/3$, (2.4) becomes

$$\left(\frac{2}{3} - \phi\right)^2 \phi'' = \left(\frac{2}{3} - \phi\right)(\phi')^2 + \frac{1}{2}\phi^3 - \frac{1}{6}(3c^2 - 6c + 4 + 6g)\phi + \frac{1}{27}(9c^2 - 18c + 8 + 18g) - G. \quad (2.5)$$

Letting $y = \phi'$, we obtain a planar system

$$\begin{aligned}\frac{d\phi}{d\xi} &= y, \\ \frac{dy}{d\xi} &= \frac{(2/3 - \phi)y^2 + (1/2)\phi^3 - (1/6)(3c^2 - 6c + 4 + 6g)\phi + (1/27)(9c^2 - 18c + 8 + 18g) - G}{(2/3 - \phi)^2},\end{aligned}\quad (2.6)$$

with first integral

$$H(\phi, y) = \left(\frac{2}{3} - \phi\right)^2 y^2 - \frac{1}{4}\phi^4 + \frac{1}{6}(3c^2 - 6c + 4 + 6g)\phi^2 - \frac{2}{27}(9c^2 - 18c + 8 + 18g)\phi + 2G\phi. \quad (2.7)$$

Note that when $G = 0$, system (2.6) and (2.7) become

$$\begin{aligned}\frac{d\phi}{d\xi} &= y, \\ \frac{dy}{d\xi} &= \frac{y^2 - (1/2)\phi^2 - (1/3)\phi + (1/6)(3c^2 - 6c + 4 + 6g) - 2/9}{2/3 - \phi}, \\ H(\phi, y) &= \left(\frac{2}{3} - \phi\right)^2 y^2 - \frac{1}{4}\phi^4 + \frac{1}{6}(3c^2 - 6c + 4 + 6g)\phi^2 - \frac{2}{27}(9c^2 - 18c + 8 + 18g)\phi,\end{aligned}\quad (2.8)$$

respectively.

Transformed by $d\xi = (2/3 - \phi)^2 d\tau$, system (2.6) becomes a Hamiltonian system

$$\begin{aligned}\frac{d\phi}{d\xi} &= \left(\frac{2}{3} - \phi\right)^2 y, \\ \frac{dy}{d\xi} &= \left(\frac{2}{3} - \phi\right)y^2 + \frac{1}{2}\phi^3 - \frac{1}{6}(3c^2 - 6c + 4 + 6g)\phi + \frac{1}{27}(9c^2 - 18c + 8 + 18g) - G.\end{aligned}\quad (2.10)$$

Since the first integral of system (2.6) is the same as that of the Hamiltonian system (2.10), system (2.6) should have the same topological phase portraits as system (2.10) except the straight line $l : \phi = 2/3$. Therefore, we should be able to obtain the topological phase portraits of system (2.6) from those of system (2.10).

Let

$$f(\phi) = \frac{1}{2}\phi^3 - \frac{1}{6}(3c^2 - 6c + 4 + 6g)\phi + \frac{1}{27}(9c^2 - 18c + 8 + 18g) - G. \quad (2.11)$$

It is easy to obtain the two extreme points of $f(\phi)$ as follows:

$$\phi_{\pm}^* = \pm \frac{1}{3} \sqrt{3c^2 - 6c + 4 + 6g}, \quad \text{for } g > -\frac{1}{2}\left(c^2 - 2c + \frac{4}{3}\right), \quad (2.12)$$

from which we can obtain a critical curve for g as follows:

$$g_0(c) = -\frac{1}{2}\left(c^2 - 2c + \frac{4}{3}\right). \quad (2.13)$$

Further, we obtain two bifurcation curves:

$$\begin{aligned} G_1 &= \frac{1}{27} \left[(3c^2 - 6c + 4 + 6g) \sqrt{3c^2 - 6c + 4 + 6g + 9c^2 - 18c + 8 + 18g} \right], \\ G_2 &= \frac{1}{27} \left[-(3c^2 - 6c + 4 + 6g) \sqrt{3c^2 - 6c + 4 + 6g + 9c^2 - 18c + 8 + 18g} \right], \end{aligned} \quad (2.14)$$

from $f(\phi_-^*) = 0$ and $f(\phi_+^*) = 0$, respectively. Note that when $g > g_0(c)$, obviously $G_1 > G_2$.

