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Research Article

Limit Cycle Bifurcations from a Nilpotent Focus or Center of Planar Systems

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We study analytic properties of the Poincaré return map and generalized focal values of analytic planar systems with a nilpotent focus or center. We use the focal values and the map to study the number of limit cycles of this kind of systems and obtain some new results on the lower and upper bounds of the maximal number of limit cycles bifurcating from the nilpotent focus or center. The main results generalize the classical Hopf bifurcation theory and establish the new bifurcation theory for the nilpotent case.

1. Introduction and Main Result

Consider an analytic system of the form:

$$\dot{x} = y + \alpha x + X(x, y, \delta), \qquad \dot{y} = -x + \alpha y + Y(x, y, \delta), \tag{1.1}$$

where $\alpha \in R$, $\delta \in R^n$, and $X, Y = O(|x, y|^2)$ for (x, y) near the origin. A Poincaré map can be defined on a cross-section with an endpoint at the origin using positive orbits of the above system and can be written in the form:

$$P(r, \alpha, \delta) = r + \sum_{j\geq 1} v_j(\alpha, \delta) r^j, \tag{1.2}$$

where the series converges for small r. A well-known fact is that for any $k \geq 1 v_{2j-1}(\alpha, \delta) = 0$ for all $j = 1, \ldots, k$ imply that $v_{2k}(\alpha, \delta) = 0$; that is, only odd values of the expansion are important for determining the behavior of trajectories near the origin. The value $v_{2k+1}(\alpha, \delta)$ is called the kth focal value or the kth Lyapunov constant. For quadratic systems, Bautin [1] proved that the Poincaré map can be written in the form:

$$P(r, \alpha, \delta) = r + \sum_{j=1}^{4} v_{2j-1}(\alpha, \delta) r^{2j-1} \left(1 + P_j(r, \alpha, \delta)\right), \tag{1.3}$$

where $P_j(r, \alpha, \delta) = O(r) \in C^{\omega}$. This implies that there are at most 3 limit cycles near the origin.

Suppose now that the origin is a nilpotent singular point, so the system is written in the form:

$$\dot{x} = y + X(x, y), \qquad \dot{y} = Y(x, y), \tag{1.4}$$

where $X, Y = O(|x, y|^2)$ for (x, y) near the origin. The following criterion for the existence of a center or a focus at the origin of (1.4) has been established in [2–4].

Theorem 1.1 (see [2–4]). *Let* (1.4) *have an isolated singular point at the origin. Let*

$$\begin{aligned} Y(x, F(x)) &= a x^{2n-1} + O\left(x^{2n}\right), \quad a \neq 0, \\ \frac{\partial X}{\partial x}(x, F(x)) + \frac{\partial Y}{\partial y}(x, F(x)) &= b x^{n-1} + O(x^n), \end{aligned} \tag{1.5}$$

where $y = F(x)$ is the solution to the equation $y + X(x, y) = 0$ satisfying $F(0) = 0$. Then the origin of (1.4) *is a center or a focus if and only if a is negative and $b^2 + 4an < 0$.*

Introducing the generalized polar coordinates:

$$x = r \operatorname{Cs}(\theta), \qquad y = r^n \operatorname{Sn}(\theta), \tag{1.6}$$

where $(\operatorname{Cs}(t), \operatorname{Sn}(t))$ is the solution of the initial problem

$$\dot{x} = y, \quad \dot{y} = -x^{2n-1}, \quad \left(x(0), y(0)\right) = (1, 0), \tag{1.7}$$

Liapunov [3, 4] proposed a method to solve the center-focus problem for (1.4). Sadovskiĭ [5] (see also [6]) and Moussu [7] investigated the problem using Lyapunov functions and normal forms, respectively. Other few approaches for computing focal values, Lyapunov constants or equivalent values and methods for studying bifurcations of local limit cycles were suggested by Chavarriga et al. [8], Giacomini et al. [9], Álvarez and Gasull [10, 11], and Liu and Li [12–15]. From [16] we know that (1.4) can be formally transformed into a formal normal form:

$$\dot{x} = y, \quad \dot{y} = -g(x) - y f(x), \tag{1.8}$$

where $g(x) = ax^m + O(x^{m+1})$, $m \geq 2$ (system (1.8) is a generalized Liénard system). Stróżyna and Żołądek [17] proved that this formal normal form can be achieved through an analytic change of variables. Thus, if (1.4) has a center or focus at the origin, then it can be changed into (1.8) with

$$g(x) = x^{2n-1}(a_{2n-1} + O(x)), \quad n \geq 2,\ a_{2n-1} > 0. \tag{1.9}$$

According to [11] under (1.9) by a change of variables x and t of the form:

$$u = \left[2n\int_0^x g(x)dx\right]^{1/2n}(\operatorname{sgn} x) \equiv u(x), \qquad \frac{dt}{dt_1} = \frac{u^{2n-1}(x)}{g(x)} \tag{1.10}$$

system (1.8) is transformed into

$$\dot{x} = y, \quad \dot{y} = -x^{2n-1} - y\overline{f}(x), \tag{1.11}$$

where

$$\begin{gathered}\overline{f}(x) = \frac{x^{2n-1}f\left(u^{-1}(x)\right)}{g\left(u^{-1}(x)\right)}, \quad n \geq 2, \\ u(x) = \left[2n\int_0^x g(x)dx\right]^{1/2n}(\operatorname{sgn} x) = (a_{2n-1})^{1/2n}\left(x + O\left(x^2\right)\right).\end{gathered} \tag{1.12}$$

Then, by Theorem 1.1 system (1.11) has a center or a focus at the origin if and only if the function $\overline{f}$ given in (1.12) satisfies

$$\overline{f}(x) = \sum_{j \geq n-1} b_j x^j, \qquad b_{n-1}^2 - 4n < 0. \tag{1.13}$$

By Filippov's theorem (see, e.g., Ye et al. [18]) under (1.13) system (1.11) has a stable (unstable) focus at the origin if there exists an integer l with $2l \geq n-1$ such that

$$b_{2l} > 0(<0), \qquad b_{2j} = 0 \quad \text{for } j < l, \tag{1.14}$$

and it has a center at the origin if $b_{2j} = 0$ for all $2j \geq n-1$.

Passing to the generalized polar coordinates $(x, y) = (r\operatorname{Cs}(\theta), r^n\operatorname{Sn}(\theta))$ we obtain from (1.11) the following equation:

$$\frac{dr}{d\theta} = \frac{\sum_{j\geq n-1} b_j(\operatorname{Sn}(\theta))^2(\operatorname{Cs}(\theta))^j r^{2-n+j}}{1 + \sum_{j\geq n-1} b_j\operatorname{Sn}(\theta)(\operatorname{Cs}(\theta))^{j+1}r^{1-n+j}}. \tag{1.15}$$

The function on the right hand side of (1.15) is periodic of the period $T = 2\sqrt{\pi/n}\Gamma(1/2n)/\Gamma((n+1)/2n)$. Let $r(\theta, r_0)$ denote the solution of (1.15) with the initial value $r(0) = r_0$. Then the Poincaré map of (1.15) has the form:

$$r(T, r_0) = \sum_{j\geq 1} V_j r_0^j. \tag{1.16}$$

Assuming that $V_1 = 1, V_2 = \cdots = V_{k-1} = 0$ Álvarez and Gasull [11] called the constant V_k the kth generalized Lyapunov constant of (1.15) (we will see that this definition is too rough since a half of the constants cannot be used to determine the stability of the origin of (1.8) or (1.11)). They also studied the normal form (1.11) and proved the following theorem.

Theorem 1.2 (see [11]). *Let* (1.13) *and* (1.14) *be satisfied. Then*

(1) $V_1 = \exp(-2b_{n-1}\pi/n\sqrt{4n - b_{n-1}^2})$ *if* $2l = n - 1$;

(2) $V_1 = 1$, $V_j = 0$ *for* $1 < j < 2 - n + 2l$, *and* $V_{2-n+2l} = -K_l b_{2l}$ *if either* $b_{n-1} = 0$ *or* $b_{n-1} \neq 0$ *and* n *is even, where* K_l *is a positive constant.*

In the case $n = 2$ Liu and Li [12] introduced different generalized polar coordinates of the form $x = r\cos\theta$, $y = r^2 \sin\theta$ to change (1.4) into the form:

$$\frac{dr}{dt} = R(\theta, r), \qquad \frac{d\theta}{dt} = Q(\theta, r), \tag{1.17}$$

where it is assumed that the origin is a center or a focus. Let $\tilde{r}(\theta, h)$ denote the solution of the 2π-periodic system

$$\frac{dr}{d\theta} = \frac{R(\theta, r)}{Q(\theta, r)} \tag{1.18}$$

satisfying $\tilde{r}(0) = h$. Note that the initial value problem is well defined also for negative h.

Let D be a simply connected domain. Denote by R the ring of analytic functions on D and by $\langle \phi_1, \ldots, \phi_k \rangle$ the ideal in R generated by the functions $\phi_1, \ldots, \phi_k$ from R. Liu and Li [12] found the following facts.

Theorem 1.3 (see [12]). *Consider system* (1.4). *Let the conditions of Theorem 1.1 be satisfied with* $n = 2$ *(or* $m = 3$*) so that the origin is a center or a focus. Then*

(1) $\tilde{r}(\theta, -\tilde{r}(\pi, h)) = -\tilde{r}(\pi - \theta, h)$.

(2) $\Delta(h) = \tilde{r}(-2\pi, h) - h = \sum_{k\geq 2} v_k h^k$, *where*

$$v_{2k+1} \in \langle v_2, v_4, \ldots, v_{2k} \rangle, \quad k \geq 1. \tag{1.19}$$

In particular, $v_{2j} = 0$, $j = 1, \ldots, k$ *imply that* $v_{2k+1} = 0$.

