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Research Article

Lie Groups Analysis and Contact Transformations for Ito System

M. M. Al-Shomrani

Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80303, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to M. M. Al-Shomrani, malshomrani@hotmail.com

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Generalized Ito systems of four coupled nonlinear evaluation equations are proposed. New classes of exact invariant solutions by using Lie group analysis are obtained. Moreover, we investigate the existence of a one-parameter group of contact transformations for a generalized Ito system. Consequently, we study the relationship between one-parameter group of a contact transformation and a one-parameter Lie point transformation for a generalized Ito system.

1. Introduction

A systematic investigation of continuous transformation groups was carried out by Lie (1882–1899). His original goal was the creation of a theory of integration for ordinary differential equations analogous to the Abelian theory for the solution of algebraic equations. He investigates the fundamental concept of the invariance group admitted by a given system of differential equations. Today, the mathematical approach whose object is the construction and analysis of the full invariance group admitted by a system of differential equations is called group analysis of differential equations. These groups now usually called Lie groups and the associated Lie algebras have important real-world applications.

For the past two decades, the Lie group method has been applied to solve a wide range of problems and to explore many physically interesting solutions of nonlinear phenomena. Recently, several extensions and modifications of the classical Lie algorithm have been proposed in order to arrive at new solutions of partial differential equations (PDE). Lie symmetry analysis is one of the most powerful methods to get particular solutions of differential equations. It is based on the study of their invariance with respect to one-parameter Lie group of point transformations whose infinitesimal operators are generated by vector fields. Once the Lie groups that leave the differential equations invariant are known, we can construct an exact solution called a group invariant solution which is invariant under the transformation.

In this work, we first find symmetry groups and obtain reduced forms and then seek some similarity solutions to the reduced forms of the following Ito coupled system. The application of one-parameter group reduces the number of independent variables, and consequently a generalized Ito system is reduced to set of ordinary differential equations (ODEs) which are solved analytically.

Now we take into consideration a generalized Ito system of four coupled nonlinear evolution equations which was introduced recently by Tam et al., [1] and Karasu-Kalkanli et al. [2]:

$$\begin{aligned} u_t &= v_x, \\ v_t &= 2v_{xxx} - 6(uv)_x - 6(wp)_x, \\ w_t &= w_{xxx} + 3uw_x, \\ p_t &= p_{xxx} + 3up_x. \end{aligned} \tag{1.1}$$

Which is the generalization of the well-known integrable Ito system [3]:

$$\begin{aligned} u_t &= v_x, \\ v_t &= -2v_{xxx} - 6(uv)_x. \end{aligned} \tag{1.2}$$

Now, we investigate the existence of a one-parameter group of contact transformations for a generalized Ito system (1.1) to obtain the Lie point transformations generators and use symmetry groups to find the same Lie point transformation generators which are obtained from contact transformations.

2. The Existence of Contact Transformations for a Generalized Ito System

Evolution equations model a wide variety of phenomena in the physical, biological, and economic sciences. Lie group theory provides a useful tool for the solution partial differential equations. Many books have been written on this aspect [4]. For Lie group theory to be useful for the solution of evolution-type partial differential equations, the Lie point transformation generators need to be determined [4]. Once the Lie point transformation generators have been determined, they can be used to obtain special solutions of the differential equations under consideration. A reduction in the number of variables and transformations to other simpler equations which may be easier to solve are also possible. The Lie theory has provided insight into many physical phenomena, which may otherwise not have been possible.

2.1. Preliminaries

We only summarize relevant aspects for the case of two independent variables (time, t, and one space variable, x). The reader is referred to [5]. The set of transformations in (t, x, u) space, namely,

$$\begin{aligned} t &= t(t, x, u, a), \\ x &= x(t, x, u, a), \\ u &= u(t, x, u, a), \end{aligned} \tag{2.1}$$

where a is a real parameter, is a one-parameter group of Lie point transformations if it satisfies the group properties. The generator of the group of transformations (2.1) is given by

$$X = \xi_1(t, x, u)\frac{\partial}{\partial t} + \xi_2(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}. \tag{2.2}$$