Additionally, we can obtain another two critical curves for g , that is,

$$g_1(c) = -\frac{1}{2}(c^2 - 2c + 1), \quad (2.15)$$

$$g_2(c) = -\frac{1}{2}(c^2 - 2c), \quad (2.16)$$

from $G_1 = 0$ and $G_2 = 0$, respectively.

Note that (2.16) can also be obtained by letting $\phi_+^* = 2/3$.

Let $(\phi^*, 0)$ be one of the singular points of system (2.10), then the characteristic values of the linearized system of system (2.10) at the singular point $(\phi^*, 0)$ are

$$\lambda_{\pm} = \pm \sqrt{\left(\frac{2}{3} - \phi^*\right)^2 f'(\phi^*)}. \quad (2.17)$$

From the qualitative theory of dynamical systems, we can determine the property of singular point $(\phi^*, 0)$ by the sign of $f'(\phi^*)$.

Based on the above analysis, we give the information of the singular points for system (2.10) and their relationship with ϕ_-^* , ϕ_+^* and $2/3$ when $g > g_2(c)$, as an illustration, in the following lemma.

Lemma 2.1. *For $g > g_2(c)$, one has $G_1 < 0 < G_2$ and the singular points of system (2.10) can be described as follows.*

- (a) If $G < G_2$, then there is only one singular point denoted as $S_1(\phi_1, 0)$ ($\phi_1 < \phi_-^* < 2/3 < \phi_+^*$). S_1 is a saddle point.
- (b) If $G = G_2$, then there are two singular points denoted as $S_1(\phi_1, 0)$ and $S_2(\phi_2, 0)$ ($\phi_1 < \phi_-^* < 2/3 < \phi_+^* = \phi_2$), respectively. S_1 is a saddle point and S_2 is a degenerate saddle point.
- (c) If $G_2 < G < 0$, then there are three singular points denoted as $S_1(\phi_1, 0)$, $S_2(\phi_2, 0)$, and $S_3(\phi_3, 0)$ ($\phi_1 < \phi_-^* < 2/3 < \phi_2 < \phi_+^* < \phi_3$), respectively. S_1 and S_3 are saddle points and S_2 is a center.

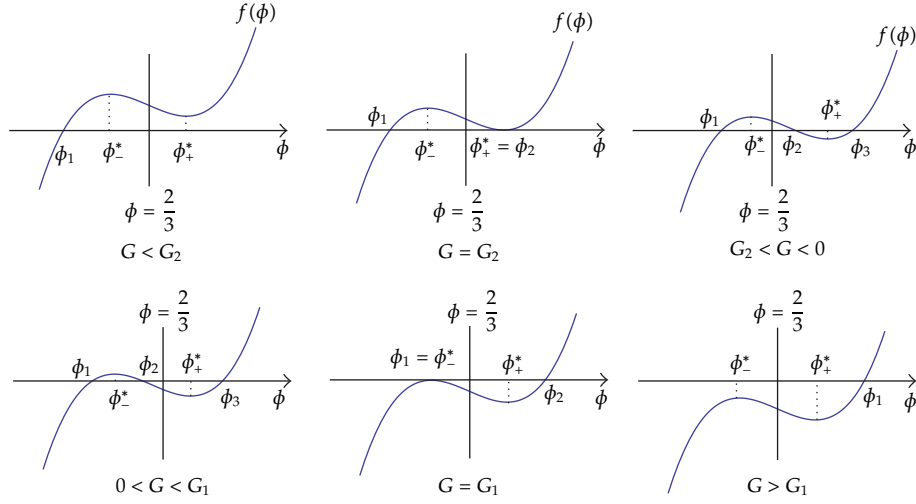


Figure 1: The graphics of $f(\phi)$ when $g > g_2(c)$.

- (d) If $0 < G < G_1$, then there are three singular points denoted as $S_1(\phi_1, 0)$, $S_2(\phi_2, 0)$, and $S_3(\phi_3, 0)$ ($\phi_1 < \phi_-^* < \phi_2 < 2/3 < \phi_+^* < \phi_3$), respectively. S_1 and S_3 are saddle points and S_2 is a center.
- (e) If $G = G_1$, then there are two singular points denoted as $S_1(\phi_1, 0)$ and $S_2(\phi_2, 0)$ ($\phi_1 = \phi_-^* < 2/3 < \phi_+^* < \phi_2$), respectively. S_1 is a degenerate saddle point and S_2 is a saddle point.
- (f) If $G > G_1$, then there is only one singular point denoted as $S_1(\phi_1, 0)$ ($\phi_-^* < 2/3 < \phi_+^* < \phi_1$). S_1 is a saddle point.

