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Research Article

Oscillation Criteria for Second-Order Nonlinear Dynamic Equations on Time Scales

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This paper is concerned with oscillation of second-order nonlinear dynamic equations of the form $(r(t)((y(t)+p(t)y(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+f_1(t,y(\delta_1(t)))+f_2(t,y(\delta_2(t)))=0$ on time scales. By using a generalized Riccati technique and integral averaging techniques, we establish new oscillation criteria which handle some cases not covered by known criteria.

1. Introduction

The theory of time scales was introduced by Stefan Hilger in his Ph.D. thesis in 1988 in order to unify continuous and discrete analysis. Not only can this theory of the so-called "dynamic equations" unify theories of differential equations and difference equations but also extend these classical cases to cases "in between", for example, to the so-called q-difference equations. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ with the topology and ordering inherited form $\mathbb{R}$, and the cases when this time scale is equal to $\mathbb{R}$ or to the integers $\mathbb{Z}$ represent the classical theories of differential and difference equations. Of course many other interesting time scales exist, and they give rise to plenty of applications. In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various dynamic equations on time scales, we refer the reader to [1–14].

In 2006, Wu et al. [1] considered the second-order nonlinear neutral dynamic equation with variable delays

$$\left(r(t)\left(\left(y(t)+p(t)y(\tau(t))\right)^{\Delta}\right)^{\gamma}\right)^{\Delta}+f\left(t,y(\delta(t))\right)=0,\quad t\in\mathbb{T},\tag{1.1}$$

where $\gamma \geq 1$ is a quotient of odd positive integers. In 2007, Saker et al. [2] also discussed (1.1) for an odd positive integer $\gamma \geq 1$. In 2010, Zhang and Wang [3] extended and complemented some results in [1, 2] for $\gamma \geq 1$ and gave some new results for $0 < \gamma < 1$. In 2011, Saker [4] considered (1.1) in different conditions. In 2010, Sun et al. [5] considered the second-order quasiliner neutral delay dynamic equation

$$\left(r(t)\left(\left(y(t)+p(t)y(\tau(t))\right)^{\Delta}\right)^{\gamma}\right)^{\Delta}+q_1x^{\alpha}(\tau_1(t))+q_2x^{\beta}(\tau_2(t))=0, \quad t\in\mathbb{T}, \tag{1.2}$$

where γ, α, and β are quotients of odd positive integers with $0 < \alpha < \gamma < \beta$.

In this paper, we study the second-order nonlinear dynamic equation

$$\left(r(t)\left(\left(y(t)+p(t)y(\tau(t))\right)^{\Delta}\right)^{\gamma}\right)^{\Delta}+f_1\left(t,y(\delta_1(t))\right)+f_2\left(t,y(\delta_2(t))\right)=0, \tag{1.3}$$

on a time scale $\mathbb{T}$, where $p \in C_{\mathrm{rd}}(\mathbb{T},[0,1))$, $f_i \in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$, $i = 1,2$, $\gamma > 0$ is a quotient of odd positive integers.

The paper is organized as follows. In the next section, we give some preliminaries and lemmas. In Section 3, we will use the Riccati transformation technique to prove our main results. In Section 4, we present two examples to illustrate our results.

2. Preliminaries and Lemmas

For convenience, we recall some concepts related to time scales. More details can be found in [6].

Definition 2.1. Let $\mathbb{T}$ be a time scale, for $t \in \mathbb{T}$ the forward jump operator is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, the backward jump operator by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, and the graininess function by $\mu(t) := \sigma(t) - t$, where $\inf\emptyset := \sup\mathbb{T}$ and $\sup\emptyset := \inf\mathbb{T}$. If $\sigma(t) > t$, t is said to be right-scattered, otherwise, it is right-dense. If $\rho(t) < t$, t is said to be left-scattered, otherwise, it is left-dense. The set $\mathbb{T}^{\kappa}$ is defined as follows. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$, otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$.

Definition 2.2. For a function $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$, one defines the delta-derivative $f^{\Delta}(t)$ of $f(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta)\cap\mathbb{T}$ for some δ) such that

$$\left|\left[f(\sigma(t))-f(s)\right]-f^{\Delta}(t)[\sigma(t)-s]\right| \leq \varepsilon|\sigma(t)-s|, \quad \forall s\in U. \tag{2.1}$$

We say that f is delta-differentiable (or in short, differentiable) on $\mathbb{T}^{\kappa}$ provided $f^{\Delta}(t)$ exists, for all $t \in \mathbb{T}^{\kappa}$.