The set of transformations in (t, x, u, u_t, u_x) space, namely,

$$\begin{aligned} \bar{t} &= \bar{t}(t, x, u, u_t, u_x, a), \\ \overline{x} &= \overline{x}(t, x, u, u_t, u_x, a), \\ \overline{u} &= \overline{u}(t, x, u, u_t, u_x, a), \\ \overline{u}_t &= \overline{u}_t(t, x, u, u_t, u_x, a), \\ \overline{u}_x &= \overline{u}_x(t, x, u, u_t, u_x, a), \end{aligned} \tag{2.3}$$

where a is a real parameter, is a one-parameter group of contact transformations if it satisfies the group properties and $\overline{u}_t = (\partial\overline{u}/\partial t)$, $\overline{u}_x = (\partial\overline{u}/\partial x)$ hold.

The generator of a group of contact transformations is

$$\begin{aligned} Y = {} & \xi_1(t, x, u, u_t, u_x)\frac{\partial}{\partial t} + \xi_2(t, x, u, u_t, u_x)\frac{\partial}{\partial x} + \eta(t, x, u, u_t, u_x)\frac{\partial}{\partial u} \\ & + \xi^1(t, x, u, u_t, u_x)\frac{\partial}{\partial u_t} + \xi^2(t, x, u, u_t, u_x)\frac{\partial}{\partial u_x}. \end{aligned} \tag{2.4}$$

The Lie characteristic function is defined by

$$W = \eta - u_t\xi_1 - u_x\ \xi_2, \tag{2.5}$$

where the functions ξ_1, ξ_2, and η can be given in terms of W as

$$\xi_1 = -W_{u_t}, \qquad \xi_2 = -W_{u_x}, \qquad \eta = W - u_t W_t - u_x W_x, \tag{2.6}$$

and the formulae for ξ_i can easily be written in terms of W as

$$\xi_1 = W_t + u_t W_u, \qquad \xi_2 = W_t + u_x W_u. \tag{2.7}$$

Higher-order prolongations can be calculated from the prolongation formula:

$$\xi_{i_1 i_2 \cdots i_s} = D_{i_1} \cdots D_{i_s} W_{u_j}(W) u_{j i_1 i_2 \cdots i_s}, \tag{2.8}$$

where D_i is the operator of total differentiation given by

$$D_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_{ij}} + \cdots . \tag{2.9}$$

If W is linear in the first derivatives u_t and u_x, then the contact transformation generator (2.4) reduces to an extended Lie point transformation generator (2.2).

2.2. Application Contact Transformations for the Generalized Ito System

In this section, we determine contact transformations for the generalized Ito system (1.1), where

$$F_1 = v_x, \tag{2.10}$$

$$F_2 = 2v_{xxx} - 6(uv)_x - 6\left(wp\right)_x, \tag{2.11}$$

$$F_3 = w_{xxx} + 3uw_x, \tag{2.12}$$

$$F_4 = p_{xxx} + 3up_x. \tag{2.13}$$

Lie point transformation generators were given by [4]. To determine contact transformations of (1.1), we solve the determining equations:

$$\widetilde{X}(u_t - v_x)|_{(2.10)-(2.13)} = 0, \tag{2.14}$$

$$\widetilde{X}(v_t - (-2v_{xxx} - 6(v\varpi)_x - 6(\omega\pi)_x))|_{(2.10)-(2.13)} = 0, \tag{2.15}$$

$$\widetilde{X}(w_t - (w_{xxx} + 3vw_x))|_{(2.10)-(2.13)} = 0, \tag{2.16}$$

$$\widetilde{X}\left(p_t - \left(p_{xxx} + 3vp_x\right)\right)|_{(2.10)-(2.13)} = 0, \tag{2.17}$$

where $\widetilde{X}$ is the prolongation of the operator (2.4) in terms of W.