Proof. Lemma 2.1 follows easily from the graphics of the function $f(\phi)$, which can be obtained directly and shown in Figure 1 (note that $f(2/3) = G$). \square

Remark 2.2. The case when $G = 0$ follows easily from the similar analysis of system (2.8), and we just omit it here for simplicity.

For the other cases, the similar analysis can be taken to make the conclusions. We just omit these processes for simplicity. However, it is worth mentioning that, when $g_0(c) < g < g_2(c)$ and $G_2 < G < \min(0, G_1)$, there exist two saddle points and one center lying on the same side of the singular line $l : \phi = 2/3$. Hence, there may exist heteroclinic orbits for system (2.6). We will show the existence of heteroclinic orbits for system (2.6) under certain conditions in the following analysis.

When $G_1 < G < G_2$, we set the three solutions of $f(\phi) = 0$ to be ϕ_s , ϕ_m , and ϕ_b ($\phi_s < \phi_m < \phi_b$), respectively. Through simple calculation, we can express ϕ_s and ϕ_b as the function of ϕ_m , that is,

$$\begin{aligned}\phi_s &= \frac{-\phi_m - \sqrt{(4/3)(3c^2 - 6c + 4 + 6g) - 3\phi_m^2}}{2}, \\ \phi_b &= \frac{-\phi_m + \sqrt{(4/3)(3c^2 - 6c + 4 + 6g) - 3\phi_m^2}}{2}.\end{aligned}\tag{2.18}$$

It follows from $\phi_s < \phi_m < \phi_b$ that ϕ_m must satisfy condition

$$\phi_m^2 < \frac{3c^2 - 6c + 4 + 6g}{9}. \quad (2.19)$$

From $H(\phi_s, 0) = H(\phi_b, 0)$, we obtain the expression of G as the function of ϕ_m ,

$$G = \frac{1}{8}\phi_m^3 + \frac{1}{27}(9c^2 - 18c + 8 + 18g). \quad (2.20)$$

Substituting (2.20) into $f(\phi_m) = 0$, we obtain the expression of ϕ_m from $f(\phi_m) = 0$ as follows:

$$\phi_{m1} = 0, \quad (2.21)$$

$$\phi_{m2} = -\frac{2}{3}\sqrt{3c^2 - 6c + 4 + 6g}, \quad (2.22)$$

$$\phi_{m3} = \frac{2}{3}\sqrt{3c^2 - 6c + 4 + 6g}.$$

We can easily know that (2.22) does not satisfy (2.19), while (2.21) satisfies (2.19), if $g > g_0(c)$.

By substituting (2.21) into (2.20), we obtain the bifurcation curve (denoted by G^*) for G as follows:

$$G^* = \frac{1}{27}(9c^2 - 18c + 8 + 18g). \quad (2.23)$$

Note that if $g > g_0(c)$, then obviously we have $G_2 < G^* < G_1$. Additionally, we have indicated that when $g_0(c) < g < g_2(c)$ and $G_2 < G < \min(0, G_1)$, there exist two saddle points and one center lying on the same side of the singular line $\phi = 2/3$. Therefore, we obtain the fourth critical curve for g from $G^* = 0$,

$$g_3(c) = -\frac{1}{2}\left(c^2 - 2c + \frac{8}{9}\right). \quad (2.24)$$

Hence, we can express the existence of the heteroclinic orbits as follows.

Lemma 2.3. (1) For $g \in (g_0(c), g_3(c))$ and $G = G^*$, there exist heteroclinic orbits for system (2.6).
(2) For $g \notin (g_0(c), g_3(c))$ or $G \neq G^*$, there exists no heteroclinic orbit for system (2.6).

Proof. Note that when $g_3(c) < g < g_2(c)$ (or $g = g_3(c)$), one saddle point and one center point lie on the left side of the singular line $l : \phi = 2/3$ and the other saddle point on the right side of (or on) the singular line $l : \phi = 2/3$. Therefore, Lemma 2.3 follows easily from the above analysis. \square

Thereby, based on the above analysis, we obtain the bifurcations of phase portraits for system (2.6) in Figures 2, 3, 4, 5, 6, 7, and 8 under the corresponding conditions.