It is easily seen that if f is continuous at $t \in \mathbb{T}$ and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}. \tag{2.2}$$

Moreover, if t is right-dense then f is differential at t if the limit

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s} \tag{2.3}$$

exists as a finite number. In this case

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}. \tag{2.4}$$

In addition, if $f^{\Delta}\geq 0$, then f is nondecreasing. A useful formula is

$$f^{\sigma}(t)=f(t)+\mu(t)f^{\Delta}(t),\quad \text{where } f^{\sigma}(t):=f(\sigma(t)). \tag{2.5}$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma}\neq 0$) of two differentiable functions f and g:

$$\begin{gathered}(fg)^{\Delta}=f^{\Delta}g+f^{\sigma}g^{\Delta}=fg^{\Delta}+f^{\Delta}g^{\sigma},\\ \left(\frac{f}{g}\right)^{\Delta}=\frac{f^{\Delta}g-fg^{\Delta}}{gg^{\sigma}}.\end{gathered} \tag{2.6}$$

Definition 2.3. Let $f:\mathbb{T}\to\mathbb{R}$ be a function, f is called right-dense continuous (rd-continuous) if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. A function $F:\mathbb{T}\to\mathbb{R}$ is called an antiderivative of f provided $F^{\Delta}(t)=f(t)$ holds for all $t\in\mathbb{T}^k$. By the antiderivative, the Cauchy integral of f is defined as $\int_a^b f(s)\Delta s = F(b)-F(a)$, and $\int_a^{\infty} f(s)\Delta s=\lim_{t\to\infty}\int_a^t f(s)\Delta s$.

Let $C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ denote the set of all rd-continuous functions mapping $\mathbb{T}$ to $\mathbb{R}$. It is shown in [6] that every rd-continuous function has an antiderivative. An integration by parts formula is

$$\int_a^b f(t)g^{\Delta}(t)\Delta t=\left[f(t)g(t)\right]\Big|_a^b-\int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t. \tag{2.7}$$

In (1.3), we assume that $\mathbb{T}$ is a time scale and

(h_1) $\tau(t),\delta_i(t)\in C_{\mathrm{rd}}(\mathbb{T},\mathbb{T})$, $\lim_{t\to\infty}\tau(t)=\infty$, $\tau(t)\leq t$, $\lim_{t\to\infty}\delta_i(t)=\infty$, and $t\leq\delta_i(t)$, $i=1,2$,

(h_2) $r(t)\in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^+)$, $\int^{\infty}(1/r(t))^{1/\gamma}\Delta t=\infty$, $p(t)\in C_{\mathrm{rd}}(\mathbb{T},[0,1))$, where $\mathbb{R}^+=(0,\infty)$,

(h_3) $f_i(t,u):\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is continuous function such that $uf_i(t,u)>0$ for all $u\neq 0$, there exist $q_i(t)\in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R}^+)$ $(i=1,2)$, quotients of odd positive integers α and β such that $|uf_1(t,u)|\geq q_1(t)|u|^{\alpha+1}$, $|uf_2(t,u)|\geq q_2(t)|u|^{\beta+1}$, and $0<\alpha<\gamma<\beta$.

Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume throughout that the time scale $\mathbb{T}$ under consideration satisfies $\inf\mathbb{T}=t_0$ and $\sup\mathbb{T}=\infty$. For $T\in\mathbb{T}$, let $[T,\infty)_{\mathbb{T}}:=\{t\in\mathbb{T}:t\geq T\}$. Throughout this paper, these assumptions will be supposed to hold. Let $\tau^*(t) = \min\{\tau(t), \delta_1(t), \delta_2(t)\}$, $T_0 = \min\{\tau^*(t) : t \geq t_0\}$ and $\tau^*_{-1}(t) = \sup\{s \geq t_0 : \tau^*(s) \leq t\}$. Clearly $\tau^*_{-1}(t) \geq t$ for $t \geq T_0$, $\tau^*_{-1}(t)$ is nondecreasing and coincides with the inverse of $\tau^*(t)$ when the latter exists.