Consequently, we find that

$$\begin{gathered}
\varphi_1^t - \varphi_2^x = 0, \\
\varphi_2^t - \left(-2\varphi_2^{xxx} - 6\eta^1 v_x + u\varphi_2^x + \eta^2 u_x + v\varphi_1^x\right) - 6\left(\eta^3 p_x + w\varphi_4^x + \eta^4 w_x + p\varphi_3^x\right) = 0, \\
\varphi_3^t - \left(\varphi_3^{xxx} + 3\left(\eta^1 w_x + u\varphi_3^x\right)\right) = 0, \\
\varphi_4^t - \left(\varphi_4^{xxx} + 3\left(\eta^1 p_x + u\varphi_4^x\right)\right) = 0,
\end{gathered} \tag{2.18}$$

where φ_1^t, φ_2^t, φ_3^t, φ_4^t, φ_1^x, φ_2^x, φ_3^x, φ_4^x, φ_2^{xxx}, φ_3^{xxx}, and φ_4^{xxx} can be determined from the following relation

$$\varphi_v^{i_1 i_2 i_3 \cdots i_s} = D_{i_1} D_{i_2} D_{i_3} \cdots D_{i_s}(W^v) - W^v_{u_j^v} u^v_{j i_1 i_2 i_3 \cdots i_s}, \tag{2.19}$$

where $v = 1, 2, 3, 4$ and u^1, u^2, u^3, u^4 are u, v, w, p, respectively.

By substituting φ_1^t, φ_2^t, φ_3^t, φ_4^t, φ_1^x, φ_2^x, φ_3^x, φ_4^x, φ_2^{xxx}, φ_3^{xxx}, and φ_4^{xxx} into (2.18) and after some calculations, we obtain the Lie characteristic functions in the following form:

$$\begin{gathered}
W_1 = u_t\left(k_3 - \frac{3}{4}\, k_4 t\right) + u_x\left(k_1 - \frac{1}{4}k_4 x\right) + \frac{1}{2}k_4 u, \\
W_2 = v_t\left(k_3 - \frac{3}{4}k_4 t\right) + v_x\left(k_1 - \frac{1}{4}k_4 x\right) + k_4 v, \\
W_3 = w_t\left(k_3 - \frac{3}{4}k_4 t\right) + w_x\left(k_1 - \frac{1}{4}k_4 x\right) + k_2 w, \\
W_4 = p_t\left(k_3 - \frac{3}{4}k_4 t\right) + p_x\left(k_1 - \frac{1}{4}k_4 x\right) + \left(-k_2 + \frac{3}{2}k_4\right)p,
\end{gathered} \tag{2.20}$$

where k_1, k_2, k_3, and k_4 are arbitrary constants. Then the Ito systems have the following infinitesimal:

$$\begin{gathered}
\xi_1 = \left(k_3 - \frac{3}{4}k_4 t\right), \qquad \xi_2 = \left(k_1 - \frac{1}{4}k_4 x\right), \\
\eta^1 = \frac{1}{2}k_4 u, \qquad \eta^2 = k_4 v, \\
\eta^3 = k_2 w, \qquad \eta^4 = \left(-k_2 + \frac{3}{2}k_4\right)p.
\end{gathered} \tag{2.21}$$

2.3. Lie Groups Analysis

Many authors applied Lie group analysis to find exact solutions, for example, in [6] the authors used Lie symmetry analysis and the method of dynamical systems for the extended mKdV equation to obtain exact solutions, in [7] the authors applied Lie symmetry analysis and Painleve analysis for the new (2+1)-dimensional KdV equation, and in [8] the authors have some analytical solutions for groundwater flow and transport equation via using Lie group analysis. Various symmetry reduction is obtained and reduce the system of partial differential equations to the system of ordinary differential equations which we can obtain the complete solutions of the system of ordinary differential equations. In this section, by requiring the invariance of the equations in (1.1) under the one-parameter group of Lie transformation, we obtain a system of partial differential equations which allows us not only to find the generator of the group but also to use the invariant surface condition and arrive at the reduced equation in all the considered cases. Now requiring the invariance of (1.1) with respect to the one-parameter Lie group of infinitesimal transformations, we investigate the similarity solution for the generalized Ito system.

