Research Article

Approximation of the Summation-Integral-Type q-Szász-Mirakjan Operators

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We introduce summation-integral-type q-Szász-Mirakjan operators and study approximation properties of these operators. We establish local approximation theorem. We give weighted approximation theorem. Also we estimate the rate of convergence of these operators for functions of polynomial growth on the interval $[0, \infty)$.

1. Introduction

First of all, we mention some concepts and notations from q-calculus. All of the results can be found in [1–5]. In what follows, q is a real number satisfying $0 < q < 1$.

For each nonnegative integer k, the q-integer $[k]_q$ and the q-factorial $[k]_q!$ are defined as

$$
[k]_q := \begin{cases} (1 - q^k)/(1 - q), & q \neq 1, \\ k, & q = 1, \end{cases}
$$

$$
[k]_q! := \begin{cases} [k]_q[k-1]_q \cdots [1]_q, & k \geq 1 \\ 1, & k = 0. \end{cases}
$$

$$(1.1)$$

Then for integers $n, k, \ n \geq k \geq 0$, we have

$$
[k+1]_q = 1 + q[k]_q, \qquad [k]_q + q^k[n-k]_q = [n]_q.
$$

$$(1.2)$$

For the integers $n, k, n \geq k \geq 0$, the q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$ (1.3)

The two q-analogues of the exponential function are defined as

$$e_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} = \frac{1}{(1-(1-q)x)_q^{\infty}}, \quad |x| < \frac{1}{1-q},$$

$$E_q(x) := \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!} = (1+(1-q)x)_q^{\infty},$$ (1.4)

where $(1+x)_q^{\infty} := \prod_{i=0}^{\infty}(1+q^i x)$.

It is easily observed that

$$e_q(x)E_q(-x) = e_q(-x)E_q(x) = 1.$$ (1.5)

The q-Jackson integral and the q-improper integral are defined as

$$\int_0^a f(t)d_qt := (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \quad a > 0,$$

$$\int_0^{\infty/A} f(t)d_qt := (1-q)\sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A}, \quad A > 0,$$ (1.6)

provided the sums converge absolutely.

For $t > 0$, the q-Gamma function is given by

$$\Gamma_q(s) = K(A,s)\int_0^{\infty/A(1-q)} t^{s-1}e_q(-t)d_qt,$$ (1.7)

where

$$K(A,s) = \frac{A^s}{1+A}\left(1+\frac{1}{A}\right)_q^s (1+A)_q^{1-s}, \quad A > 0.$$ (1.8)

It was observed in [5] that $K(x,s)$ is a q-constant, that is, $K(qx,s) = K(x,s)$. In particular, for $s \in \mathbb{N}$, $K(A,s) = q^{s(s-1)/2}$ and $K(A,0) = 1$. $\Gamma_q(s+1) = [s]_q\Gamma_q(s)$, $\Gamma_q(1) = 1$.

In 1997, Phillips [6] firstly introduced and studied q analogue of Bernstein polynomials. After this, the applications of q-calculus in the approximation theory become one of the main areas of research; many authors studied new classes of q-generalized

operators (for instance, see [7–12]). In order to approximate integrable functions, in 2008 Gupta and Heping [13] introduced and studied the q-Durrmeyer-type operators as follows:

$$M_{n,q}(f;x) = [n+1]_q \sum_{k=1}^{n} q^{1-k} p_{n,k}(q;x) \int_0^1 p_{n,k-1}(q;qt) f(t) d_q t + f(0) p_{n,0}(q;x), \qquad (1.9)$$

where $f \in C[0,1]$, $x \in [0,1]$, $n = 1,2,\ldots$, $0 < q < 1$, and

$$p_{n,k}(q;x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x). \qquad (1.10)$$

In 2010, Mahmudov and Kaffaoğlu [14] defined and studied q-Szász-Durrmeyer operators as follows:

$$D_{n,q}(f;x) = [n]_q \sum_{k=0}^{\infty} q^k s_{n,k}(q;x) \int_0^{\infty/(1-q)} s_{n,k}(q;t) f(t) d_q t, \qquad (1.11)$$

where $f \in \mathbb{R}^{[0,\infty)}$, $0 < q < 1$, $n \in \mathbb{N}$, and

$$s_{n,k}(q;x) = \frac{1}{E_q([n]_q x)} q^{k(k-1)/2} \frac{[n]_q^k x^k}{[k]_q!}. \qquad (1.12)$$

In 2011, Aral and Gupta [15] introduced and studied q-generalized Szász-Durrmeyer operators by means of the q-integral as follows:

$$\mathfrak{D}_n^q(f(t),x) = \frac{[n]_q}{b_n} \sum_{k=0}^{\infty} s_{n,k}^q(x) \int_0^{q b_n/(1-q^n)} s_{n,k}^q(t) f(t) d_q t, \qquad (1.13)$$

where $f \in C([0,\infty))$, $0 < q < 1$, $n \in \mathbb{N}$, $\{b_n\}$ is a sequence of positive numbers such that $\lim_{n \to \infty} b_n = \infty$, $0 \le x < \alpha_q(n)$, $\alpha_q(n) := b_n/(1-q)[n]_q$ and

$$s_{n,k}^q(x) = \frac{([n]_q x)^k}{q^{(k+1)/2}[k]_q!(b_n)^k} E_q\left(-[n]_q \frac{x}{b_n}\right). \qquad (1.14)$$

In the present paper, we will introduce a kind of q-analogue of the summation-integral-type Szász-Mirakjan operators as follows.