(3) *The origin is a stable (unstable) focus if*

$$v_{2k} < 0(> 0), \qquad v_{2j} = 0 \quad \text{for } j < k. \tag{1.20}$$

In the latter case the origin is called a kth order weak focus of (1.4).

We remark that the conclusions of Theorem 1.3 provide new and useful information on the property of the coefficients v_k. Liu and Li [12] also gave some new methods to compute the focal values $v_2, v_4, \ldots, v_{2k}$, or equivalent values and studied the problem of limit cycle bifurcations near the origin (using the second conclusion of the above theorem). They found a new phenomenon: a node can generate a limit cycle when its stability changes.

In this paper we study the problem of limit cycle bifurcations near the origin of the analytic system

$$\dot{x} = y + X(x, y, \delta), \qquad \dot{y} = Y(x, y, \delta), \tag{1.21}$$

where $\delta = (\delta_1, \ldots, \delta_m) \in D \subset \mathbb{R}^m$, D is a simply connected domain, and $X, Y = O(|x, y|^2)$ for $|x|$ small and $\delta \in D$.

To perform our analysis we first introduce a novel Poincaré map using a specific transversal section, study its analytical properties (Theorems 1.5 and 1.6), and then give a new definition of generalized focal values (or generalized Lyapunov constants) following [12]. Second, using the Poincaré map together with the generalized focal values we establish new bifurcation theory of limit cycles from a nilpotent focus or center and obtain conditions for finding a lower bound and an upper bound of the maximal number of limit cycles bifurcated directly from the nilpotent point (Theorems 1.7 and 1.8). Third, we provide a new method to compute the generalized focal values using the normal form (Theorems 1.10 and 1.11). Moreover, we prove that the normal form and the original system have the same generalized focal values when the higher-order term has a sufficiently high order (Theorem 1.12 and Corollary 1.13). For polynomial systems we prove that the maximal order of a nilpotent focus is uniformly bounded (Theorem 1.14). All these results directly generalize the classical Hopf bifurcation theory and establish the new bifurcation theory in the nilpotent case.

We now state our main results more precisely. Let $y = F(x, \delta)$ be the solution to the equation $y + X(x, y, \delta) = 0$. We define the following two functions:

$$g(x, \delta) = -Y(x, F(x, \delta), \delta), \qquad f(x, \delta) = -\left[\frac{\partial X}{\partial x}(x, F(x, \delta), \delta) + \frac{\partial Y}{\partial y}(x, F(x, \delta), \delta)\right]. \tag{1.22}$$

By Theorem 1.1, if

$$g(x, \delta) = \sum_{j \geq 2n-1} a_j(\delta) x^j, \quad n \geq 2, \; a_{2n-1}(\delta) > 0, \tag{1.23}$$

$$f(x, \delta) = \sum_{j \geq n-1} b_j(\delta) x^j, \qquad b_{n-1}^2(\delta) - 4n a_{2n-1}(\delta) < 0, \tag{1.24}$$

then the origin is a center or a focus of (1.21) for all $\delta \in D$. For convenience, we introduce the following definition.

Definition 1.4. Let for all $\delta \in D$ (1.23) and (1.24) be satisfied for some $n \geq 2$ so that the the origin is a center or a focus of (1.21). In this case we say that (1.21) has a singular point of multiplicity n at the origin.

Next, let us define a Poincaré return map for the planar system (1.21). For each $\delta \in D$ and $x_0 \neq 0$ with $|x_0|$ small consider the solution $(x(t, x_0, \delta), y(t, x_0, \delta))$ of (1.21) with the initial condition $(x(0), y(0)) = (x_0, F(x_0, \delta))$. Then there is a unique least positive number $\tau = \tau(x_0, \delta) > 0$ such that $y(\tau, x_0, \delta) = F(x(\tau, x_0, \delta), \delta)$ and $x_0 x(\tau, x_0, \delta) > 0$ (see Figure 1 for the case of small $x_0 > 0$).

Assume that for all $\delta \in D$ (1.23) and (1.24) are satisfied, so that the solution $x(\tau, x_0, \delta)$ exists for $0 < |x_0| < \varepsilon_0$, where $\varepsilon_0 = \varepsilon_0(D)$ is a small positive constant. Define

$$P(x_0, \delta) = \begin{cases} x(\tau, x_0, \delta), & 0 < |x_0| < \varepsilon_0, \\ 0, & x_0 = 0. \end{cases} \tag{1.25}$$

The map $P(x_0, \delta)$ is the Poincaré map we will use for the remainder of the paper.

Obviously, the function $P(x_0, \delta)$ is continuous at $x_0 = 0$ if (1.23) and (1.24) hold. It is easily seen that (1.21) has a periodic orbit near the origin if and only if the map has two fixed points near zero: one positive and the other one negative. Moreover, we note that the function is uniquely defined since the Poincaré section is chosen to be on the curve $y = F(x, \delta)$. This enables us to obtain some nice analytical properties of this function at $x_0 = 0$, as stated in the following theorem.

Theorem 1.5. *Let* (1.21) *satisfy* (1.23) *and* (1.24) *for all* $\delta \in D$. *Then there is a unique analytic function* $\overline{P}(x_0, \delta)$ *in* x_0 *at* $x_0 = 0$, *satisfying* $(\partial \overline{P}/\partial x_0)(0, \delta) > 0$ *and such that the displacement function* $\overline{d}(x_0, \delta)$ *has the expansion*

$$\overline{d}(x_0, \delta) = \overline{P}(x_0, \delta) - x_0 = \sum_{j \geq 1} v_j(\delta) x_0^j \tag{1.26}$$

for $|x_0|$ *sufficiently small, where*

(1) *if* n *is odd, then* $P(x_0, \delta) = \overline{P}(x_0, \delta)$ *for all* $|x_0|$ *small;*

(2) *if* n *is even, then for all* $|x_0|$ *small*

$$P(x_0, \delta) = \begin{cases} \overline{P}(x_0, \delta) & \text{for } x_0 > 0, \\ \overline{P}^{-1}(x_0, \delta) & \text{for } x_0 < 0, \end{cases} \tag{1.27}$$

where $\overline{P}^{-1}$ *denotes the inverse of* $\overline{P}$ *in* x_0.

It follows from the theorem that system (1.21) has a periodic orbit near the origin if and only if the analytic function $\overline{d}$ defined by (1.26) has two zeros in x_0 near $x_0 = 0$, among

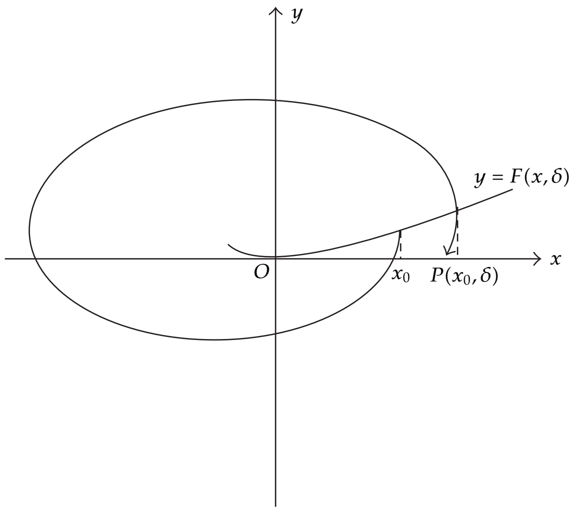

Figure 1: The Poincaré map of (1.21) with $x_0 > 0$.

which one is positive and the other one is negative. The function $\overline{d}$ is called the displacement function or the bifurcation function of (1.21).

The above theorem tells us that the function $P(x_0, \delta)$ is analytic in x_0 at $x_0 = 0$ if n is odd, and not analytic in x_0 at $x_0 = 0$ if n is even unless the origin is a center (in this case P is the identity). The theorem is a natural generalization of the case of elementary center or focus to the nilpotent case ($n > 1$), but it deals with two different cases (odd and even n), and the phenomenon in the case of n even is new.

For the property of the coefficients v_j in (1.26) we have the following theorem which is more general than Theorem 1.3.

Theorem 1.6. *Let* (1.21) *satisfy* (1.23) *and* (1.24) *for all* $\delta \in D$*. Then*

(1) *for n odd we have* $v_{2k} \in \langle v_1, v_3, \ldots, v_{2k-1} \rangle$, $k \geq 1$;

(2) *for n even we have* $v_1 = 0$, $v_{2k+1} \in \langle v_2, v_4, \ldots, v_{2k} \rangle$, $k \geq 1$.

Define $p_n = [1 + (-1)^n]/2$. Then the conclusions of the above theorem can be written uniformly as

$$v_{2k+p_n} \in \left\langle v_{1+p_n}, v_{3+p_n}, \ldots, v_{2k-1+p_n} \right\rangle, \quad k \geq 1. \tag{1.28}$$

From the proof of Theorem 1.6 we will see that v_{2k+p_n} depends on $v_{1+p_n}, v_{3+p_n}, \ldots, v_{2k-1+p_n}$ smoothly. Using Theorem 1.6 we derive the following two statements on limit cycle bifurcations near the origin.

Theorem 1.7 (bifurcations from the focus). *Let* (1.21) *satisfy* (1.23) *and* (1.24) *for all* $\delta \in D$. *Denote that* $p_n = [1 + (-1)^n]/2$.

(1) *If there is an integer* $k \geq 1$ *such that*

$$\sum_{j=1}^{k+1} \left| v_{2j-1+p_n}(\delta) \right| > 0, \quad \forall \delta \in D, \tag{1.29}$$

then there exists a neighborhood U *of the origin such that* (1.21) *has at most* k *limit cycles in* U *for all* $\delta \in \overline{D}$, *where* $\overline{D}$ *is any compact subset of* D.