Let us consider a one-parameter Lie group of infinitesimal transformations [9–11] of the form:

$$\begin{gathered}
x \longrightarrow X = x + \varepsilon\xi_1(t,x,u,v,w,p) + O\left(\varepsilon^2\right),\\
t \longrightarrow T = t + \varepsilon\xi_2\ (t,x,u,v,w,p) + O\left(\varepsilon^2\right),\\
u \longrightarrow U = u + \varepsilon\eta^1(t,x,u,v,w,p) + O\left(\varepsilon^2\right),\\
v \longrightarrow V = v + \varepsilon\eta^2(t,x,u,v,w,p) + O\left(\varepsilon^2\right),\\
w \longrightarrow W = w + \varepsilon\eta^3(t,x,u,v,w,p) + O\left(\varepsilon^2\right),\\
p \longrightarrow P = p + \varepsilon\eta^4(t,x,u,v,w,p) + O\left(\varepsilon^2\right). \quad \varepsilon \ll 1.
\end{gathered} \tag{2.22}$$

The functions ξ_1, ξ_2, η^1, η^2, η^3, and η^4 are the infinitesimal of transformations for the variables t, x, u, v, w, and p, respectively. In order to obtain these infinitesimal functions we have to construct a third-extended vector field $\widetilde{X}$ that is defined by

$$\begin{gathered}
\widetilde{X} = \xi_1\frac{\partial}{\partial t} + \xi_2\frac{\partial}{\partial x} + \eta^1\frac{\partial}{\partial u} + \frac{\partial}{\partial u_t} + \eta^1\frac{\partial}{\partial v} + \eta^2\frac{\partial}{\partial u} + \eta^3\frac{\partial}{\partial p}\\
+ \eta^4\frac{\partial}{\partial \rho} + \varphi_1^x\frac{\partial}{\partial u_x} + \varphi_1^t\frac{\partial}{\partial u_t} + \varphi_2^x\frac{\partial}{\partial v_x} + \varphi_2^t\frac{\partial}{\partial v_t} + \cdots
\end{gathered} \tag{2.23}$$

and the symmetry vector field X given by (2.2). The equations in (1.1) can be written in the form:

$$\begin{gathered}
H_1 = (u_t - v_x),\\
H_2 = (v_t\ - (-2v_{xxx} - 6(v\varpi)_x - 6(\omega\pi)_x)),\\
H_3 = (w_t - (w_{xxx} + 3vw_x)),\\
H_4 = (p_t - (p_{xxx} + 3vp_x)).
\end{gathered} \tag{2.24}$$

The invariance of (2.24) under the infinitesimal transformations (2.23) needs applying the extended operator to the system of PDEs (2.24), and we have

$$
\begin{gathered}
\widetilde{X} H_1=\varphi_1^t-\varphi_2^x=0, \\
\widetilde{X} H_2=\varphi_2^t-\left[-2 \varphi_2^{xxx}-6\left(\eta^1 \varpi \xi+\upsilon \varphi_2^x+\eta^2 \upsilon \xi+\varpi \varphi_1^x\right)-6\left(\eta^3 \pi \xi+\varphi_4^x \omega+\eta^4 \omega \xi+\varphi_3^x \pi\right)\right]=0, \\
\widetilde{X} H_3=\varphi_3^t-\left(\varphi_3^{xxx}+3\left(\eta^1 w x+\varphi_3^x u\right)\right)=0, \\
\widetilde{X} H_4=\varphi_4^t-\left(\varphi_4^{xxx}+3\left(\eta^1 p x+\varphi_4^x u\right)\right)=0,
\end{gathered}
\tag{2.25}
$$

under $H_1 = 0$, $H_2 = 0$, $H_3 = 0$, and $H_4 = 0$. By using symbolic software Math Lie and equating the different coefficients of the various monomials in the first-, second- and third-order partial derivatives of u, v, w, and p into (2.25) and after some calculation, we obtain the following system of partial differential equations for ξ_1, ξ_2, η^1, η^2, η^3, and η^4 [12–15]:

$$
\begin{aligned}
& -3\eta^1+3u\xi_2-3u\xi_{1t}+\xi_{2xxx}-3\eta^3_{xxw}=0, \\
& -3\eta^1+3u\xi_2-3u\xi_{1t}+\xi_{2xxx}-3\eta^4_{xxp}=0, \\
& 3\eta^1-3u\xi_2+3u\xi_{1t}-\xi_{2xxx}+3\eta^2_{xxv}=0, \\
& \eta^4_t-3u\eta^4_x-\eta^4_{xxx}=0, \qquad \eta^3_t-3u\eta^3_x-\eta^3_{xxx}=0, \\
& 6v\eta^1_x+\eta^2_t+6u\eta^2_x+6p\eta^3_x+6w\eta^4_x+2\eta^2_{xxx}=0, \\
& -\xi_2+\eta^3_{xw}=0, \qquad -\xi_2+\eta^4_{xp}=0, \\
& -\xi_2+\eta^2_{xv}=0, \\
& \eta^3-w\xi_2+w\xi_{1t}-w\xi_{1t}-w\eta^2_v+p\eta^3_p+w\eta^4_p=0, \\
& \eta^2-v\xi_{2x}+v\xi_{1t}+v\eta^1_u-v\eta^2_v=0, \\
& \eta^4-p\xi_{2x}+p\xi_{1t}-p\eta^2_v+p\eta^3_w+w\eta^4_w=0, \\
& 3\xi_{2x}-3\xi_{1t}=0, \qquad \xi_{2x}-3\xi_{1t}-\eta^1_u-\eta^2_v=0, \\
& \eta^1_t-\eta^2_x=0, \qquad \eta^1_v=\eta^1_w=\eta^1_p=0, \\
& \eta^2_p=\eta^2_w=\eta^2_{vv}=\eta^2_u=0, \qquad \eta^3_u=\eta^3_v=\eta^3_{xp}=\eta^3_{pp}=\eta^3_{ww}=\eta^3_{wp}=0, \\
& \eta^4_u=\eta^4_v=\eta^4_{xw}=\eta^4_{pp}=\eta^4_{ww}=\eta^4_{xw}=0, \\
& \xi_{2p}=\xi_{2w}=\xi_{2v}=\xi_{2t}=\xi_2=0, \qquad \xi_{1p}=\xi_{1w}=\xi_{1v}=\xi_{1u}=\xi_{1x}=0.
\end{aligned}
\tag{2.26}
$$

Table 1

	$\mathcal{X}_1$	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$
$\mathcal{X}_1$	0	0	0	$-\mathcal{X}_4/4$
$\mathcal{X}_2$	0	0	0	0
$\mathcal{X}_3$	0	0	0	$-3\mathcal{X}_3/4$
$\mathcal{X}_4$	$\mathcal{X}_4/4$	0	$3\mathcal{X}_3/4$	0

Now, we solve this system of linear partial differential (2.26) for the infinitesimal ξ_1, ξ_2, η^1, η^2, η^3, and η^4 and we obtain

$$\xi_1 = \left(k_3 - \frac{3}{4}k_4 t\right), \qquad \xi_2 = \left(k_1 - \frac{1}{4}k_4 x\right),$$
$$\eta^1 = \frac{1}{2}k_4 u, \qquad \eta^2 = k_4 v, \tag{2.27}$$
$$\eta^3 = k_2 w, \qquad \eta^4 = \left(-k_2 + \frac{3}{2}k_4\right)p,$$

where k_1, k_2, k_3 and k_4 are arbitrary constants. The above equations are the same as (2.21). We obtain from (2.21) that the Lie point transformation generators are

$$\mathcal{X}_1 = \frac{\partial}{\partial x},$$
$$\mathcal{X}_2 = w\frac{\partial}{\partial w} - p\frac{\partial}{\partial p},$$
$$\mathcal{X}_3 = \frac{\partial}{\partial t}, \tag{2.28}$$
$$\mathcal{X}_4 = -\frac{x}{4}\frac{\partial}{\partial x} - \frac{3t}{4}\frac{\partial}{\partial t} + \frac{u}{2}\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + \frac{3p}{2}\frac{\partial}{\partial p}.$$

The corresponding Lie algebra of infinitesimal symmetries of (1.1) is spanned by the infinitesimal generators $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$, and $\mathcal{X}_4$. Thus, corresponding commutator table of $\{\mathcal{X}_i; (i = 1, 2, 3, 4)\}$ can be constructed Table 1.

It is easy to check that $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4\}$ are closed under the Lie bracket. Thus, a basis for the Lie algebra is $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4\}$, which is a 4-dimensional Lie group algebra.