Let $f \in C([0, \infty))$. For every $n \in \mathbb{N}$, $q \in (0, 1)$, $x \in [0, \infty)$, and $A > 0$, the summation-integral-type q-Szász Mirakjan operators are defined as

$$S_{n,q}(f; x) = [n]_q \sum_{k=0}^{\infty} s_{n,k}(x, q) \int_0^{\infty/A(1-q)} q^{k^2} f(t) s_{n,k}(t, q) d_q t, \qquad (1.15)$$

where

$$s_{n,k}(x, q) = \frac{\left([n]_q x\right)^k}{[k]_q!} e_q\left(-[n]_q x\right). \qquad (1.16)$$

We will study the approximation properties of the operators (1.15). We point out that the operators (1.15) considered in this paper are more general than the operators (1.11) considered in [14]. It is easy to verify when $A = 1$, the operators (1.15) reduce to the operators (1.11). Thus, the operators (1.11) are the special case of the operators (1.15). The literature [14] studied the local approximation and the Voronovskaja-type theorem for the operators (1.11). In the present paper, besides the local approximation of the operators (1.15), our research work is different from the literature in [14]; we study the rate of convergence, the weighted Korovkin-type theorem, and the weighted approximation error of the operators (1.15) and obtain some new results. As regards the operators (1.13) considered in [15], it is obvious that these operators are quite different from the operators (1.15). In [15], for the operators (1.13), a direct approximation result in weighted function space with the help of a weighted Korovkin type theorem on the finite interval was obtained. The weighted approximation error of these operators in terms of weighted modulus of continuity was given. Also, an asymptotic formula was established. In the present paper, we introduce a new weighted modulus of continuity which is different from that in [15]. We obtain some results of the weighted approximation with the help of the new weighted modulus of continuity on the infinite interval.

2. Two Lemmas

For the operators $S_{n,q}(f; x)$ defined by (1.15), we give the following two lemmas.

Lemma 2.1. *For $S_{n,q}(t^i; x)$, $i = 0, 1, 2, 3, 4$, we have*

   (i) $S_{n,q}(1; x) = 1$;

   (ii) $S_{n,q}(t; x) = x/q^2 + 1/q[n]_q$;

   (iii) $S_{n,q}(t^2; x) = x^2/q^6 + ([3]_q + q)x/q^5[n]_q + [2]_q/q^3[n]_q^2$;

   (iv) $S_{n,q}(t^3; x) = x^3/q^{12} + (x^2/q^{11}[n]_q)([5]_q + q[3]_q + q^2) + (x/q^9[n]_q^2)([6]_q + 2q[4]_q + 2q^2[2]_q) + [2]_q[3]_q/q^6[n]_q^3$;

   (v) $S_{n,q}(t^4; x) = x^4/q^{20} + (x^3/q^{19}[n]_q)([7]_q + q[5]_q + q^2[3]_q + q^3) + (x^2/q^{17}[n]_q^2)([10]_q + 2q[8]_q + 4q^2[6]_q + 4q^3[4]_q + 3q^4[2]_q) + (x/q^{14}[n]_q^3)([10]_q + 3q[8]_q + 5q^2[6]_q + 6q^3[4]_q + 4q^4[2]_q) + [2]_q[3]_q[4]_q/q^{10}[n]_q^4$.

Proof. In view of the concepts of the q-improper integral and q-exponential function, for nonnegative integer i, we have

$$
\int_0^{\infty/A(1-q)} t^i s_{n,k}(t,q) d_q t = \int_0^{\infty/A(1-q)} t^i \frac{\left([n]_q t\right)^k}{[k]_q!} e_q\left(-[n]_q t\right) d_q t
$$

$$
= \frac{1}{[n]_q^{i+1}[k]_q!} \int_0^{\infty/(A/[n]_q)(1-q)} u^{k+i} e_q(-u) d_q u \tag{2.1}
$$

$$
= \frac{\Gamma_q(k+i+1)}{[n]_q^{i+1}[k]_q! K\left(A/[n]_q, k+i+1\right)} = \frac{[k+i]_q!}{[n]_q^{i+1}[k]_q! q^{(k+i+1)(k+i)/2}}.
$$

(i) When $i = 0$, by the formulas (1.5) and (2.1), we can get

$$
S_{n,q}(1;x) = \sum_{k=0}^{\infty} \frac{\left([n]_q x\right)^k}{[k]_q!} e_q\left(-[n]_q x\right) q^{k(k-1)/2} = e_q\left(-[n]_q x\right) E_q\left([n]_q x\right) = 1. \tag{2.2}
$$

(ii) When $i = 1$, using $[k+1]_q = [k]_q + q^k$, by the formulas (1.5) and (2.1), we can get

$$
S_{n,q}(t;x) = \sum_{k=0}^{\infty} \frac{\left([n]_q x\right)^k}{[k]_q!} e_q\left(-[n]_q x\right) q^{(k^2-3k-2)/2} \frac{[k]_q + q^k}{[n]_q}
$$

$$
= \frac{1}{[n]_q} \sum_{k=1}^{\infty} \frac{\left([n]_q x\right)^k}{[k-1]_q!} e_q\left(-[n]_q x\right) q^{(k^2-3k-2)/2} + \frac{1}{q[n]_q} \tag{2.3}
$$