(2) *If there is* $\delta_0 \in D$ *such that* $v_{2k+1+p_n}(\delta_0) \neq 0$, *then for all* $\delta \in D$ *near* δ_0 (1.21) *has at most* k *limit cycles in a neighborhood of the origin. If further,*

$$\begin{gathered} v_{2j-1+p_n}(\delta_0) = 0, \quad j = 1, \ldots, k, \\ \operatorname{rank} \frac{\partial(v_{1+p_n}, v_{3+p_n}, \ldots, v_{2k-1+p_n})}{\partial(\delta_1, \delta_2, \ldots, \delta_m)}(\delta_0) = k, \end{gathered} \tag{1.30}$$

then for an arbitrary sufficiently small neighborhood of the origin there are some $\delta \in D$ *near* δ_0 *such that* (1.21) *has exactly* k *limit cycles in the neighborhood.*

Theorem 1.8 (bifurcations from the center). *Let* (1.21) *satisfy* (1.23) *and* (1.24) *for all* $\delta \in D$. *Assume*

(i) *there exist* $\delta_0 \in D$ *and an integer* $k \geq 1$ *such that* (1.30) *is satisfied,*

(ii) *the origin is a center of* (1.21) *if* $v_{2j-1+p_n}(\delta_0) = 0$, $j = 1, \ldots, k$,

then there exists a neighborhood U *of the origin such that* (1.21) *has at most* $k-1$ *limit cycles in* U *for all* $\delta \in D$ *near* δ_0, *and also, for an arbitrary sufficiently small neighborhood of the origin, there are some* $\delta \in D$ *near* δ_0 *such that* (1.21) *has exactly* $k-1$ *limit cycles in the neighborhood. Hence, the cyclicity of the system at the point* δ_0 *is equal to* $k-1$.

Now, different from [11–15], we give the following definition.

Definition 1.9. We call $v_{2k+1+p_n}(\delta)$ the generalized focal value of order k of (1.21) at the origin and call the origin a focus of order k if $v_{2k+1+p_n}(\delta) \neq 0$ and $v_{2j-1+p_n}(\delta) = 0$ for $j = 1, \ldots, k$.

The above definition is very reasonable and natural, since by Theorem 1.7, we see that a nilpotent focus of order k generates at most k limit cycles under perturbations which satisfy (1.23) and (1.24).

By the above definition, condition (ii) of Theorem 1.8 means that the origin is a focus of (1.21) of order at most $k-1$. This condition alone is not enough to ensure the conclusion of the theorem. For example, using Theorem 1.10 stated below one can prove that the system

$$\dot{x} = y - \left(a_1 x^3 - a_2^2 x^5 + a_2 x^7\right), \qquad \dot{y} = -x^3 \tag{1.31}$$

has exactly two limit cycles near the origin for $0 < a_1 \ll a_2 \ll 1$. But, the focus at the origin has the order at most 1 for $(a_1, a_2) \neq 0$ (the origin is a center for $a_1 = a_2 = 0$).

The generalized focal values $v_{1+p_n}, v_{3+p_n}, \ldots, v_{2k+1+p_n}, \ldots$ can be calculated using the normal form of system (1.21). We will give a different method to compute them. From a result of Stróżyna and Żołądek [17] we know that (1.21) has the analytic normal form:

$$\dot{x} = y, \quad \dot{y} = -g(x,\delta) - yf(x,\delta). \tag{1.32}$$

Note that f and g in (1.32) may be different from the ones given by (1.22). As before, let $\delta \in D \subset \mathbb{R}^m$ where D is a domain. Also, suppose that for small $|x|$ the function $g(x,\delta)$ satisfies (1.23). Define

$$F(x,\delta) = \int_0^x f(x,\delta)dx, \qquad G(x,\delta) = \int_0^x g(x,\delta)dx. \tag{1.33}$$

It is easy to see that for $xy < 0$ the equation $G(x,\delta) = G(y,\delta)$ defines a unique analytic function $y = \alpha(x,\delta) = -x + O(x^2)$. Set

$$F(\alpha(x,\delta),\delta) - F(x,\delta) = \sum_{j\geq 1} B_j(\delta)x^j. \tag{1.34}$$

By Theorem 1.1, if (1.32) satisfies (1.23) and (1.24), then it has a center or focus at the origin. Thus, under (1.23) and (1.24) the Poincaré map for (1.32) is well defined near the origin.

Theorem 1.10. *Let* (1.32) *satisfy* (1.23) *and* (1.24) *for all $\delta \in D$. Then, for $x_0 > 0$ small, the Poincaré map $P(x_0,\delta)$ has the form:*

$$P(x_0,\delta) - x_0 = \sum_{j\geq 0} v_{2j+1+p_n}(\delta)x_0^{2j+1+p_n}\left(1 + P_j^*(x_0,\delta)\right), \tag{1.35}$$

where $P_j^*(x_0,\delta) = O(x_0)$,

$$\begin{aligned} v_{1+p_n}(\delta) &= K_l^* B_{2l+1}(\delta) + (1-p_n)O\left(B_{2l+1}^2\right), \\ v_{2j+1+p_n}(\delta) &= K_{l+j}^* B_{2l+2j+1}(\delta) + \widetilde{B}_{2l+2j+1}(\delta), \quad j \geq 1, \end{aligned} \tag{1.36}$$

$l = [n/2]$, K_{l+j}^*, $j \geq 0$ are positive constants and $\widetilde{B}_{2l+2j+1} \in \langle B_{2l+1}, B_{2l+3}, \ldots, B_{2l+2j-1}\rangle$. Thus, Theorems 1.7 and 1.8 hold if v_{2j+1+p_n} is replaced by $B_{2l+2j+1}$, $j \geq 0$.

Let

$$f(x,\delta) = \sum_{j\geq 0} b_j(\delta)x^j. \tag{1.37}$$

For system (1.32) we have the following result.

Theorem 1.11. *Let* (1.32) *satisfy* (1.23), (1.34), *and* (1.37) *for all* $\delta \in D$. *Assume there exist* $\delta_0 \in D$ *and* $k \geq [n/2]$ *such that*

$$B_{2k+1}(\delta_0) < 0(> 0), \quad B_{2j-1}(\delta_0) = 0, \quad j = 1, \ldots, k. \tag{1.38}$$

Let one of the following conditions be satisfied:

(a) $n = 2$, *and*

$$b_0(\delta_0) = 0, \qquad b_1^2(\delta_0) - 8a_3(\delta_0) < 0; \tag{1.39}$$

(b) $n > 2$, $g(-x, \delta) = -g(x, \delta)$, $f(-x, \delta) = f(x, \delta)$, *and*

$$b_j(\delta_0) = 0 \quad \text{for } j = 0, \ldots, n-2, \qquad b_{n-1}^2(\delta_0) - 4na_{2n-1}(\delta_0) < 0. \tag{1.40}$$

Then

(1) *for* $\delta = \delta_0$ (1.32) *has a stable (unstable) focus at the origin.*

(2) *If further*

$$\operatorname{rank} \frac{\partial(B_1, B_3, \ldots, B_{2k-1})}{\partial(\delta_1, \delta_2, \ldots, \delta_m)}(\delta_0) = k, \tag{1.41}$$

then for an arbitrary sufficiently small neighborhood of the origin there are some $\delta \in D$ *near* δ_0 *such that* (1.32) *has at least* k *limit cycles in the neighborhood.*

From Theorems 1.5–1.10, it seems that under (1.23) and (1.24) we have solved the problem of limit cycle bifurcation for generic systems. Theoretically it is, but in practice it is not. The reason is that in general we do not know what is the transformation from (1.21) to its normal form (1.32). Here we give a method to solve the problem completely both theoretically and in practice. It includes three main steps described below.

Under (1.23) and (1.24) by the normal form theory (see, e.g., [16]) for any integer $m > 2n - 1$ there is a change of variables of the form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} + H_m(u, v, \delta), \tag{1.42}$$

where $H_m(u, v, \delta) = O(|u, v|^2)$ is a polynomial in u, v of degree at most m, such that it transforms (1.21) into (1.43) (called the normal form of order m of (1.21), or the Takens normal form; we still use (x, y) for the new variables u, v)

$$\dot{x} = y + X_{m+1}(x, y, \delta), \qquad \dot{y} = -g_m(x, \delta) - y f_{m-1}(x, \delta) + Y_{m+1}(x, y, \delta), \tag{1.43}$$

where

$$g_m(x,\delta) = \sum_{j=2n-1}^{m} a_j(\delta)x^j, \qquad f_{m-1}(x,\delta) = \sum_{j=n-1}^{m-1} b_j(\delta)x^j \tag{1.44}$$

with $a_{2n-1}(\delta) > 0$ and $b_{n-1}^2(\delta) - 4na_{2n-1}(\delta) < 0$, and $X_{m+1}(x,y,\delta)$, $Y_{m+1}(x,y,\delta)$ being analytic functions satisfying $X_{m+1},\ Y_{m+1} = O(|x,y|^{m+1})$. Here, we should mention that the functions g_m and f_{m-1} depend only on the terms of degree at most m of the expansions of the functions X and Y in (1.21) at the origin.

The Poincaré maps of (1.21) and (1.43) are essentially the same. Denote the Poincaré map of (1.43) by $P(x_0,\delta)$, and then the displacement function has the expansion

$$P(x_0,\delta) - x_0 = \sum_{j\geq 1} v_j(\delta)x_0^j, \tag{1.45}$$

where the series converge for small $|x_0|$.

Truncating the series (1.43) at terms of order m we obtain the polynomial system

$$\dot{x} = y, \quad \dot{y} = -g_m(x,\delta) - yf_{m-1}(x,\delta). \tag{1.46}$$

In practice, for given system (1.21) it is not difficult to find the corresponding system (1.46). For (1.46) we can use Theorem 1.10 to find its focal values at the origin up to any large order. Let $P_m(x_0,\delta)$ denote the Poincaré map of (1.46). We write the expansion of the displacement function as

$$P_m(x_0,\delta) - x_0 = \sum_{j\geq 1} \overline{v}_j(\delta)x_0^j, \tag{1.47}$$

where the series converge for small $|x_0|$.