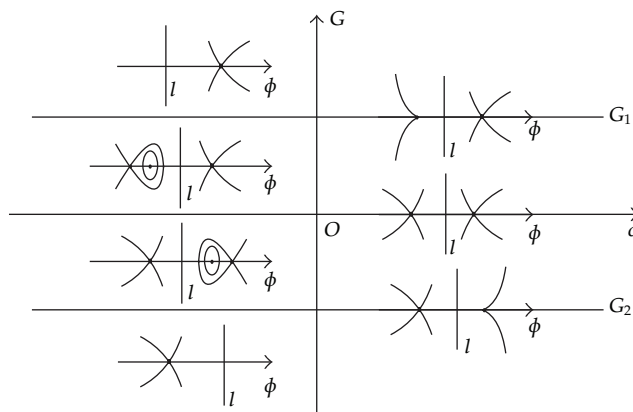


Figure 2: The phase portraits of system (2.6) when $g > g_2(c)$.

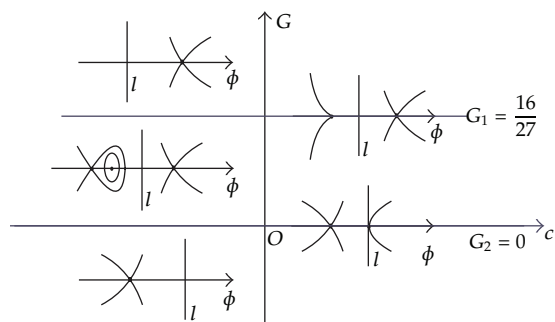


Figure 3: The phase portraits of system (2.6) when $g = g_2(c)$.

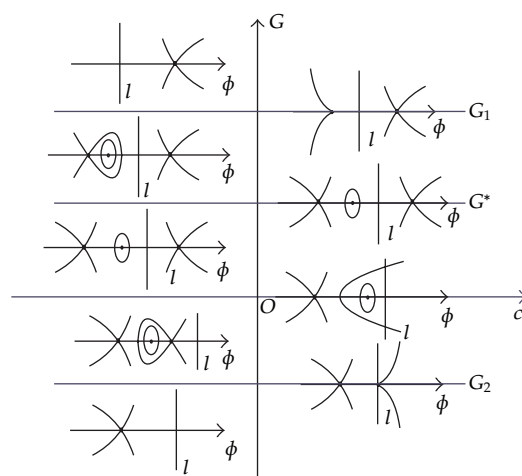


Figure 4: The phase portraits of system (2.6) when $g_3(c) < g < g_2(c)$.

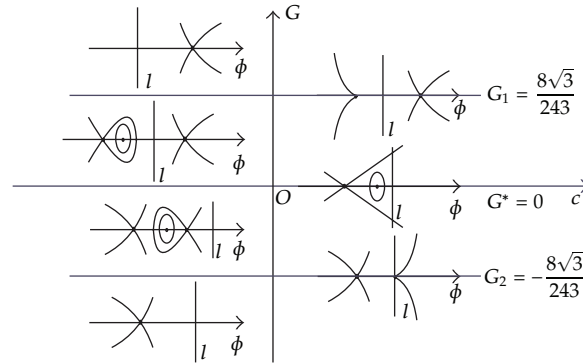


Figure 5: The phase portraits of system (2.6) when $g = g_3(c)$.

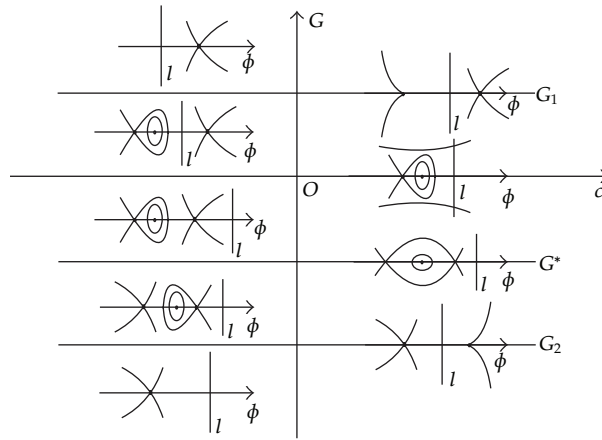


Figure 6: The phase portraits of system (2.6) when $g_1(c) < g < g_3(c)$.