$$
= \frac{x}{q^2} + \frac{1}{q[n]_q}.
$$

(iii) When $i = 2$, using $[k+2]_q[k+1]_q = [k]_q^2 + q^k(1+[2]_q)[k]_q + q^{2k}[2]_q$, by the formulas (1.5) and (2.1), we can get

$$
S_{n,q}\left(t^2;x\right) = \sum_{k=0}^{\infty} \frac{\left([n]_q x\right)^k}{[k]_q!} e_q\left(-[n]_q x\right) q^{(k^2-5k-6)/2} \frac{[k]_q^2 + q^k\left(1+[2]_q\right)[k]_q + q^{2k}[2]_q}{[n]_q^2}
$$

$$= \frac{1}{[n]_q^2} \sum_{k=2}^{\infty} \frac{\left([n]_q x\right)^k}{[k-2]_q!} e_q\left(-[n]_q x\right) q^{(k^2-5k-6)/2}$$

$$+ \frac{[3]_q + q}{[n]_q^2} \sum_{k=1}^{\infty} \frac{\left([n]_q x\right)^k}{[k-1]_q!} e_q\left(-[n]_q x\right) q^{(k^2-3k-8)/2} + \frac{[2]_q}{q^3 [n]_q^2}$$

$$= \frac{x^2}{q^6} + \frac{\left([3]_q + q\right) x}{q^5 [n]_q} + \frac{[2]_q}{q^3 [n]_q^2}.$$

(2.4)

(iv) When $i = 3$, by the formulas (1.5) and (2.1), we can get

$$S_{n,q}\left(t^3; x\right) = \sum_{k=0}^{\infty} \frac{\left([n]_q x\right)^k}{[k]_q!} e_q\left(-[n]_q x\right) q^{(k^2-7k-12)/2} \frac{[k+3]_q [k+2]_q [k+1]_q}{[n]_q^3}. \qquad (2.5)$$

Using $[k+3]_q [k+2]_q [k+1]_q = [k]_q^3 + q^k (1+[2]_q+[3]_q)[k]_q^2 + q^{2k}([2]_q+[3]_q+[2]_q[3]_q)[k]_q + q^{3k}[2]_q[3]_q$, we have

$$S_{n,q}\left(t^3; x\right)$$

$$= \frac{1}{[n]_q^3} \sum_{k=3}^{\infty} \frac{\left([n]_q x\right)^k}{[k-3]_q!} e_q\left(-[n]_q x\right) q^{(k^2-7k-12)/2}$$

$$+ \frac{1}{[n]_q^3} \sum_{k=2}^{\infty} \frac{\left([n]_q x\right)^k}{[k-2]_q!} e_q\left(-[n]_q x\right) q^{(k^2-5k-16)/2} \left[1 + 2q + q^2 \left(1 + [2]_q + [3]_q\right)\right]$$

(2.6)

$$+ \frac{1}{[n]_q^3} \sum_{k=1}^{\infty} \frac{\left([n]_q x\right)^k}{[k-1]_q!} e_q\left(-[n]_q x\right) q^{(k^2-3k-16)/2}$$

$$\times \left[1 + q\left(1 + [2]_q + [3]_q\right) + q^2 \left([2]_q + [3]_q + [2]_q[3]_q\right)\right] + \frac{[2]_q[3]_q}{q^6 [n]_q^3}$$

$$= \frac{x^3}{q^{12}} + \frac{x^2}{q^{11}[n]_q}\left([5]_q + q[3]_q + q^2\right) + \frac{x}{q^9 [n]_q^2}\left([6]_q + 2q[4]_q + 2q^2[2]_q\right) + \frac{[2]_q[3]_q}{q^6 [n]_q^3}.$$

(v) When $i = 4$, similar to the case of $i = 3$, by simple calculation we can get the desired result.

$\square$

Lemma 2.2. *Let $q \in (0,1)$, $x \in [0,\infty)$, we have*

(i) $S_{n,q}(t - x; x) = ((1 - q^2)/q^2)x + 1/q[n]_q$;

(ii) $S_{n,q}((t - x)^2; x) \leq (2/q^6)(1 - q^4 + 2/[n]_q)x(1 + x) + [2]_q/q^3[n]_q^2$.

Proof. By Lemma 2.1, we have

$$S_{n,q}(t - x; x) = S_{n,q}(t; x) - x = \frac{1 - q^2}{q^2}x + \frac{1}{q[n]_q}. \tag{2.7}$$

By Lemma 2.1, we have

$$
\begin{aligned}
S_{n,q}\left((t - x)^2; x\right) &= S_{n,q}\left(t^2; x\right) - 2x S_{n,q}(t; x) + x^2 \\
&= x^2\left(\frac{1}{q^6} - \frac{2}{q^2} + 1\right) + x\left(\frac{[3]_q + q}{q^5[n]_q} - \frac{2}{q[n]_q}\right) + \frac{[2]_q}{q^3[n]_q^2} \\
&\leq \left(\frac{1}{q^6} - \frac{2}{q^2} + 1 + \frac{[3]_q + q}{q^5[n]_q} - \frac{2}{q[n]_q}\right)x(1 + x) + \frac{[2]_q}{q^3[n]_q^2} \\
&\leq \frac{2}{q^6}\left(1 - q^4 + \frac{2}{[n]_q}\right)x(1 + x) + \frac{[2]_q}{q^3[n]_q^2}.
\end{aligned}
\tag{2.8}
$$