We intend to use $\overline{v}_j(\delta)$ instead of $v_j(\delta)$. To this end, we have to solve the following problem. For any given $k > 1$ find $m > 2n-1$ such that $v_j(\delta) = \overline{v}_j(\delta)$ for $1 \leq j \leq k$. The following theorem gives a solution.

Theorem 1.12. *Consider* (1.43) *and* (1.46)*. Then, for any integer* $k \geq 1$*, if* $m \geq (k+2)n-2$*, then*

$$v_j(\delta) = \overline{v}_j(\delta) \quad \text{for } 1 \leq j \leq kn. \tag{1.48}$$

Therefore, we have the following.

Corollary 1.13. *Under* (1.23) *and* (1.24) *for any integer* $k \geq 1$ *for* (1.21) *the coefficients* $v_1, v_2, \ldots, v_{kn}$ *in* (1.26) *depend only on the terms of degree at most* $(k+2)n-2$ *of the expansions of the functions* X *and* Y *at the origin.*

In the case of elementary center or focus, the above conclusion is well-known.

Finally, we consider the following polynomial system of degree k

$$\dot{x} = y + \sum_{2 \le i+j \le k} a_{ij} x^i y^j, \qquad \dot{y} = \sum_{2 \le i+j \le k} b_{ij} x^i y^j. \tag{1.49}$$

Theorem 1.14. *For any $k \ge 3$ there is an integer $N_k > 0$ such that an arbitrary polynomial system of the form* (1.49) *has a singular point of multiplicity at most N_k at the origin (i.e., one must have $2 \le n \le N_k$ if* (1.49) *satisfies* (1.23) *and* (1.24)*, see Definition 1.4). Further, for each $2 \le n \le N_k$, if* (1.49) *satisfies* (1.23) *and* (1.24)*, then there exists an integer $K_n(k) > 0$ such that for* (1.49) *the origin is a focus of order at most $K_n(k)$. Hence, the origin as a nilpotent focus of* (1.49) *generates at most $K_n(k)$ limit cycles.*

We organize the paper as follows. In Section 2 we give preliminary lemmas. In Section 3 we prove our main results. In Section 4 few examples are provided.

2. Preliminaries

Consider system (1.21). In this section we will always suppose that (1.23) and (1.24) are satisfied. Introducing the new variable $v = y - F(x, \delta)$ we obtain from (1.21) (reusing y for v)

$$\begin{aligned} \dot{x} &= y(1 + Z_1(x, y, \delta)), \\ \dot{y} &= -g(x, \delta) - y f(x, \delta) + y^2 Z_2(x, y, \delta), \end{aligned} \tag{2.1}$$

where the functions f and g are given by (1.22), Z_1 and Z_2 are analytic functions near the origin with $Z_1(x, y, \delta) = O(|x, y|)$. In the discussion below for convenience we will often omit δ. As suggested by Liu and Li in [15] we pass in (2.1) to the generalized polar coordinates

$$x = r\cos\theta, \qquad y = r^n \sin\theta, \quad r > 0. \tag{2.2}$$

Lemma 2.1. *Let* (1.23) *and* (1.24) *be satisfied. Then the substitution* (2.2) *transforms* (2.1) *into the system*

$$\begin{aligned} \dot{\theta} &= S(\theta, r) = \frac{r^{n-1}}{H(\theta)}[S_0(\theta) + O(r)], \\ \dot{r} &= R(\theta, r) = \frac{r^n}{H(\theta)}[R_0(\theta) + O(r)], \end{aligned} \tag{2.3}$$

where S and R are 2π-periodic in θ and have the properties

$$S\left(\pi + (-1)^{n-1}\theta, -r\right) = (-1)^{n-1} S(\theta, r), \qquad R\left(\pi + (-1)^{n-1}\theta, -r\right) = -R(\theta, r), \tag{2.4}$$

$$\begin{aligned} H(\theta) &= \cos^2\theta + n\sin^2\theta > 0, \\ S_0(\theta) &= -\left[n\sin^2\theta + b_{n-1}\cos^n\theta\sin\theta + a_{2n-1}\cos^{2n}\theta\right] < 0, \\ R_0(\theta) &= \cos\theta\sin\theta\left(1 - a_{2n-1}\cos^{2n-2}\theta - b_{n-1}\sin\theta\cos^{n-2}\theta\right). \end{aligned} \tag{2.5}$$

Proof. From (2.2) we have

$$\dot{x} = \cos\theta\dot{r} - r\sin\theta\dot{\theta}, \qquad \dot{y} = nr^{n-1}\sin\theta\dot{r} + r^n\cos\theta\dot{\theta}. \tag{2.6}$$

We solve the above equations for $\dot{\theta}$ and $\dot{r}$ and obtain (2.3) with

$$\begin{aligned} S(\theta, r) &= \frac{\cos\theta\dot{y} - nr^{n-1}\sin\theta\dot{x}}{r^n\left(\cos^2\theta + n\sin^2\theta\right)}, \\ R(\theta, r) &= \frac{\sin\theta\dot{y} + r^{n-1}\cos\theta\dot{x}}{r^{n-1}\left(\cos^2\theta + n\sin^2\theta\right)}. \end{aligned} \tag{2.7}$$

Then noting that

$$\cos(\pi \pm \theta) = -\cos\theta, \qquad \sin(\pi \pm \theta) = \mp\sin\theta \tag{2.8}$$

and that (2.2) is invariant as (θ, r) is replaced by $(\pi + (-1)^{n-1}\theta, -r)$ one can easily prove (2.4). The other conclusions are direct. This ends the proof. □

By (2.3) and (2.4) we obtain the following analytic 2π-periodic equation

$$\frac{dr}{d\theta} = \overline{R}(\theta, r), \tag{2.9}$$

where

$$\begin{aligned} \overline{R}(\theta, r) &= r\frac{\sin\theta\dot{y} + r^{n-1}\cos\theta\dot{x}}{\cos\theta\dot{y} - nr^{n-1}\sin\theta\dot{x}} \\ &= r\left[\frac{R_0(\theta)}{S_0(\theta)} + O(r)\right], \\ \overline{R}\left(\pi + (-1)^{n-1}\theta, -r\right) &= (-1)^n R(\theta, r). \end{aligned} \tag{2.10}$$

Let $r(\theta, h)$ denote the solution of (2.9) with the initial value $r(0) = h$. For properties of the solution we have the following.

Lemma 2.2. *The solution $r(\theta, h) = O(h)$ is analytic in (θ, h) for $|h|$ small and satisfies the following:*

(1) $r(\theta, -r(\pi, h)) = -r(\pi + (-1)^{n-1}\theta, h)$;

(2) $r(\theta \pm 2\pi, h) = r(\theta, r(\pm 2\pi, h))$.

Proof. Let $\tilde{r}(\theta) = -r(\pi + (-1)^{n-1}\theta, h)$. Then by (2.9) and (2.10) we have

$$\begin{aligned} \frac{d\tilde{r}}{d\theta} &= (-1)^n \overline{R}\Big(\pi + (-1)^{n-1}\theta, r\Big(\pi + (-1)^{n-1}\theta, h\Big)\Big) \\ &= (-1)^n \overline{R}\Big(\pi + (-1)^{n-1}\theta, -\tilde{r}(\theta)\Big) \\ &= \overline{R}(\theta, \tilde{r}(\theta)). \end{aligned} \tag{2.11}$$

This means that $\tilde{r}(\theta)$ is also a solution to (2.9). Then the first conclusion follows from the uniqueness of the solution to the initial problem. The second one follows in the same way. This completes the proof. □

Further we have the following.

Lemma 2.3. *Let $P(x_0, \delta)$ be the Poincaré return map of* (1.21) *defined in Section 1. Then for $|x_0| > 0$ small we have $P(x_0, \delta) = r(-2\pi, x_0)$ for $x_0 > 0$, and $P(x_0, \delta) = r((-1)^n 2\pi, x_0)$ for $x_0 < 0$.*

Proof. First, it is easy to see that (1.21) and (2.1) have the same Poincaré map $P(x_0, \delta)$. Then, noting from (2.3) that $\dot{\theta} < 0$ for small $r > 0$, by the definition of P and (2.2), we see that

$$P(x_0, \delta) = x(\tau, x_0) = r(-2\pi, x_0) \tag{2.12}$$

for $x_0 > 0$ small. Now consider the case of $x_0 < 0$. Let $r^*(\theta, h)$ denote the solution of (2.9) satisfying $r^*(\pi) = h$. Similarly as above we have

$$P(x_0, \delta) = x(\tau, x_0) = -r^*(-\pi, -x_0), \tag{2.13}$$

since under (2.2) the points $(x_0, 0)$ and $(P(x_0, \delta), 0)$ on the (x, y) plane correspond to the points $(\pi, -x_0)$ and $(-\pi, -P(x_0, \delta))$ on the (θ, r) plane, respectively.

Further, by Lemma 2.2(1), we have

$$r^*(\theta, -h) = -r(\pi - \theta, h) \quad \text{for } n \text{ even}, \tag{2.14}$$

$$r^*(\theta, -r(2\pi, h)) = -r(\pi + \theta, h) \quad \text{for } n \text{ odd}. \tag{2.15}$$

Noting that, by Lemma 2.2(2), $x_0 = r(2\pi, h)$ if and only if $h = r(-2\pi, x_0)$, we see that (2.15) becomes

$$r^*(\theta, -x_0) = -r(\pi + \theta, r(-2\pi, x_0)) \quad \text{for } n \text{ odd}. \tag{2.16}$$

Therefore, by (2.14) and (2.16) we have for $x_0 < 0$

$$P(x_0, \delta) = \begin{cases} r(2\pi, x_0) & \text{for } n \text{ even}, \\ r(-2\pi, x_0) & \text{for } n \text{ odd}. \end{cases} \tag{2.17}$$

This ends the proof. □

Lemma 2.4. *Let $d(x_0, \delta) = P(x_0, \delta) - x_0$. Then there exists an analytic function $K(h, \delta)$ convergent for small $|h|$ with $K(0, \delta) = (\partial r/\partial x_0)(\pi, 0) > 0$ such that*

$$d(\tilde{x}_0, \delta) = -K(x_0, \delta) d(x_0, \delta) \tag{2.18}$$

for $x_0 > 0$ small, where $\tilde{x}_0 = -r(\pi, x_0)$.