3. Main Results and the Theoretic Derivations of Main Results

In this section, we state our results about solitons, kink (antikink) solutions, peakons, and periodic cusp waves for the first component of system (1.1). To relate conveniently, we omit $\varphi = \phi + c - 2/3$ and the expression of the second component of system (1.1) in the following theorems.

Theorem 3.1. *For constant wave speed c , integral constants g and G , one has the following.*

(1) *If c , g , and G satisfy one of the following conditions:*

- (i) $g \geq g_2(c)$ and $0 < G < G_1$;
- (ii) $g_3(c) \leq g < g_2(c)$ and $G^* < G < G_1$;
- (iii) $g_1(c) < g < g_3(c)$ and $0 \leq G < G_1$;
- (iv) $g_0(c) < g < g_3(c)$ and $G^* < G < 0$,

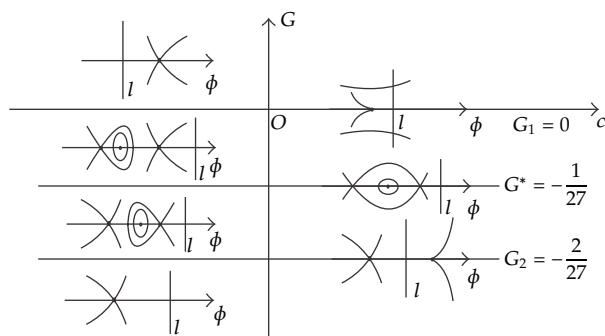


Figure 7: The phase portraits of system (2.6) when $g = g_1(c)$.

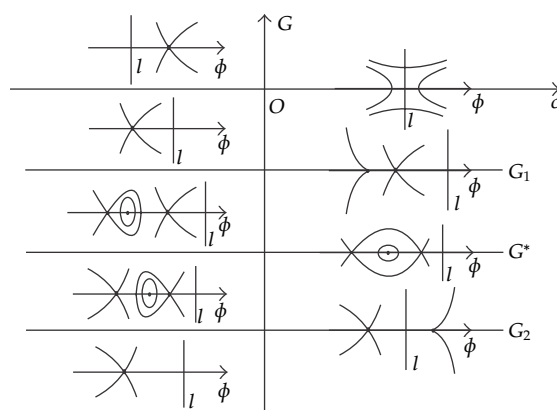


Figure 8: The phase portraits of system (2.6) when $g_0(c) < g < g_1(c)$.

then there exist soliton solutions for (1.1), which can be implicitly expressed as

$$\begin{aligned}
 & \left(2\sqrt{(\phi_{12} - \phi_1)(\phi_{13} - \phi_1)(\phi_{12} - \phi)(\phi_{13} - \phi)} + 4\phi_1\phi + 2\phi_{12}\phi_{13} + 2\phi_1^2 \right)^{\alpha_1} \\
 & \times \left(2\sqrt{(\phi_{12} - \phi)(\phi_{13} - \phi)} + 2\phi + 2\phi_1 \right) \\
 & \times \left((\phi - \phi_1)^{\alpha_1} (\phi_{13} - \phi_{12})^{\alpha_1+1} \right)^{-1} = e^{|\xi|/2},
 \end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
 \phi_{12} &= -\phi_1 - \sqrt{\frac{2}{3}(3c^2 - 6c + 4 + 6g) - 2\phi_1^2}, \\
 \phi_{13} &= -\phi_1 + \sqrt{\frac{2}{3}(3c^2 - 6c + 4 + 6g) - 2\phi_1^2}, \\
 \alpha_1 &= \frac{2/3 - \phi_1}{\sqrt{(\phi_{12} - \phi_1)(\phi_{13} - \phi_1)}}.
 \end{aligned} \tag{3.2}$$

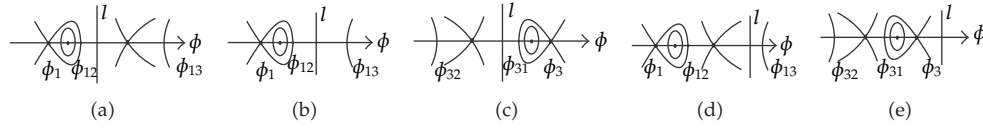


Figure 9: The different kinds of homoclinic orbits for system (2.6).