$\square$

3. Local Approximation

Let $C_B[0,\infty)$ denote the class of all real valued continuous bounded functions f on $[0,\infty)$ endowed with the norm $\|f\| = \sup\{|f(x)| : x \in [0,\infty)\}$. The K-functional is defined as

$$K_2(f, \delta) = \inf_{g \in W^2}\{\|f - g\| + \delta\|g''\|\}, \tag{3.1}$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By [16, page 177, Theorem 2.4] there exists an absolute constant $C > 0$ such that

$$K_2(f, \delta) \leq C\omega_2\left(f, \sqrt{\delta}\right), \tag{3.2}$$

where

$$\omega_2\left(f, \sqrt{\delta}\right) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in [0,\infty)} |f(x + 2h) - 2f(x + h) + f(x)| \tag{3.3}$$

is the second order modulus of smoothness of $f \in C_B[0, \infty)$. By

$$\omega(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0, \infty)} |f(x + h) - f(x)|, \tag{3.4}$$

we denote the usual modulus of continuity of $f \in C_B[0, \infty)$.

Theorem 3.1. *Let $f \in C_B[0, \infty)$. For every $x \in [0, \infty)$, $q \in (0, 1)$, we have*

$$|S_{n,q}(f; x) - f(x)| \leq C\omega_2(f, \delta_n(q, x)) + \omega(f, \delta_n(q, x)), \tag{3.5}$$

where C is an absolute constant, $\delta_n(q, x) = ((2/q^6)(1 - q^4 + 2/[n]_q)x(1 + x) + [2]_q/q^3[n]_q^2)^{1/2}$.

Proof. For $f \in C_B[0, \infty)$, $x \in [0, \infty)$, we define

$$\widehat{S}_{n,q}(f; x) = S_{n,q}(f; x) - f\left(\frac{x}{q^2} + \frac{1}{q[n]_q}\right) + f(x). \tag{3.6}$$

By Lemma 2.1, we get $\widehat{S}_{n,q}(1; x) = 1$, $\widehat{S}_{n,q}(t; x) = x$. Let $g \in W^2$, $x, t \in [0, \infty)$, by Taylor's formula

$$g(t) = g(x) + (t - x)g'(x) + \int_x^t (t - u)g''(u)du, \tag{3.7}$$

we obtain

$$\widehat{S}_{n,q}(g; x) = g(x) + \widehat{S}_{n,q}\left(\int_x^t (t - u)g''(u)du; x\right). \tag{3.8}$$

By the definition given by (3.6), for $x \in [0, \infty)$, we have

$$\begin{aligned}
\left|\widehat{S}_{n,q}(g; x) - g(x)\right| &\leq \left|S_{n,q}\left(\int_x^t (t - u)g''(u)du; x\right)\right| \\
&\quad + \left|\int_x^{x/q^2 + 1/q[n]_q} \left(\frac{x}{q^2} + \frac{1}{q[n]_q} - u\right)g''(u)du\right| \\
&\leq S_{n,q}\left(\left|\int_x^t |t - u||g''(u)|du\right|; x\right) \\
&\quad + \int_x^{x/q^2 + 1/q[n]_q} \left|\frac{x}{q^2} + \frac{1}{q[n]_q} - u\right||g''(u)|du \\
&\leq \left[S_{n,q}\left((t - x)^2; x\right) + \left(\frac{x}{q^2} + \frac{1}{q[n]_q} - x\right)^2\right]\|g''\|.
\end{aligned} \tag{3.9}$$

Since $(x/q^2 + 1/q[n]_q - x)^2 \leq \delta_n^2(q, x)$, so, by Lemma 2.2, we have

$$\left| \widehat{S}_{n,q}(g; x) - g(x) \right| \leq 2\delta_n^2(q, x) \|g''\|. \tag{3.10}$$

By the definition given by (1.15) and Lemma 2.1, we have

$$\left| S_{n,q}(f; x) \right| \leq S_{n,q}(1; x) \|f\| = \|f\|. \tag{3.11}$$

So, by the definition given by (3.6), we obtain

$$\left| \widehat{S}_{n,q}(f; x) \right| \leq \left| S_{n,q}(f; x) \right| + 2\|f\| \leq 3\|f\|. \tag{3.12}$$

Thus, for $x \in [0, \infty)$, we have

$$\begin{aligned}
\left| S_{n,q}(f; x) - f(x) \right| &\leq \left| \widehat{S}_{n,q}(f - g; x) \right| + \left| \widehat{S}_{n,q}(g; x) - g(x) \right| \\
&\quad + \left| g(x) - f(x) \right| + \left| f\left(\frac{x}{q^2} + \frac{1}{q[n]_q} \right) - f(x) \right| \\
&\leq 4\|f - g\| + 2\delta_n^2(q, x)\|g''\| + \omega(f, \delta_n(q, x)).
\end{aligned} \tag{3.13}$$

Hence, taking infimum on the right hand side over all $g \in W^2$, we can get

$$\left| S_{n,q}(f; q; x) - f(x) \right| \leq 4K_2\left(f, \delta_n^2(q, x) \right) + \omega(f, \delta_n(q, x)). \tag{3.14}$$