Proof. By Lemma 2.2, we have

$$r\left((-1)^n 2\pi, \tilde{x}_0\right) = -r(-\pi, x_0) = -r(\pi, r(-2\pi, x_0)). \tag{2.19}$$

Hence, by Lemma 2.3 for $x_0 > 0$

$$\begin{aligned} d(\tilde{x}_0, \delta) &= r\left((-1)^n 2\pi, \tilde{x}_0\right) - \tilde{x}_0 \\ &= -r(\pi, r(-2\pi, x_0)) + r(\pi, x_0) \\ &= -K(x_0, \delta)[r(-2\pi, x_0) - x_0] \\ &= -K(x_0, \delta) d(x_0, \delta), \end{aligned} \tag{2.20}$$

where

$$K(x_0, \delta) = \int_0^1 \frac{\partial r}{\partial x_0}(\pi, x_0 + s(r(-2\pi, x_0) - x_0)) ds. \tag{2.21}$$

It is obvious that K is analytic for $|x_0|$ small and $K(0, \delta) = (\partial r/\partial x_0)(\pi, 0) > 0$. This completes the proof. □

3. Proof of the Main Results

In this section we prove our main results presented in Theorems 1.5–1.12.

Proof of Theorem 1.5. We take $\overline{P}(x_0, \delta) = r(-2\pi, x_0)$ for $|x_0|$ small. Then by Lemma 2.2 $\overline{P}$ is analytic. Note that by Lemma 2.2, $r(2\pi, x_0)$ is the inverse of $r(-2\pi, x_0)$ in x_0. Then Theorem 1.5 follows directly from Lemma 2.3. The proof is complete. □

Proof of Theorem 1.6. There are two cases to consider separately.
Case A (n odd). By (1.26) and Theorem 1.5(1), we have

$$d(x_0,\delta) = \overline{d}(x_0,\delta) = \sum_{j\geq 1} v_j(\delta)x_0^j \tag{3.1}$$

for all $|x_0|$ small.

By Lemma 2.4, we can suppose that

$$K(x_0,\delta) = \sum_{j\geq 0} k_j x_0^j, \qquad \widetilde{x}_0 = -r(\pi,x_0) = \sum_{j\geq 1} l_j x_0^j, \tag{3.2}$$

where $k_0 > 0$, $l_1 = -k_0$. Substituting (3.1) and (3.2) into (2.18), we obtain

$$\sum_{j\geq 1} v_j \left(\sum_{i\geq 1} l_i x_0^i\right)^j = -\sum_{i\geq 0, j\geq 1} k_i v_j x_0^{i+j}. \tag{3.3}$$

Comparing the coefficients of the terms x_0^2, x_0^4 and x_0^{2j} on both sides yields

$$\begin{aligned}
&v_2 l_1^2 + v_1 l_2 = -(k_0 v_2 + k_1 v_1),\\
&v_4 l_1^4 + 3v_3 l_1^2 l_2 + v_2\left(l_2^2 + 2l_1 l_3\right) + v_1 l_4 = -(k_0 v_4 + k_1 v_3 + k_2 v_2 + k_3 v_1),\\
&\cdots\\
&v_{2j} l_1^{2j} + v_{2j-1} L_{1,j}(l_1,l_2) + \cdots + v_2 L_{2j-2,j}\left(l_1,l_2,\ldots,l_{2j-1}\right) + v_1 l_{2j} = -\sum_{i=0}^{2j-1} k_i v_{2j-i},\\
&\cdots,
\end{aligned} \tag{3.4}$$

where $L_{i,j}(l_1,l_2,\ldots,l_{i+1})$, $i = 1,2,\ldots,2j-2$, are polynomials. Thus, from the above equations we obtain

$$v_{2k} \in \langle v_1, v_3, \ldots, v_{2k-1}\rangle, \quad k \geq 1. \tag{3.5}$$

Case B (n even). By (1.26) and $x_0 = \overline{P}^{-1}(\overline{P}(x_0,\delta),\delta)$ we find that

$$\overline{P}^{-1}(x_0,\delta) = \widetilde{v}_1 x_0 + \widetilde{v}_2 x_0^2 + \widetilde{v}_3 x_0^3 + \cdots, \tag{3.6}$$

where

$$
\begin{aligned}
\tilde{v}_1 &= (v_1+1)^{-1},\\
\tilde{v}_2 &= -v_2(v_1+1)^{-3},\\
&\cdots\\
\tilde{v}_j &= -v_j(v_1+1)^{-(j+1)} + L_j(v_2, v_3, \dots, v_{j-1}),\\
&\cdots
\end{aligned}
\tag{3.7}
$$

and each L_j is a polynomial of degree at least 2. Now we suppose that $x_0 > 0$. Then (3.1) holds by Theorem 1.5. Further, noting that $\tilde{x}_0 < 0$ by Theorem 1.5 again

$$d(\tilde{x}_0, \delta) = P(\tilde{x}_0, \delta) - \tilde{x}_0 = \overline{P}^{-1}(\tilde{x}_0, \delta) - \tilde{x}_0. \tag{3.8}$$

Then, inserting (3.1), (3.2), (3.6), and (3.8) into (2.18) we obtain

$$
\begin{aligned}
&(\tilde{v}_1 - 1)l_1 = -k_0 v_1,\\
&(\tilde{v}_1 - 1)l_2 + \tilde{v}_2 l_1^2 = -(k_0 v_2 + k_1 v_1),\\
&\cdots\\
&(\tilde{v}_1 - 1)l_j + L_j(\tilde{v}_2, \tilde{v}_3, \dots, \tilde{v}_{j-1}) + \tilde{v}_j l_1^j = -(k_0 v_j + k_1 v_{j-1} + \cdots + k_{j-1} v_1),\\
&\cdots,
\end{aligned}
\tag{3.9}
$$

where

$$L_j(\tilde{v}_2, \tilde{v}_3, \dots, \tilde{v}_{j-1}) \in \langle \tilde{v}_2, \tilde{v}_3, \dots, \tilde{v}_{j-1} \rangle. \tag{3.10}$$

Finally, noting that $l_1 = -k_0$ and substituting (3.7) into (3.9) we easily see that

$$v_{2j+1} \in \langle v_2, v_4, \dots, v_{2j} \rangle, \quad j \geq 1. \tag{3.11}$$

This ends the proof. □

Proof of Theorem 1.7. For the first part, suppose that the conclusion is not true. Then there exists a sequence $\{\delta_m\}$ in $\overline{D}$ such that for $\delta = \delta_m$ (1.21) has $k+1$ limit cycles $L_{m,1}, L_{m,2}, \dots, L_{m,k+1}$ which approach the origin as $m \to \infty$. Then by Theorem 1.6, the function $\overline{d}(x_0, \delta_m)$ has $2k+2$ nonzero roots in x_0 which approach zero as $m \to \infty$.

Since $\overline{D}$ is compact, we can assume $\delta_m \to \delta_0 \in \overline{D}$ as $m \to \infty$. By our assumption, $\sum_{j=1}^{k+1} |v_{2j-1+p_n}(\delta_0)| > 0$. Thus, for some $1 \leq l \leq k+1$,

$$v_{2l-1+p_n}(\delta_0) \neq 0, \quad v_{2j-1+p_n}(\delta_0) = 0 \quad \text{for } 1 \leq j \leq l-1. \tag{3.12}$$

Therefore, by (1.26) and Theorem 1.6, we have

$$\overline{d}(x_0,\delta_0)=v_{2l-1+p_n}(\delta_0)x_0^{2l-1+p_n}+O\left(x_0^{2l+p_n}\right). \tag{3.13}$$

Note that $\overline{d}(0,\delta)=0$. It follows from Rolle's theorem (see [1]) that for some $\varepsilon_0>0$ the function $\overline{d}(x_0,\delta)$ has at most $2l-2+p_n$ nonzero roots in $(-\varepsilon_0,\varepsilon_0)$ for all $|\delta-\delta_0|<\varepsilon_0$. We have proved that the function $\overline{d}(x_0,\delta_m)$ has $2k+2$ nonzero roots which approach zero as $m\to\infty$. It then follows that $2k+2\le 2l-2+p_n$, contradicting to $2l-2+p_n\le 2k+p_n\le 2k+1$. The first conclusion follows.

For the second one, from the above proof one can see that for all $\delta\in D$ near δ_0 (1.21) has at most k limit cycles in a neighborhood of the origin. Then, by Theorem 1.6, the displacement function $\overline{d}$ can be written as

$$\overline{d}(x_0,\delta)=\sum_{j\ge1}v_{2j-1+p_n}(\delta)x_0^{2j-1+p_n}\left(1+P_j(x_0,\delta)\right), \tag{3.14}$$

where $P_j(0,\delta)=0$. Like in [19] one can show that P_j are series convergent in a neighborhood of δ_0 (see also, e.g., [20, 21]). Further, by (1.30), we can take $v_{1+p_n},v_{3+p_n},\dots,v_{2k-1+p_n}$ as free parameters, varying near zero. Precisely, if we change them such that

$$\begin{gathered}0<\left|v_{1+p_n}\right|\ll\left|v_{3+p_n}\right|\ll\cdots\ll\left|v_{2k-1+p_n}\right|\ll1,\qquad v_{1+p_n}v_{3+p_n}<0,\dots,\\ v_{2k-1+p_n}v_{2k+1+p_n}<0,\end{gathered} \tag{3.15}$$

then by (3.14) the function $\overline{d}$ has exactly k positive zeros in x_0 near $x_0=0$, which give k limit cycles. This finishes the proof. □

By Theorem 1.5 and (3.14) we immediately have the following.