(2) If c , g , and G satisfy condition:

(v) $g > g_2(c)$ and $G_2 < G < 0$,

then there exist soliton solutions for (1.1), which can be implicitly expressed as

$$\begin{aligned} & \left(2\sqrt{(\phi_3 - \phi_{31})(\phi_3 - \phi_{32})(\phi - \phi_{31})(\phi - \phi_{32}) + 4\phi_3\phi + 2\phi_{31}\phi_{32} + 2\phi_3^2} \right)^{\alpha_3} \\ & \times \left(2\sqrt{(\phi - \phi_{31})(\phi - \phi_{32}) + 2\phi + 2\phi_3} \right) \\ & \times \left((\phi_3 - \phi)^{\alpha_3} (\phi_{31} - \phi_{32})^{\alpha_3+1} \right)^{-1} = e^{-|\xi|/2}, \end{aligned} \quad (3.3)$$

where

$$\phi_{31} = -\phi_3 - \sqrt{\frac{2}{3}(3c^2 - 6c + 4 + 6g) - 2\phi_3^2}, \quad (3.4)$$

$$\phi_{32} = -\phi_3 + \sqrt{\frac{2}{3}(3c^2 - 6c + 4 + 6g) - 2\phi_3^2}, \quad (3.5)$$

$$\alpha_3 = \frac{(2/3) - \phi_3}{\sqrt{(\phi_3 - \phi_{31})(\phi_3 - \phi_{32})}}. \quad (3.6)$$

If c , g , and G satisfy one of the following conditions:

(vi) $g_3(c) \leq g < g_2(c)$ and $G_2 < G < 0$;

(vii) $g_0(c) < g < g_3(c)$ and $G_2 < G < G^*$,

then there exist solitons solution for (1.1), which can be implicitly expressed as

$$\begin{aligned} & \left(2\sqrt{(\phi_3 - \phi_{31})(\phi_3 - \phi_{32})(\phi - \phi_{31})(\phi - \phi_{32}) + 4\phi_3\phi + 2\phi_{31}\phi_{32} + 2\phi_3^2} \right)^{\alpha_3} \\ & \times \left(2\sqrt{(\phi - \phi_{31})(\phi - \phi_{32}) + 2\phi + 2\phi_3} \right) \\ & \times \left((\phi_3 - \phi)^{\alpha_3} (\phi_{31} - \phi_{32})^{\alpha_3+1} \right)^{-1} = e^{|\xi|/2}, \end{aligned} \quad (3.7)$$

where ϕ_{31} , ϕ_{32} , and α_3 are given in (3.4), (3.5), and (3.6) respectively.

Remark 3.2. (3.1) and (3.7) are the same as those given in [1]; however, (3.3) is not shown in [1].

Remark 3.3. We give all possible homoclinic orbits in Figure 9, while it seems that Figures 9(b), 9(d), and 9(e) are not given in [1].

Proof. (1) From the phase portraits in Figures 2, 3, 4, 5, 6, 7, and 8, we see that when c , g , and G satisfy one of the conditions, that is, (i), (ii), (iii), or (iv), there exist homoclinic orbits as showed individually in Figures 9(a), 9(b), or 9(d). The expressions of the homoclinic orbits can be given as follows:

$$y = \pm \frac{(\phi - \phi_1) \sqrt{(\phi_{12} - \phi)(\phi_{13} - \phi)}}{2(2/3 - \phi)}, \quad \phi_1 \leq \phi \leq \phi_{12} < \frac{2}{3} < \phi_{13}. \quad (3.8)$$

Substituting (3.8) into the first equation of system (2.6) and integrating along the homoclinic orbits, it follows that

$$\int_{\phi}^{\phi_{12}} \frac{(2/3 - s) ds}{(s - \phi_1) \sqrt{(\phi_{12} - s)(\phi_{13} - s)}} = \frac{1}{2} |\xi|. \quad (3.9)$$

From (3.9), we obtain the soliton solutions (3.1).