2.4. Reduction to Ordinary Differential Equations (ODEs)

Theoretically, all of the similarity variables associated with Lie symmetries (2.23) can be derived by solving the following characteristic equation:

$$\frac{dt}{\xi_1} = \frac{dx}{\xi_2} = \frac{du}{\eta^1} = \frac{dv}{\eta^2} = \frac{dw}{\eta^3} = \frac{dp}{\eta^4}. \tag{2.29}$$

Consequently we get the following:

$$\frac{dt}{(k_3-(3/4)k_4t)}=\frac{dx}{(k_1-(1/4)k_4x)}=\frac{du}{(1/2)k_4u}=\frac{dv}{k_4v}=\frac{dw}{k_2w}=\frac{dp}{(-k_2+(3/4)k_4)p}, \tag{2.30}$$

from

$$\frac{dt}{(k_3-(3/4)k_4t)}=\frac{dx}{(k_1-(1/4)k_4x)}. \tag{2.31}$$

Solving (2.31), we obtain the similarity variable is

$$Z=\frac{(4k_1-k_4x)}{(4k_3-3k_4t)^{1/3}}. \tag{2.32}$$

From (2.30), we get

$$\begin{gathered}\frac{dt}{(k_3-(3/4)k_4t)}=\frac{du}{(1/2)k_4u}, \qquad \frac{dt}{(k_3-(3/4)k_4t)}=\frac{dv}{k_4v}\\ \frac{dt}{(k_3-(3/4)k_4t)}=\frac{dw}{k_2w}, \qquad \frac{dt}{(k_3-(3/4)k_4t)}=\frac{dp}{(-k_2+(3/4)k_4)p}.\end{gathered} \tag{2.33}$$

By solving (2.33), we obtain the similarity solutions take the form:

$$\begin{aligned}u(t,x)&=\left(k_3-\frac{3}{4}k_4t\right)^{-2/3}F_1(Z),\\ v(t,x)&=\left(k_3-\frac{3}{4}k_4t\right)^{-4/3}F_2(Z),\\ w(t,x)&=\left(k_3-\frac{3}{4}k_4t\right)^{(-4k_2/3k_4)}F_3(Z),\\ p(t,x)&=\left(k_3-\frac{3}{4}k_4t\right)^{((-4k_2/3k_4)-2)}F_4(Z),\end{aligned} \tag{2.34}$$

where $Z=(4k_1-k_4x)/(4k_3-3k_4t)^{1/3}$ and $F_1(Z)$, $F_2(Z)$, $F_3(Z)$, and $F_4(Z)$ are arbitrary functions.

Substituting (2.34) into the equations in (1.1), we finally obtain the system of nonlinear ordinary differential equations for $F_1(Z)$, $F_2(Z)$, $F_3(Z)$, and $F_4(Z)$ takes the form:

$$
\begin{gathered}
F_2'(Z) + 2F_1'(Z) + 2F_1(Z) = 0, \\
2k_4^2 F_2''''(Z) + 6(F_1F_2)' + 6(F_3F_4)' - ZF_2' - 4F_2 = 0, \\
k_4^3 F_3'''(Z) + 3k_4F_1F_3' + k_4ZF_3' - 4k_2F_2 = 0, \\
k_4^3 F_4'''(Z) + 3k_4F_1F_4' + k_4ZF_4' + 2(-2k_2 + 3k_4)F_4 = 0.
\end{gathered} \tag{2.35}
$$

Solving a system of an ordinary differential equations (2.4), we have four cases of solutions for $F_1(Z)$, $F_2(Z)$, $F_3(Z)$, and $F_4(Z)$.

Case 1.

$$
\begin{gathered}
F_1(Z) = -Z, \qquad F_2(Z) = \frac{3}{2}Z^2, \\
F_3(Z) = A_1Z, \qquad F_4(Z) = B_1Z^2,
\end{gathered} \tag{2.36}
$$

where $k_4 = 2k_2$, A_1 and B_1 are arbitrary constants with $A_1B_1 = 2$.

Case 2.

$$
\begin{gathered}
F_1(Z) = -Z, \qquad F_2(Z) = \frac{3}{2}Z^2, \\
F_3(Z) = A_2Z^2, \qquad F_4(Z) = B_2Z,
\end{gathered} \tag{2.37}
$$

where $k_4 = k_2$, A_2 and B_2 are arbitrary constants with $A_2B_2 = 2$.