Corollary 3.1. *Let* (1.21) *satisfy* (1.23) *and* (1.24) *for a fixed $\delta\in D$. Then, if*

$$v_{2k+1+p_n}(\delta)<0(>0),\qquad v_{2j-1+p_n}(\delta)=0\quad\text{for } j=1,\dots,k, \tag{3.16}$$

then the origin is a stable (unstable) focus of order k of (1.21)*. If*

$$v_{2j-1+p_n}(\delta)=0\quad\forall j\ge1, \tag{3.17}$$

the origin is a center of (1.21)*.*

Proof of Theorem 1.8. Under (1.30) the values $v_{1+p_n},v_{3+p_n},\dots,v_{2k-1+p_n}$ can be taken as free parameters. Further, by our assumption, the origin is a center of (1.21) as $v_{2j-1+p_n}(\delta)=0$, $j=1,\dots,k$. It then follows that

$$v_{2j-1+p_n}(\delta)\in\left\langle v_{1+p_n},v_{3+p_n},\dots,v_{2k-1+p_n}\right\rangle\quad\forall j\ge k+1. \tag{3.18}$$

Therefore, (3.14) can be further written in the form:

$$\overline{d}(x_0,\delta) = \sum_{j=1}^{k} v_{2j-1+p_n}(\delta)x_0^{2j-1+p_n}\left(1+\overline{P}_j(x_0,\delta)\right), \tag{3.19}$$

where $\overline{P}_j(0,\delta) = 0$ and P_j are series convergent in a neighborhood of δ_0 [19]. Using the reasoning of Bautin [1] (see also e.g., [20–22]) one can easily see that the conclusion of the theorem holds. The proof is completed. □

Proof of Theorem 1.10. Now we consider (1.32), where g satisfies (1.23). Let

$$F(x,\delta) = \int_0^x f(x,\delta)dx, \qquad G(x,\delta) = \int_0^x g(x,\delta)dx. \tag{3.20}$$

If f satisfies (1.24), then the origin is a center or focus of (1.32), and

$$F(\alpha(x,\delta),\delta) - F(x,\delta) = \sum_{j\geq n} B_j(\delta)x^j = \sum_{j\geq n_1} B_j(\delta)x^j, \tag{3.21}$$

where

$$B_n = \frac{(-1)^n - 1}{n} b_{n-1}, \qquad n_1 = 2l+1, \quad l = \left[\frac{n}{2}\right], \tag{3.22}$$

and $\alpha(x,\delta) = -x + O(x^2)$ satisfies $G(\alpha(x,\delta),\delta) = G(x,\delta)$ for $|x|$ small. Note that (1.32) is equivalent to the following system:

$$\dot{x} = y - F(x,\delta), \qquad \dot{y} = -g(x,\delta) \tag{3.23}$$

which has the same Poincaré return map $P(x_0,\delta)$ as (1.32). Introducing the change of variables x and t

$$u = [2nG(x,\delta)]^{1/2n}(\operatorname{sgn} x) = (a_{2n-1})^{1/2n}\left(x + O\left(x^2\right)\right) \equiv \varphi(x), \qquad \frac{dt}{dt_1} = \frac{\varphi^{2n-1}(x)}{g(x,\delta)} \tag{3.24}$$

system (3.23) becomes

$$\dot{u} = y - \overline{F}(u,\delta), \qquad \dot{y} = -u^{2n-1}, \tag{3.25}$$

which is equivalent to

$$\dot{u} = y, \qquad \dot{y} = -u^{2n-1} - y\overline{f}(u,\delta), \tag{3.26}$$

where

$$\overline{F}(u,\delta) = F\left(\varphi^{-1}(u),\delta\right), \qquad \overline{f}(u,\delta) = \frac{\partial \overline{F}}{\partial u}(u,\delta). \tag{3.27}$$

The systems (3.25) and (3.26) have the same Poincaré return map, denoted by $P_1(u_0,\delta)$. One can see that the maps P and P_1 have the relation $P_1 \circ \varphi = \varphi \circ P$. Hence,

$$P(x_0,\delta) - x_0 = K(u_0)(P_1(u_0,\delta) - u_0), \tag{3.28}$$

where $K(u_0) = (a_{2n-1})^{-1/2n} + O(u_0)$ is analytic. By (1.26) and (3.14), for $u_0 > 0$ small we have

$$P_1(u_0,\delta) - u_0 = \sum_{j\ge 1} v_j(\delta) u_0^j = \sum_{j\ge 1} v_{2j-1+p_n}(\delta) u_0^{2j-1+p_n}(1 + P_j(u_0,\delta)). \tag{3.29}$$

Hence,

$$\begin{aligned} P(x_0,\delta) - x_0 &= \sum_{j\ge 1} v_{2j-1+p_n}(\delta)(a_{2n-1})^{-1/2n} u_0^{2j-1+p_n}\left(1 + \widetilde{P}_j(u_0,\delta)\right) \\ &= \sum_{j\ge 1} v_{2j-1+p_n}(\delta)(a_{2n-1})^{(2j-2+p_n)/2n} x_0^{2j-1+p_n}\left(1 + P_j^*(x_0,\delta)\right), \end{aligned} \tag{3.30}$$

where $\widetilde{P}_j(u_0,\delta) = O(u_0)$, $P_j^*(x_0,\delta) = O(x_0)$.

Since α satisfies $G(\alpha(x,\delta),\delta) = G(x,\delta)$ and $x\alpha < 0$ for $|x|$ small, we have $\varphi(\alpha) = -\varphi(x)$ or $\alpha = \varphi^{-1}(-\varphi(x)) = \varphi^{-1}(-u)$, where $u = \varphi(x)$. Thus, we have

$$F(\alpha(x,\delta),\delta) - F(x,\delta) = F\left(\varphi^{-1}(-u),\delta\right) - F\left(\varphi^{-1}(u),\delta\right) = \overline{F}(-u,\delta) - \overline{F}(u,\delta). \tag{3.31}$$

Let

$$\overline{f}(u,\delta) = \sum_{j\ge n-1} \overline{b}_j(\delta) u^j. \tag{3.32}$$

Then

$$\overline{F}(u,\delta) = \sum_{j\ge n} \frac{\overline{b}_{j-1}(\delta)}{j} u^j. \tag{3.33}$$

Thus, by (3.31) we have

$$F(\alpha(x,\delta),\delta) - F(x,\delta) = -2 \sum_{j\ge [n/2]} \frac{\overline{b}_{2j}(\delta)}{2j+1} u^{2j+1}. \tag{3.34}$$

Substituting $u = \varphi(x) = (a_{2n-1})^{1/2n}(x + O(x^2))$ into the above equality and comparing with (3.21) we obtain

$$\begin{gathered} B_{2l+1} = -\overline{K}_l \overline{b}_{2l}, \qquad B_{2l+2} = O\left(\overline{b}_{2l}\right), \\ B_{2l+2j+1} = -\overline{K}_{l+j}\overline{b}_{2l+2j} + \widetilde{B}_{2l+2j+1}, \\ B_{2l+2j+2} \in \left\langle \overline{b}_{2l}, \overline{b}_{2l+2}, \dots, \overline{b}_{2l+2j} \right\rangle, \quad j \geq 1, \end{gathered} \tag{3.35}$$

where $\overline{K}_l, \overline{K}_{l+1}, \dots$ are positive constants and $\widetilde{B}_{2l+2j+1} \in \langle \overline{b}_{2l}, \overline{b}_{2l+2}, \dots, \overline{b}_{2l+2j-2} \rangle$.

Then by Theorem 1.5 for $u_0 > 0$ small we clearly have

$$P_1(u_0, \delta) = u_0 + \sum_{j \geq 1} v_j(\delta) u_0^j = \sum_{j \geq 1} V_j(\delta) u_0^j, \tag{3.36}$$

where V_j are introduced before Theorem 1.2. Thus, by Theorem 1.2, we have

$$v_1 = -K_l \overline{b}_{2l} + O\left(\overline{b}_{2l}^2\right), \qquad v_{2j+1}\big|_{v_1 = \cdots = v_{2j-1} = 0} = -K_{l+j}\overline{b}_{2l+2j}, \quad j \geq 1 \tag{3.37}$$

for $n = 2l + 1$ odd, and

$$v_2 = -K_l \overline{b}_{2l}, \qquad v_{2j+2}\big|_{v_2 = \cdots = v_{2j} = 0} = -K_{l+j}\overline{b}_{2l+2j}, \quad j \geq 1 \tag{3.38}$$

for $n = 2l$ even, where K_{l+j}, $j \geq 0$ are positive constants. Hence,

$$\begin{gathered} v_{1+p_n} = -K_l \overline{b}_{2l} + (1 - p_n) O\left(\overline{b}_{2l}^2\right), \\ v_{2j+1+p_n} = -K_{l+j}\overline{b}_{2l+2j} + \overline{\varphi}\left(\overline{b}_{2l}, \overline{b}_{2l+2}, \dots, \overline{b}_{2l+2j-2}\right), \quad j \geq 1, \end{gathered} \tag{3.39}$$

where $\overline{\varphi}(0, 0, \dots, 0) = 0$. Note that (1.32) is analytic in each $\overline{b}_j$. It follows from Theorem 1.5 that $\overline{\varphi}$ is analytic in $(\overline{b}_{2l}, \overline{b}_{2l+2}, \dots, \overline{b}_{2l+2j-2})$, which yields $\overline{\varphi} \in \langle \overline{b}_{2l}, \overline{b}_{2l+2}, \dots, \overline{b}_{2l+2j-2} \rangle$. Then (3.35) and (3.39) together give

$$\begin{gathered} v_{1+p_n} = \frac{K_l}{\overline{K}_l} B_{2l+1} + (1 - p_n) O\left(B_{2l+1}^2\right), \\ v_{2j+1+p_n} = \frac{K_{l+j}}{\overline{K}_{l+j}} B_{2l+2j+1} + \widetilde{B}_{2l+2j+1}, \quad j \geq 1, \end{gathered} \tag{3.40}$$

where $\widetilde{B}_{2l+2j+1} \in \langle B_{2l+1}, B_{2l+3}, \dots, B_{2l+2j-1} \rangle$.