(2) When c , g , and G satisfy one of the conditions, that is, (v), (vi), or (vii), there exist homoclinic orbits as showed individually in Figures 9(c) or 9(e). The expressions of the homoclinic orbits can be given as follows:

$$y = \pm \frac{(\phi_3 - \phi) \sqrt{(\phi - \phi_{31})(\phi - \phi_{32})}}{2(\phi - 2/3)}, \quad \phi_{32} < \frac{2}{3} < \phi_{31} \leq \phi \leq \phi_3, \quad (3.10)$$

or

$$y = \pm \frac{(\phi_3 - \phi) \sqrt{(\phi - \phi_{31})(\phi - \phi_{32})}}{2(2/3 - \phi)}, \quad \phi_{32} < \phi_{31} \leq \phi \leq \phi_3 < \frac{2}{3}. \quad (3.11)$$

Substituting (3.10) and (3.11) into the first equation of system (2.6), and integrating along the homoclinic orbits, it follows that

$$\begin{aligned} \int_{\phi_{31}}^{\phi} \frac{(s - 2/3) ds}{(\phi_3 - s) \sqrt{(s - \phi_{31})(s - \phi_{32})}} &= \frac{1}{2} |\xi|, \\ \int_{\phi_{31}}^{\phi} \frac{(2/3 - s) ds}{(\phi_3 - s) \sqrt{(s - \phi_{31})(s - \phi_{32})}} &= \frac{1}{2} |\xi|. \end{aligned} \quad (3.12)$$

From (3.12), we obtain the soliton solutions (3.3) and (3.7). \square

Theorem 3.4. *If integral constants g and G satisfy $g_0(c) < g < g_3(c)$ and $G = G^*$, then there exist kink and antikink solutions.*

Proof. We have showed that when $g_0(c) < g < g_3(c)$ and $G = G^*$, there exist heteroclinic orbits for system (2.6). The heteroclinic can be expressed as

$$y = \pm \frac{(\phi - \phi_s)(\phi_b - \phi)}{2/3 - \phi}, \quad \phi_s \leq \phi \leq \phi_b, \quad (3.13)$$

where

$$\begin{aligned} \phi_s &= -\sqrt{\frac{1}{3}(3c^2 - 6c + 4 + 6g)}, \\ \phi_b &= \sqrt{\frac{1}{3}(3c^2 - 6c + 4 + 6g)}, \end{aligned} \quad (3.14)$$

which can be obtained by substituting (2.23) into (2.11).

Substituting (3.13) into the first equation of system (2.6) and integrating along the heteroclinic orbits, it follows that

$$\int_{\phi_0}^{\phi} \frac{(2/3 - t)dt}{(t - \phi_s)(\phi_b - t)} = \pm \frac{1}{2}\xi, \quad (3.15)$$

where $\phi_0 \in (\phi_s, \phi_b)$ is the initial value.

From (3.15), we have

$$\begin{aligned} &(\phi - \phi_s)^{(2/3 - \phi_s)/(\phi_b - \phi_s)} \cdot (\phi_b - \phi)^{(2/3 - \phi_b)/(\phi_b - \phi_s)} \\ &= (\phi_0 - \phi_s)^{(2/3 - \phi_s)/(\phi_b - \phi_s)} \cdot (\phi_b - \phi_0)^{(2/3 - \phi_b)/(\phi_b - \phi_s)} e^{\pm \xi/2}. \end{aligned} \quad (3.16)$$

If we take $\phi_0 = (\phi_s + \phi_b)/2$, (3.16) becomes

$$(\phi - \phi_s)^{(2/3 - \phi_s)/(\phi_b - \phi_s)} \cdot (\phi_b - \phi)^{(2/3 - \phi_b)/(\phi_b - \phi_s)} = \left(\frac{\phi_b - \phi_s}{2} \right)^{4/3/(\phi_b - \phi_s)} e^{\pm \xi/2}. \quad (3.17)$$