Case 3.

$$
\begin{gathered}
F_1(Z) = -Z, \qquad F_2(Z) = \frac{3}{2}Z^2, \\
F_3(Z) = \frac{2}{d}Z^3 - \frac{8k_2^2}{9d}, \qquad F_4(Z) = d,
\end{gathered} \tag{2.38}
$$

where $K_4 = (2/3)K_2$ and d is arbitrary constant.

Case 4.

$$
\begin{gathered}
F_1(Z) = \frac{1}{3}Z, \qquad F_2(Z) = -\frac{1}{2}Z^2, \\
F_3(Z) = cZ, \qquad F_4(Z) = 0,
\end{gathered} \tag{2.39}
$$

where, $K_4 = 2K_2$ and C is arbitrary constant.

Substituting from (2.36)–(2.39) into (2.34) we obtain the solutions for the generalized Ito system (1.1) in the following

Family 1.

$$u(t,x) = -Z\left(k_3 - \frac{3}{4}k_4 t\right)^{-2/3}, \qquad v(t,x) = \frac{3}{2}Z^2\left(k_3 - \frac{3}{4}k_4 t\right)^{-4/3},$$
$$w(t,x) = A_1 Z\left(k_3 - \frac{3}{4}k_4 t\right)^{(-4k_2/3k_4)}, \qquad p(t,x) = B_1 Z^2\left(k_3 - \frac{3}{4}k_4 t\right)^{((-4k_2/3k_4)-2)}, \tag{2.40}$$

where $k_4 = 2k_2$, A_1 and B_1 are arbitrary constants with $A_1 B_1 = 2$.

Family 2.

$$u(t,x) = -Z\left(k_3 - \frac{3}{4}k_4 t\right)^{-2/3}, \qquad v(t,x) = \frac{3}{2}Z^2\left(k_3 - \frac{3}{4}k_4 t\right)^{-4/3},$$
$$w(t,x) = A_2 Z^2\left(k_3 - \frac{3}{4}k_4 t\right)^{(-4k_2/3k_4)}, \qquad p(t,x) = B_2 Z\left(k_3 - \frac{3}{4}k_4 t\right)^{((-4k_2/3k_4)-2)}, \tag{2.41}$$

where $k_4 = k_2$, A_2 and B_2 are arbitrary constants with $A_2 B_2 = 2$.

Family 3.

$$u(t,x) = -Z\left(k_3 - \frac{3}{4}k_4 t\right)^{-2/3}, \qquad v(t,x) = \frac{3}{2}Z^2\left(k_3 - \frac{3}{4}k_4 t\right)^{-4/3},$$
$$w(t,x) = \left(\frac{2}{d}Z^3 - \frac{8k_2^2}{9d}\right)\left(k_3 - \frac{3}{4}k_4 t\right)^{(-4k_2/3k_4)}, \qquad p(t,x) = d\left(k_3 - \frac{3}{4}k_4 t\right)^{((-4k_2/3k_4)-2)}, \tag{2.42}$$

where $k_4 = (2/3)k_2$ and d is arbitrary constant.

Family 4.

$$u(t,x) = \frac{1}{2}Z\left(k_3 - \frac{3}{4}k_4 t\right)^{-2/3}, \qquad v(t,x) = -\frac{1}{2}Z^2\left(k_3 - \frac{3}{4}k_4 t\right)^{-4/3},$$
$$w(t,x) = cZ\left(k_3 - \frac{3}{4}k_4 t\right)^{(-4k_2/3k_4)}, \qquad p(t,x) = 0, \tag{2.43}$$

where $k_4 = 2k_2$ and c is arbitrary constant and $Z = ((4k_1 - k_4 x)/(4k_3 - 3k_4 t)^{1/3})$.

3. Conclusion

In this paper, we proved the existence of a one-parameter group of contact transformations for a generalized Ito system (1.1). Moreover, we obtained the relation between the Lie point transformations generators and contact transformations for a generalized Ito system. Also, we used the symmetry groups to find the same Lie point transformation generators which are obtained from contact transformations.

Finally, applying one-parameter group, we explored several new solutions for the Ito system through the Lie symmetry analysis which have not been reported in the literature for this model.

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