Then (1.36) follows from (3.30) and (3.40). This finishes the proof. □

Proof of Theorem 1.11. Let $|\delta-\delta_0|$ be small. For $n=2$ we have $p_n=1$. Then the first conclusion follows directly from Corollary 3.1 and (1.36)–(1.39). In fact, we have by Theorem 1.10

$$v_{2k}(\delta_0)=K_k^*B_{2k+1}(\delta_0),\quad K_k^*>0,\quad v_{2j}(\delta_0)=0\quad \text{for } j=1,\ldots,k-1. \tag{3.41}$$

For the second conclusion, we first keep $B_1(\delta)=0$ and vary $B_3(\delta),\ldots,B_{2k-1}(\delta)$ near zero to obtain exactly $k-1$ simple limit cycles near the origin. These limit cycles are bifurcated by changing the stability of the focus at the origin $k-1$ times. Then we vary B_1 such that $0<|B_1|\ll|B_3|$, and $B_1B_3<0$. This step produces one more limit cycle bifurcated from the origin by changing the stability of the origin which is a node now by [23]. The theorem is proved for the case $n=2$.

For $n>2$ since $g(-x,\delta)=-g(x,\delta)$, $f(-x,\delta)=f(x,\delta)$ we have

$$b_j(\delta)=0\quad \text{for } j=0,\ldots,n-2,\quad b_{n-1}^2(\delta)-4na_{2n-1}(\delta)<0 \tag{3.42}$$

if

$$B_{2l+1}(\delta)<0(>0),\quad B_{2j-1}(\delta)=0,\quad j=1,\ldots,l \tag{3.43}$$

for some $[n/2]\le l\le k$. In this case the origin is a stable (unstable) focus of (3.23) by Theorem 1.10. If (3.43) holds for some $0\le l<[n/2]$, then by [23] again the origin is a stable (unstable) node of (3.23). Thus, we can proceed similarly as above. The proof is complete. □

We remark that if $g(-x,\delta)=-g(x,\delta)$, then $\alpha(x,\delta)=-x$.

Proof of Theorem 1.12. Consider (1.43). Without loss of generality, we can assume $X_{m+1}=0$ in (1.43) (otherwise, introduce the change of variables $v=y+X_{m+1}(x,y)$). In this case, we can write (1.43) into the form:

$$\dot{x}=y,\quad \dot{y}=-g(x)-f(x)y+y^2\sum_{j\ge 0}\varphi_j(x)y^j, \tag{3.44}$$

where

$$g(x)=g_m(x)+O\Big(|x|^{m+1}\Big),\qquad f(x)=f_{m-1}(x)+O(|x|^m),\qquad \varphi_j(x)=O\Big(|x|^{m-1-j}\Big). \tag{3.45}$$

For the sake of convenience we rewrite the functions g, f, and φ_j as follows:

$$\begin{aligned}
g(x)&=x^{2n-1}\Big[g_0(x)+x^ng_1(x)+x^{2n}g_2(x)+\cdots\Big],\\
f(x)&=x^{n-1}\Big[f_0(x)+x^nf_1(x)+x^{2n}f_2(x)+\cdots\Big],\\
\varphi_j(x)&=\varphi_{j0}(x)+x^{n-1}\varphi_{j1}(x)+x^{2n-1}\varphi_{j2}(x)+\cdots,
\end{aligned} \tag{3.46}$$

where f_j, g_j, and φ_{jl}, $j\ge 0$, $l\ge 1$, are polynomials in x of degree at most $n-1$, and φ_{j0}, $j\ge 0$, are polynomials in x with degree at most $n-2$.

Now we change (3.44) using (2.2) to the system

$$\dot{x} = r^n \sin\theta, \qquad \dot{y} = r^{2n-1}\sum_{j\geq 0} V_j(\theta, r) r^{jn}, \tag{3.47}$$

where by (3.46)

$$\begin{aligned} V_0(\theta, r) &= -\cos^{2n-1}\theta g_0(r\cos\theta) - \sin^n\theta f_0(r\cos\theta) + r\sin^2\theta\varphi_{00}(r\cos\theta), \\ V_j(\theta, r) &= -\cos^{2n-1+jn}\theta g_j(r\cos\theta) - \sin^n\theta\cos^{jn}\theta f_j(r\cos\theta) + r\sin^{2+j}\theta\varphi_{j0}(r\cos\theta) \\ &\quad + \sum_{k=0}^{j-1}\sin^{2+k}\theta\cos^{(j-k)n-1}\theta\varphi_{k,j-k}(r\cos\theta), \quad j \geq 1. \end{aligned} \tag{3.48}$$

This yields the equation

$$\frac{dr}{d\theta} = r\frac{\sum_{j\geq 0} R_j(\theta, r) r^{jn}}{\sum_{j\geq 0} S_j(\theta, r) r^{jn}}, \tag{3.49}$$

where

$$\begin{aligned} S_0(\theta, r) &= -n\sin^2\theta + \cos\theta\, V_0(\theta, r), \qquad R_0(\theta, r) = \sin\theta\cos\theta + \sin\theta V_0(\theta, r), \\ S_j(\theta, r) &= \cos\theta V_j(\theta, r), \qquad R_j(\theta, r) = \sin\theta V_j(\theta, r), \quad j \geq 1. \end{aligned} \tag{3.50}$$

By (3.48) and (3.50) we can further expand S_j and R_j in r to obtain for $j \geq 0$

$$S_j(\theta, r) = \sum_{l=0}^{n-1}\overline{S}_{l+jn}(\theta) r^l, \qquad R_j(\theta, r) = \sum_{l=0}^{n-1}\overline{R}_{l+jn}(\theta) r^l, \tag{3.51}$$

so that the above differential equation can be written as

$$\frac{dr}{d\theta} = r\frac{\sum_{j\geq 0}\overline{R}_j(\theta) r^j}{\sum_{j\geq 0}\overline{S}_j(\theta) r^j}. \tag{3.52}$$

Further, letting

$$\frac{1}{\sum_{j\geq 0}\overline{S}_j(\theta) r^j} = \sum_{j\geq 0}\widetilde{S}_j(\theta) r^j, \tag{3.53}$$

$$\widetilde{R}_j(\theta) = \sum_{k+l=j}\overline{R}_k(\theta)\widetilde{S}_l(\theta), \quad j \geq 0 \tag{3.54}$$

we obtain

$$\frac{dr}{d\theta} = r\sum_{j\geq 0}\widetilde{R}_j(\theta)r^j. \tag{3.55}$$

Note that for any $j \geq 0$, $\widetilde{S}_j$ depends only on $\overline{S}_k$ with $0 \leq k \leq j$. Then by (3.54) one can see that,

$$\text{for any} \quad j \geq 0, \quad \widetilde{R}_j \text{ depends only on } \overline{R}_k \text{ and } \overline{S}_k \text{ with } 0 \leq k \leq j. \tag{3.56}$$

Let $r(\theta, r_0)$ denote the solution of (3.55) with the initial value r_0. The for r_0 small we have

$$r(\theta, r_0) = \sum_{j\geq 1} r_j(\theta)r_0^j, \tag{3.57}$$

where $r_1, r_2, r_3, \ldots$ satisfy $r_1(0) = 1$, $r_2(0) = r_3(0) = \cdots = 0$, and

$$\begin{aligned}
r_1' &= \widetilde{R}_0 r_1,\\
r_2' &= \widetilde{R}_0 r_2 + \widetilde{R}_1 r_1^2,\\
r_3' &= \widetilde{R}_0 r_3 + 2\widetilde{R}_1 r_1 r_2 + \widetilde{R}_2 r_1^3,\\
&\cdots
\end{aligned} \tag{3.58}$$

which implies that, for any $j \geq 1$, the function r_j depends only on $\widetilde{R}_k$ with $0 \leq k \leq j-1$. Hence, by Lemma 2.3, (1.45), and (3.56) we come to the following conclusion.

$$\text{For any } j \geq 1, \quad v_j(\delta) \text{ depends only on } \overline{R}_k \text{ and } \overline{S}_k \text{ with } 0 \leq k \leq j-1. \tag{3.59}$$

Further, by (3.48)–(3.51), one can observe that, for $0 \leq l \leq n-1$, $\overline{S}_l$ and $\overline{R}_l$ depend only on the coefficients of degree l of the polynomials g_0, f_0, and $x\varphi_{00}$ in x. Hence, by (3.59) we see that for $1 \leq j \leq n$, v_j depends only on the coefficients of degree at most $j-1$ of the polynomials g_0, f_0, and $x\varphi_{00}$ in x.