By inequality (3.2), for every $q \in (0, 1)$, we have

$$\left| S_{n,q}(f; x) - f(x) \right| \leq C\omega_2(f, \delta_n(q, x)) + \omega(f, \delta_n(q, x)). \tag{3.15}$$

$$\square$$

4. Rate of Convergence

Let $H_{x^m}[0, \infty)$ be the set of all functions f defined on $[0, \infty)$ satisfying the condition $|f(x)| \leq M_f(1 + x^m)$, where $m > 0$, M_f is a constant depending only on f. Let $C_{x^m}[0, \infty)$ denote the subspace of all continuous functions in $H_{x^m}[0, \infty)$. Also let $C_{x^m}^*[0, \infty)$ be the subspace of all functions $f \in C_{x^m}[0, \infty)$, for which $\lim_{|x| \to \infty}(f(x)/(1 + x^m))$ is finite. The norm on $C_{x^m}^*[0, \infty)$ is $\|f\|_{x^m} = \sup_{x \in [0, \infty)}(|f(x)|/(1 + x^m))$. We denote the modulus of continuity of f on the closed interval $[0, a]$, $a > 0$ by $x \in [0, \infty)$ as

$$\omega_a(f, \delta) = \sup_{|t - x| \leq \delta} \sup_{x, t \in [0, a]} \left| f(t) - f(x) \right|, \quad \delta > 0. \tag{4.1}$$

We observe that for $f \in C_{x^m}^*[0, \infty)$, the modulus of continuity $\omega_a(f, \delta) \to 0$ as $\delta \to 0^+$.

Theorem 4.1. *Let $f \in C_{x^2}[0, \infty)$, $q \in (0, 1)$, and $\omega_{a+1}(f, \delta)$ be the modulus of continuity of f on the finite interval $[0, a + 1] \subset [0, \infty)$, where $a > 0$. Then we have*

$$\left\| S_{n,q}(f; \cdot) - f \right\|_{C[0,a]} \leq 5M_f \left(1 + a^2\right) \eta_n^2(q) + 2\omega_{a+1}(f, \eta_n(q)), \tag{4.2}$$

where $\eta_n(q) = ((2/q^6)(1 - q^4 + 2/[n]_q)a(1 + a) + [2]_q/(q^3[n]_q^2))^{1/2}$, $\|f\|_{C[0,a]} = \max\{|f(x)|, x \in [0, a]\}$.

Proof. For $x \in [0, a]$ and $t > a + 1$, since $t - x > 1$, we have $|f(t) - f(x)| \leq M_f(2 + x^2 + t^2) \leq M_f[2 + 3x^2 + 2(t - x)^2] \leq 5M_f(1 + a^2)(t - x)^2$.

For $x \in [0, a]$ and $0 \leq t \leq a + 1$, we have $|f(t) - f(x)| \leq \omega_{a+1}(f, |t - x|) \leq (1 + |t - x|/\delta)\omega_{a+1}(f, \delta)$ with $\delta > 0$.

So, for $x \in [0, a]$ and $t \geq 0$, we may write

$$\left| f(t) - f(x) \right| \leq 5M_f \left(1 + a^2\right)(t - x)^2 + \left(1 + \frac{|t - x|}{\delta}\right)\omega_{a+1}(f, \delta). \tag{4.3}$$

Thus, by Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
\left| S_{n,q}(f; x) - f(x) \right| &\leq S_{n,q}\left(|f(t) - f(x)|; x\right) \leq 5M_f \left(1 + a^2\right) S_{n,q}\left((t - x)^2; x\right) \\
&\quad + \omega_{a+1}(f, \delta)\left(1 + \frac{1}{\delta}\left(S_{n,q}\left((t - x)^2; x\right)\right)^{1/2}\right).
\end{aligned} \tag{4.4}$$

By Lemma 2.2, for $x \in [0, a]$ and $\eta_n(q) = ((2/q^6)(1 - q^4 + 2/[n]_q)a(1 + a) + [2]_q/q^3[n]_q^2)^{1/2}$, we have $S_{n,q}((t - x)^2; x) \leq \eta_n^2(q)$. Hence, for every $x \in [0, a]$, $q \in (0, 1)$, we obtain

$$\left| S_{n,q}(f; x) - f(x) \right| \leq 5M_f \left(1 + a^2\right)\eta_n^2(q) + \omega_{a+1}(f, \delta)\left(1 + \frac{\eta_n(q)}{\delta}\right). \tag{4.5}$$

By taking $\delta = \eta_n(q)$, we immediately get the desired result. $\square$

Corollary 4.2. *Assume that $f \in C_{x^2}[0, \infty)$, $q \in (0, 1)$. Let $M > 0$, $\alpha \in (0, 1]$, $f \in Lip_M^\alpha$ on $[0, a + 1]$, where $a > 0$. Then we have*

$$\left\| S_{n,q}(f; \cdot) - f \right\|_{C[0,a]} \leq 5M_f \left(1 + a^2\right)\eta_n^2(q) + M\eta_n^\alpha(q), \tag{4.6}$$

where $\eta_n(q)$ and $\|f\|_{C[0,a]}$ are given in Theorem 4.1.