(3.16) or (3.17) are kink (antikink) solutions. \square

Theorem 3.5. (1) *If $g = g_3(c)$ and $G = 0$, then there exist peakons for (1.1), which can be explicitly expressed as*

$$\phi = \frac{2}{3} \left(2e^{-(1/2)|x-ct|} - 1 \right). \quad (3.18)$$

(2) If $g_3(c) < g < g_2(c)$ and $G = 0$, then (1.1) has periodic cusp waves

$$u(x, t) = \phi(\xi - 2iT) + c - \frac{2}{3}, \quad (3.19)$$

where $i = 0, \pm 1, \pm 2, \dots$, $\xi = x - ct \in [(2i - 1)T, (2i + 1)T]$, and

$$\begin{aligned} \phi(\xi) = & \frac{1}{2} \left(\frac{4}{3} - \frac{1}{3} \sqrt{6(6c - 3c^2 - 6g)} \right) e^{(1/2)|x-ct|} \\ & + \frac{1}{2} \left(\frac{4}{3} + \frac{1}{3} \sqrt{6(6c - 3c^2 - 6g)} \right) e^{-(1/2)|x-ct|} - \frac{2}{3}, \end{aligned} \quad (3.20)$$

with

$$T = 2 \ln \left(\frac{4 + \sqrt{6(6c - 3c^2 - 6g)}}{\sqrt{2(9c^2 - 18c + 8 + 18g)}} \right). \quad (3.21)$$

Remark 3.6. When $G = 0$, system (2.8) is the exact planar system of the Fornberg-Whitham equation [8, 9], and peakons (3.18) and periodic cusp waves (3.19) are the same as those in [9].

Proof. (1) When $g = g_3(c)$ and $G = 0$, from Figure 5, we see that there is a triangle orbit, which can be expressed as

$$y = \pm \frac{1}{2} \left(\phi + \frac{2}{3} \right), \quad \text{for } -\frac{2}{3} \leq \phi \leq \frac{2}{3}, \quad (3.22)$$

$$\phi = \frac{2}{3}, \quad \text{for } -\frac{2}{3} \leq y \leq \frac{2}{3}. \quad (3.23)$$

Substituting (3.22) into the first equation of system (2.6) and integrating along the triangle orbits, it follows that

$$\int_{\phi}^{2/3} \frac{dt}{s + 2/3} = \frac{1}{2} |\xi|. \quad (3.24)$$

From (3.24), we obtain peakons (3.18).

(2) When $g_3(c) < g < g_2(c)$ and $G = 0$, from Figure 4, we see that there is an semiellipse orbit, which can be expressed as

$$y = \pm \frac{1}{2} \sqrt{\phi^2 + \frac{4}{3}\phi + \frac{4}{3} - \frac{2}{3}(3c^2 - 6c + 4 + 6g)}, \quad \text{for } \frac{1}{3} \left(\sqrt{2(9c^2 - 18c + 8 + 18g)} - 2 \right) \leq \phi \leq \frac{2}{3}, \quad (3.25)$$

$$\phi = \frac{2}{3}, \quad \text{for } -\frac{1}{6} \sqrt{6(6c - 3c^2 - 6g)} \leq y \leq \frac{1}{6} \sqrt{6(6c - 3c^2 - 6g)}. \quad (3.26)$$

Substituting (3.25) into the first equation of system (2.6) and integrating along the semi-ellipse orbit, it follows that

$$\int_{\phi}^{2/3} \frac{dt}{\sqrt{\phi^2 + (4/3)\phi + (4/3) - (2/3)(3c^2 - 6c + 4 + 6g)}} = \frac{1}{2}|\xi|. \quad (3.27)$$

From (3.27), we obtain periodic cusp waves (3.19) along with (3.20) and (3.21). \square

Remark 3.7. Our results are more general than those in [10] from the perspective of parameters, because the results in [10] are obtained by setting integral constant to a special value ($h = (3c/32)(c^2 - 8g)$ in [10]), while our results are obtained under all possible parametric conditions.

Remark 3.8. The system investigated in this paper is relatively complicated for all parameters. That is why Fan et al. [1] did not show the exact parametric conditions for kink (antikink) solutions. However, we figure out a new way to obtain the exact explicit parametric conditions for kink (antikink) solutions and obtain solitons under more general conditions.

4. Conclusions

Based on a previous paper [1], we further study the bifurcations of traveling wave solutions for the two-component Fornberg-Whitham equation, present all possible phase portraits determinately, and show all the exact explicit parametric conditions under which there exist solitons and kink (or antikink) solutions for (1.1). In addition, we obtain peakons and periodic cusp waves with explicit expressions for (1.1). Our results extend the previous study [1], which only gives some possible phase portraits, some undetermined parametric conditions, and some implicit expressions of solitons and kink (or antikink) solutions.

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