Similarly, for $j \geq 1$ and $0 \leq l \leq n-1$, or $jn \leq l + jn \leq (j+1)n-1$, $\overline{S}_{l+jn}$ and $\overline{R}_{l+jn}$ depend only on the coefficients of degree l of the polynomials g_j, f_j, $x\varphi_{j0}$ and $\varphi_{i,j-i}$ with $i = 0, \ldots, j-1$ in x. In other words, for $jn+1 \leq u \leq (j+1)n$, $\overline{S}_{u-1}$, and $\overline{R}_{u-1}$ depend only on the coefficients of degree $u-1-jn$ of the polynomials g_j, f_j, $x\varphi_{j0}$, and $\varphi_{i,j-i}$ with $i = 0, \ldots, j-1$ in x. Let $N_{[a,b]}$ denote the set of integers in the interval $[a, b]$. Then, for $jn+1 \leq u \leq (j+1)n$, we have

$$N_{[0,u-1]} = \bigcup_{i=0}^{j-1} N_{[in,(i+1)n-1]} \bigcup N_{[jn,u-1]}. \tag{3.60}$$

Thus, for all $k \in N_{[in,(i+1)n-1]}$, $\overline{S}_k$ and $\overline{R}_k$ depend only on g_i, f_i, $x\varphi_{i0}$, and $\varphi_{l,i-l}$ with $l = 0, \dots, i-1$. And for $k \in N_{[jn,u-1]}$, $\overline{S}_k$ and $\overline{R}_k$ depend only on the coefficients of degree $k - jn$ of the polynomials g_j, f_j, $x\varphi_{j0}$, and $\varphi_{l,j-l}$ with $l = 0, \dots, j-1$ in x.

Therefore, by (3.59) for $jn+1 \le u \le (j+1)n$, $v_u(\delta)$ depends only on the functions g_i, f_i, $x\varphi_{i0}$ and $\varphi_{l,i-l}$ with $l = 0, \dots, i-1$, $i = 0, \dots, j-1$ and the coefficients of degree at most $u-1-jn$ of the polynomials g_j, f_j, $x\varphi_{j0}$, and $\varphi_{l,j-l}$ with $l = 0, \dots, j-1$ in x.

We claim that if $j \ge 0$, $m \ge (j+1)n$, then, for $jn+1 \le u \le (j+1)n$, $v_u(\delta)$ depends only on the functions g_i, f_i, with $i = 0, \dots, j-1$ and the coefficients of degree at most $u-1-jn$ of the polynomials g_j, f_j in x.

In fact, by the above discussion, we need only to prove $\varphi_{00} = 0$ in the case $j = 0$ and $\varphi_{ls} = 0$ for $l + s \le j$ and $0 \le l \le j-1$ in the case $j > 0$. This can be shown easily since

$$\begin{gathered} \varphi_{j0} = O\left(|x|^{m-1-j}\right), \qquad \varphi_{js} = O\left(|x|^{m-j-sn}\right) \quad \text{for } s \ge 1, \\ \deg \varphi_{j0} \le n-2, \qquad \deg \varphi_{js} \le n-1 \quad \text{for } s \ge 1 \end{gathered} \tag{3.61}$$

by (3.45) and (3.46).

By (3.46), the above claim can be restated as the claim that if $j \ge 0$, $m \ge (j+1)n$, then for $jn+1 \le u \le (j+1)n$, $v_u(\delta)$ depends only on the coefficients of degree at most $2n+u-2$ of g and the coefficients of degree at most $n+u-2$ of f in x. Thus, for any integers k and m satisfying $k \ge 1$ and $m \ge (k+1)n$, taking $j = 0, \dots, k$ we see that, for all $1 \le u \le (k+1)n$, $v_u(\delta)$ depends only on the coefficients of degree at most $2n+u-2$ of g and the coefficients of degree at most $n+u-2$ of f in x.

Finally, by (3.45), if $m \ge (k+3)n-2$, then

$$2n+u-2 \le m, \quad n+u-2 \le m-1 \quad \text{for } u \le (k+1)n. \tag{3.62}$$

In this case, for all $1 \le u \le (k+1)n$, $v_u(\delta)$ depends only on g_m and f_{m-1} in (3.45). Then the conclusion of Theorem 1.12 follows. □

Proof of Theorem 1.14. Let

$$X(x,y,\delta) = \sum_{2\le i+j\le k} a_{ij}x^i y^j, \qquad Y(x,y,\delta) = \sum_{2\le i+j\le k} b_{ij}x^i y^j. \tag{3.63}$$

Then (1.49) is equivalent to an analytic system of the form (2.1), where the functions f and g are given by (1.22). We can write

$$g(x,\delta) = \sum_{j\ge 2} a_j x^j, \tag{3.64}$$

where all a_j are polynomials in a_{ij} and b_{ij}. Thus, by the well-known Hilbert basis theorem there exists $N_k > 0$ such that $a_j = 0$ for all $j \ge 2N_k$ if $a_2 = \cdots = a_{2N_k-1} = 0$. That is, $a_2 = \cdots = a_{2N_k-1} = 0$ imply that $g = 0$. In this case, all points on the line $y = 0$ and near the origin are singular for (2.1). Thus, the origin has a multiplicity at most N_k as a nilpotent focus or center.

Further, for any $2 \le n \le N_k$ let (1.49) have a nilpotent focus or center at the origin, having multiplicity n. Then for any sufficiently large m, (1.49) is equivalent to a system of the form (1.43), where the coefficients a_j in g_m and b_j in f_{m-1} are all polynomials in a_{ij} and b_{ij}. Moreover, $a_{2n-1} > 0$ and $b_{n-1}^2 - 4na_{2n-1} < 0$. Let

$$F(x) = \int_0^x f_{m-1} dx, \qquad G(x) = \int_0^x g_m dx, \tag{3.65}$$

and $\alpha(x) = -x + \sum_{j\ge 2} \alpha_j x^j$ satisfies $G(\alpha) = G(x)$. Then, noting that

$$G^{1/2n} = \left[\frac{a_{2n-1}}{(2n)}\right]^{1/2n} |x| \left(1 + \sum_{j\ge 1} \beta_j x^j\right), \tag{3.66}$$

where all β_j are polynomials in a_{ij}, b_{ij}, and $c \equiv a_{2n-1}^{-1}$, one can see that all α_j are polynomials in a_{ij}, b_{ij}, and c. Therefore, if we let

$$F(\alpha(x)) - F(x) = \sum_{j\ge n} B_j^m x^j = \sum_{j\ge 2l+1} B_j^m x^j, \tag{3.67}$$

where $l = [n/2]$, then all B_j^m are polynomials in a_{ij}, b_{ij}, and c. By Theorems 1.10 and 1.12, for any integer $k_0 > 0$ there exists an integer $m_0 > 0$ such that B_j^m, $j = 2l+1, \dots, k_0$ are independent of m for all $m \ge m_0$, denoting B_j^m by B_j. Then, we have defined a series of coefficients B_j for all $j \ge 2l + 1$, which are all polynomials in a_{ij}, b_{ij}, and c. Again, by the Hilbert basis theorem there exists $K_n(k) > 0$ such that

$$B_{2l+1} = B_{2l+2} = B_{2l+3} = \cdots = B_{2l+2K_n(k)+1} = 0 \Longrightarrow B_{2l+j+1} = 0, \quad j \ge 2K_n(k) + 1. \tag{3.68}$$

Now, we take $\overline{K}$ and m such that $2K_n(k) + 1 + p_n \le \overline{K}n$, $m > (\overline{K} + 2n) - 2$. Then by Theorems 1.10 and 1.12 again, for (1.49) we have

$$v_{1+p_n} = v_{3+p_n} = \cdots = v_{2K_n(k)+1+p_n} = 0 \tag{3.69}$$

yielding $v_{j+p_n} = 0$, $j \ge 1$. Then the conclusion follows. □

4. Application Examples

In this section we provide some application examples based on the ones given in [11]. Consider a Kukles type system of the form

$$\dot{x} = y, \quad \dot{y} = -\left(a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3\right). \tag{4.1}$$

The authors of [11] proved that if $a_{30} > 0$ and $a_{11}^2 - 8a_{30} < 0$, then for (4.1) $v_2 = v_4 = v_6 = v_8 = 0$ if and only if $a_{21} = a_{03} = a_{11}a_{02} = 0$, which implies that the origin is a center. Moreover, there can be 3 limit cycles near the origin. See Theorem 4.1 in [11] and its proof.

From this conclusion and Theorem 1.7 we have immediately the following.

Proposition 4.1. *Let a_{11}, a_{02}, a_{30}, a_{21}, a_{12}, and a_{03} be bounded parameters satisfying*

$$a_{30} > 0, \qquad a_{11}^2 - 8a_{30} < 0, \qquad |a_{21}| + |a_{03}| + |a_{11}a_{02}| > 0. \tag{4.2}$$

Then there exists a neighborhood V of the origin such that the system (4.1) *has at most* 3 *limit cycles in V.*

Consider now the system

$$\dot{x} = -y + Ax^2 + Bxy + Cy^2, \qquad \dot{y} = x^3 + xy^2 + y^3. \tag{4.3}$$

By Theorem 4.2 in [11] and its proof if $A^2 < 2$, then the origin of (4.3) is always a focus with $|v_2| + |v_4| + |v_6| + |v_8| > 0$. Moreover, there are systems inside (4.3) with at least 3 limit cycles around the origin. Then by Theorem 1.7 again we have the following.

Proposition 4.2. *Let A, B, and C be bounded parameters with $A^2 < 2$. Then there exists a neighborhood V of the origin such that the system* (4.3) *has at most* 3 *limit cycles in V.*

Finally, consider the system

$$\dot{x} = y, \quad \dot{y} = -\left(x^3 + x^5\right) - \sum_{j=0}^{k} b_{2j}x^{2j}y, \tag{4.4}$$

where $k \geq 2$. By Theorems 1.8–1.11, we obtain the following.

Proposition 4.3. *Let b_{2j} be bounded parameters. Then*

(1) *if $b_0 = 0$, the system* (4.4) *has at most $k - 1$ limit cycles near the origin, and $k - 1$ limit cycles can appear.*

(2) *If $b_0 \neq 0$, there are systems inside* (4.4) *which have at least k limit cycles near the origin.*

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