Proof. Let $q \in (0, 1)$, $M > 0$, $\alpha \in (0, 1]$, $f \in Lip_M^\alpha$ on $[0, a + 1]$. Then for any $x \in [0, a]$, $t \in [0, a + 1]$, we have $|f(t) - f(x)| \leq M|t - x|^\alpha$. So, according to the proof of Theorem 4.1, for $x \in [0, a]$, $t \geq 0$, we have

$$\left| f(t) - f(x) \right| \leq 5M_f \left(1 + a^2\right) S_{n,q}\left((t - x)^2; x\right) + M|t - x|^\alpha. \tag{4.7}$$

Using the Hölder inequality with $m = 2/\alpha, n = 2/(2 - \alpha)$, for any $x \in [0, a], t \geq 0$, we get

$$\left|S_{n,q}(M|t - x|^\alpha; x)\right| \leq M\left[S_{n,q}\left((t - x)^2; x\right)\right]^{\alpha/2}. \tag{4.8}$$

So, for any $x \in [0, a]$, $t \geq 0$, we have

$$
\begin{aligned}
\left|S_{n,q}(f; x) - f(x)\right| &\leq 5M_f\left(1 + a^2\right)S_{n,q}\left((t - x)^2; x\right) + M\left[S_{n,q}\left((t - x)^2; x\right)\right]^{\alpha/2} \\
&\leq 5M_f\left(1 + a^2\right)\eta_n^2(q) + M\eta_{ln}^\alpha(q).
\end{aligned}
\tag{4.9}
$$

The desired result follows immediately. $\qquad\square$

5. Weighted Approximation

Now we give the weighted approximation result for the operators $S_{n,q}(f; x)$.

Theorem 5.1. *Let the sequence $q = \{q_n\}$ satisfies $q_n \in (0, 1)$ and $q_n \to 1$ as $n \to \infty$. For $f \in C_{x^2}^*[0, \infty)$, we have*

$$\lim_{n\to\infty} \left\|S_{n,q_n}(f; \cdot) - f\right\|_{x^2} = 0. \tag{5.1}$$

Proof. Using the Theorem in [17], we see that it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty} \left\|S_{n,q_n}(e_i; \cdot) - e_i\right\|_{x^2} = 0, \tag{5.2}$$

where $e_i(x) = x^i$, $i = 0, 1, 2$.

Since $S_{n,q_n}(e_0; x) = 1$, it is clear that $\lim_{n\to\infty}\|S_{n,q_n}(e_0; \cdot) - e_0\|_{x^2} = 0$.

By Lemma 2.2, we have

$$
\begin{aligned}
\left\|S_{n,q_n}(e_1; \cdot) - e_1\right\|_{x^2} &= \sup_{x\in[0,\infty)} \frac{\left|S_{n,q_n}(t; x) - x\right|}{1 + x^2} \\
&\leq \frac{1 - q_n^2}{q_n^2} \sup_{x\in[0,\infty)} \frac{x}{1 + x^2} + \frac{1}{q_n[n]_{q_n}} \leq \frac{1 - q_n^2}{q_n^2} + \frac{1}{q_n[n]_{q_n}}.
\end{aligned}
\tag{5.3}
$$

Since $q_n \in (0, 1)$ and $\lim_{n\to\infty} q_n = 1$, we have $[n]_{q_n} \to \infty$ as $n \to \infty$ (see [18]), so, we can obtain $\lim_{n\to\infty}\|S_{n,q_n}(e_1; \cdot) - e_1\|_{x^2} = 0$.

For $i = 2$, by Lemma 2.1, we have

$$
\begin{aligned}
\left\| S_{n,q_n}(e_2; \cdot) - e_2 \right\|_{x^2} &= \sup_{x \in [0,\infty)} \frac{\left| S_{n,q_n}(t^2; x) - x^2 \right|}{1 + x^2} \leq \frac{1 - q_n^6}{q_n^6} \sup_{x \in [0,\infty)} \frac{x^2}{1 + x^2} \\
&\quad + \frac{\left([3]_{q_n} + q_n\right)}{q_n^5 [n]_{q_n}} \sup_{x \in [0,\infty)} \frac{x}{1 + x^2} + \frac{[2]_{q_n}}{q_n^3 [n]_{q_n}^2} \\
&\leq \frac{1 - q_n^6}{q_n^6} + \frac{\left([3]_{q_n} + q_n\right)}{q_n^5 [n]_{q_n}} + \frac{[2]_{q_n}}{q_n^3 [n]_{q_n}^2},
\end{aligned}
\tag{5.4}
$$

which implies that $\lim_{n \to \infty} \left\| S_{n,q_n}(e_2; \cdot) - e_2 \right\|_{x^2} = 0$. In a word, we complete the proof. $\square$

It is known that, if f is not uniformly continuous on the interval $[0, \infty)$, then the usual first modulus of continuity $\omega(f, \delta)$ does not tend to zero as $\delta \to 0^+$. For every $f \in C_{x^2}^*[0, \infty)$, we would like to take a weighted modulus of continuity $\Omega(f, \delta)$ which tends to zero as $\delta \to 0^+$.

For every $f \in C_{x^2}^*[0, \infty)$, let

$$
\Omega(f, \delta) = \sup_{0 < h \leq \delta, x \geq 0} \frac{\left| f(x + h) - f(x) \right|}{(1 + h^2)(1 + x^2)}.
\tag{5.5}
$$

The weighed modulus of continuity $\Omega(f, \delta)$ was defined by İspir in [19]. It is known that $\Omega(f, \delta)$ has the following properties.

Lemma 5.2 (see [19]). *Let $f \in C_{x^2}^*[0, \infty)$. Then*

(i) $\Omega(f, \delta)$ *is a monotone increasing function of δ;*

(ii) *for each $f \in C_{x^2}^*[0, \infty)$, $\lim_{\delta \to 0^+} \Omega(f, \delta) = 0$;*

(iii) *for each $m \in \mathbb{N}$, $\Omega(f, m\delta) \leq m\Omega(f, \delta)$;*

(iv) *for each $\lambda \in \mathbb{R}^+$, $\Omega(f, \lambda\delta) \leq (1 + \lambda)\Omega(f, \delta)$.*

Theorem 5.3. *Let $f \in C_{x^2}^*[0, \infty)$, $q \in (0, 1)$. Then we have the inequality*

$$
\left\| S_{n,q}(f; \cdot) - f \right\|_{x^5} \leq M(q)\Omega\left(f, \left(1 - q^4 + \frac{3}{[n]_q}\right)^{1/2} \right),
\tag{5.6}
$$

where $M(q)$ is a positive constant independent f and n.

Proof. By the definition of $\Omega(f, \delta)$ and Lemma 5.2 (iv), for every $\delta > 0$, we have

$$
\left| f(t) - f(x) \right| \leq \left(1 + (t - x)^2\right)\left(1 + x^2\right)\left(1 + \frac{|t - x|}{\delta}\right)\Omega(f, \delta).
\tag{5.7}
$$

Then, we obtain

$$\left|S_{n,q}(f;x) - f(x)\right| \le \left(1 + x^2\right)\Omega(f,\delta)S_{n,q}\left(\left(1 + (t-x)^2\right)\left(1 + \frac{|t-x|}{\delta}\right);x\right)$$

$$\le \left(1 + x^2\right)\Omega(f,\delta)\left\{S_{n,q}\left(1 + (t-x)^2;x\right)\right. \tag{5.8}$$

$$\left. + S_{n,q}\left(\left(1 + (t-x)^2\right)\frac{|t-x|}{\delta};x\right)\right\}.$$

Also by the Cauchy-Schwartz inequality, we have

$$S_{n,q}\left(\left(1 + (t-x)^2\right)\frac{|t-x|}{\delta};x\right) \le \left\{S_{n,q}\left(\left(1 + (t-x)^2\right)^2;x\right)\right\}^{1/2}\left\{S_{n,q}\left(\frac{|t-x|^2}{\delta^2};x\right)\right\}^{1/2}.$$

$$\tag{5.9}$$

Consequently,

$$\left|S_{n,q}(f;x) - f(x)\right|$$

$$\le \left(1 + x^2\right)\Omega(f,\delta)$$

$$\times \left\{S_{n,q}\left(1 + (t-x)^2;x\right) + \left(S_{n,q}\left(\left(1 + (t-x)^2\right)^2;x\right)\right)^{1/2}\left(S_{n,q}\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{1/2}\right\}.$$

$$\tag{5.10}$$

In view of Lemmas 2.1 and 2.2, we get

$$S_{n,q}\left(1 + (t-x)^2;x\right) \le 1 + \frac{2}{q^6}\left(1 - q^4 + \frac{2}{[n]_q}\right)x(1+x) + \frac{[2]_q}{q^3[n]_q^2}$$

$$\le \frac{2}{q^6}\left(2 - q^4 + \frac{3}{[n]_q}\right)(1+x)^2 \tag{5.11}$$

$$\le M_1(q)(1+x)^2.$$

Also,

$$S_{n,q}\left(\left(1 + (t-x)^2\right)^2;x\right)$$

$$= 1 + 2S_{n,q}\left((t-x)^2;x\right) + S_{n,q}\left(t^4;x\right)$$

$$- 4xS_{n,q}\left(t^3;x\right) + 6x^2S_{n,q}\left(t^2;x\right) - 4x^3S_{n,q}(t;x) + x^4$$

$$= x^4 \left(\frac{1}{q^{20}} - \frac{4}{q^{12}} + \frac{6}{q^6} - \frac{4}{q^2} + 1 \right)$$

$$+ x^3 \left[\frac{1}{q^{19}[n]_q} \left([7]_q + q[5]_q + q^2[3]_q + q^3 \right) \right.$$

$$- \frac{4}{q^{11}[n]_q} \left([5]_q + q[3]_q + q^2 \right) + \frac{6}{q^5[n]_q} \left([3]_q + q \right) - \frac{4}{q[n]_q} \Bigg]$$

$$+ x^2 \left[\frac{2}{q^6} - \frac{4}{q^2} + 2 + \frac{1}{q^{17}[n]_q^2} \left([10]_q + 2q[8]_q + 4q^2[6]_q + 4q^3[4]_q + 3q^4[2]_q \right) \right.$$

$$\left. - \frac{4}{q^9[n]_q^2} \left([6]_q + 2q[4]_q + 2q^2[2]_q \right) + \frac{6[2]_q}{q^3[n]_q^2} \right]$$

$$+ x \left[\frac{2\left([3]_q + q\right)}{q^5[n]_q} - \frac{4}{q[n]_q} \right.$$

$$\left. + \frac{1}{q^{14}[n]_q^3} \left([10]_q + 3q[8]_q + 5q^2[6]_q + 6q^3[4]_q + 4q^4[2]_q \right) - \frac{4[2]_q[3]_q}{q^6[n]_q^3} \right]$$

$$+ \left(1 + \frac{2[2]_q}{q^3[n]_q^2} + \frac{[2]_q[3]_q[4]_q}{q^{10}[n]_q^4} \right) \leq \frac{16}{q^{20}}x^4 + \frac{80}{q^{19}}x^3 + \frac{164}{q^{17}}x^2 + \frac{84}{q^{14}}x + 1 + \frac{28}{q^{10}}$$

$$\leq M_2(q)\left(1 + x^2\right)^2,$$

$$\left\{ S_{n,q}\left(\frac{|t-x|^2}{\delta^2}; x \right) \right\}^{1/2} \leq \frac{1}{\delta} \left[\frac{2}{q^6}\left(1 - q^4 + \frac{2}{[n]_q} \right)x(1+x) + \frac{[2]_q}{q^3[n]_q^2} \right]^{1/2}$$

$$\leq \frac{1}{\delta} M_3(q)\left(1 - q^4 + \frac{3}{[n]_q} \right)^{1/2}(1+x).$$

$$(5.12)$$

Now from inequalities (5.10)–(5.12), we have

$$|S_{n,q}(f;x) - f(x)| \leq \Omega(f,\delta) \left[M_1(q)\left(1 + x^2 \right)(1+x)^2 \right.$$

$$\left. + \sqrt{M_2(q)}\, M_3(q)\frac{1}{\delta}\left(1 - q^4 + \frac{3}{[n]_q} \right)^{1/2}\left(1 + x^2 \right)^2(1+x) \right]$$

$$\leq \Omega(f, \delta) \left[M_1(q) \left(1 + x^2 \right) (1 + x)^2 \right.$$

$$\left. + \sqrt{M_2(q)} M_3(q) M_4 \frac{1}{\delta} \left(1 - q^4 + \frac{3}{[n]_q} \right)^{1/2} \left(1 + x^5 \right) \right],$$

(5.13)

where $M_4 = \sup_{x \geq 0} ((1 + x^2)^2 (1 + x) / (1 + x^5))$.

Taking $\delta = (1 - q^4 + 3/[n]_q)^{1/2}$, from the above inequality we can obtain the desired result. □

Acknowledgments

References

[1] G. Gasper and M. Rahman, *Basic Hypergeometrik Series, Encyclopedia of Mathematics and its Applications*, vol. 35, Cambridge University Press, Cambridge, UK, 1990.

[2] V. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer, New York, NY, USA, 2002.

[3] F. H. Jackson, "On a q-definite integrals," *Quarterly Journal of Pure and Applied Mathematics*, vol. 41, pp. 193–203, 1910.

[4] H. T. Koelink and T. H. Koornwinder, "q-special functions, a tutorial," in *Deformation Theory and Quantum Groups with Applications to Mathematical Physics (Amherst, MA, 1990)*, vol. 134 of *Contemporary Mathematics*, pp. 141–142, American Mathematical Society, Providence, RI, USA, 1992.

[5] A. De Sole and V. G. Kac, "On integral representations of q-gamma and q-beta functions," *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei*, vol. 16, no. 1, pp. 11–29, 2005.

[6] G. M. Phillips, "Bernstein polynomials based on the q-integers," *Annals of Numerical Mathematics*, vol. 4, no. 1–4, pp. 511–518, 1997.

[7] O. Agratini and G. Nowak, "On a generalization of Bleimann, Butzer and Hahn operators based on q-integers," *Mathematical and Computer Modelling*, vol. 53, no. 5-6, pp. 699–706, 2011.

[8] O. Doğru and M. Örkcü, "Statistical approximation by a modification of q-Meyer-König and Zeller operators," *Applied Mathematics Letters*, vol. 23, no. 3, pp. 261–266, 2010.

[9] V. Gupta and C. Radu, "Statistical approximation properties of q-Baskakov-Kantorovich operators," *Central European Journal of Mathematics*, vol. 7, no. 4, pp. 809–818, 2009.

[10] S. G. Gal, "Voronovskaja's theorem, shape preserving properties and iterations for complex q-Bernstein polynomials," *Studia Scientiarum Mathematicarum Hungarica*, vol. 48, no. 1, pp. 23–43, 2011.

[11] N. I. Mahmudov, "Approximation by genuine q-Bernstein-Durrmeyer polynomials in compact disks," *Hacettepe Journal of Mathematics and Statistics*, vol. 40, no. 1, pp. 77–89, 2011.

[12] I. Yüksel, "Approximation by q-Phillips operators," *Hacettepe Journal of Mathematics and Statistics*, vol. 40, no. 2, pp. 191–201, 2011.

[13] V. Gupta and W. Heping, "The rate of convergence of q-Durrmeyer operators for $0 < q < 1$," *Mathematical Methods in the Applied Sciences*, vol. 31, no. 16, pp. 1946–1955, 2008.

[14] N. I. Mahmudov and H. Kaffaoğlu, "On q-Szász-Durrmeyer operators," *Central European Journal of Mathematics*, vol. 8, no. 2, pp. 399–409, 2010.

[15] A. Aral and V. Gupta, "Generalized Szász Durrmeyer operators," *Lobachevskii Journal of Mathematics*, vol. 32, no. 1, pp. 23–31, 2011.

[16] R. A. DeVore and G. G. Lorentz, *Constructive Approximation*, vol. 303, Springer, Berlin, Germany, 1993.

[17] A. D. Gadzhiev, "Theorems of the type of P. P. Korovkin type theorems," *Matematicheskie Zametki*, vol. 20, no. 5, pp. 781–786, 1976, English Translation in *Mathematical Notes*, vol. 20, no. 5-6, pp. 996–998, 1976.

[18] V. S. Videnskii, "On q-Bernstein polynomials and related positive linear operators," in *Problems of Modern Mathematics and Mathematical Education*, pp. 118–126, St.-Pertersburg, Russia, 2004.

[19] N. İspir, "On modified Baskakov operators on weighted spaces," *Turkish Journal of Mathematics*, vol. 25, no. 3, pp. 355–365, 2001.