By a solution of (1.3), we mean a nontrivial real-valued function $y(t)$ which has the properties $[y(t) + p(t)y(\tau(t))] \in C^1_{\mathrm{rd}}[\tau^*_{-1}(t_0), \infty)$ and $r(t)([y(t) + p(t)y(\tau(t))])^{\Delta})^{\gamma} \in C^1_{\mathrm{rd}}[\tau^*_{-1}(t_0), \infty)$. Our attention is restricted to those solutions of (1.3) that exist on some half line $[t_y, \infty)$ and satisfy $\sup\{|y(t)| : t \geq t_1\} > 0$ for any $t_1 \geq t_y$. A solution $y(t)$ of (1.3) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

For convenience, we use the notation $x(\sigma(t)) = x^{\sigma}(t)$, $x(\delta_i(t)) = x^{\delta_i}(t)$ $(i = 1, 2)$ and $x^{\Delta}(\sigma(t)) = (x^{\Delta}(t))^{\sigma}$, and set

$$x(t) := y(t) + p(t)y(\tau(t)). \tag{2.8}$$

Then (1.3) becomes

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + f_1(t, y(\delta_1(t))) + f_2(t, y(\delta_2(t))) = 0. \tag{2.9}$$

Now, we give the first lemma. Set

$$R_T(t) = \int_T^t \frac{\Delta s}{(r(s))^{1/\gamma}}. \tag{2.10}$$

Lemma 2.4. *Let conditions (h_1)–(h_3) hold. If $y(t)$ is an eventually positive solution of (1.3), then there exists $T \in \mathbb{T}$ sufficiently large such that $x(t) > 0$, $x^{\Delta}(t) > 0$, $(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} < 0$, $x(t) > R_T(t)r^{1/\gamma}(t)x^{\Delta}(t)$, and $(x^{\delta_i}(t)/x^{\sigma}(t)) > (R_T(t)r^{1/\gamma}(t))/(R_T(t)r^{1/\gamma}(t) + \mu(t))$ $(i = 1, 2)$ for $t \in [T, \infty)_{\mathbb{T}}$.*

Proof. If $y(t)$ is an eventually positive solution of (1.3), then by (h_1) there exists a $T \in [t_0, \infty)_{\mathbb{T}}$ such that

$$y(t) > 0, \quad y(\tau(t)) > 0, \quad y(\delta_i(t)) > 0, \quad \text{for } t \geq T,\ i = 1, 2. \tag{2.11}$$

From (2.8), (1.3), and (h_2), we see that $x(t) \geq y(t)$. Also by (1.3) and (h_3), we have

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} \leq -q_1(t)y^{\alpha}(\delta_1(t)) - q_2(t)y^{\beta}(\delta_2(t)) < 0, \quad \text{for } t \geq T, \tag{2.12}$$

which implies that $r(t)(x^{\Delta}(t))^{\gamma}$ is decreasing on $[T, \infty)_{\mathbb{T}}$.

We claim that $r(t)(x^{\Delta}(t))^{\gamma} > 0$ on $[T, \infty)_{\mathbb{T}}$. Assume not, there is a $t_1 \in [T, \infty)_{\mathbb{T}}$ such that $r(t_1)(x^{\Delta}(t_1))^{\gamma} < 0$. Since $r(t)(x^{\Delta}(t))^{\gamma} \leq r(t_1)(x^{\Delta}(t_1))^{\gamma}$ for $t \geq t_1$, we have

$$x^{\Delta}(t) \leq (r(t_1))^{1/\gamma} x^{\Delta}(t_1)\left(\frac{1}{r(t)}\right)^{1/\gamma}. \tag{2.13}$$

Integrating the inequality above form t_1 to t $(\geq t_1)$, by (h_2) we get

$$x(t) \leq x(t_1) + (r(t_1))^{1/\gamma} x^{\Delta}(t_1) \int_{t_1}^{t} \left(\frac{1}{r(s)}\right)^{1/\gamma} \Delta s \longrightarrow -\infty \quad (t \longrightarrow \infty), \tag{2.14}$$

and this contradicts the fact that $x(t) > 0$, for all $t \geq T$. Thus we have $r(t)(x^{\Delta}(t))^{\gamma} > 0$ on $[T,\infty)_{\mathbb{T}}$ and so $x^{\Delta}(t) > 0$ on $[T,\infty)_{\mathbb{T}}$.

Note that

$$\begin{aligned} x(t) > x(t) - x(T) &= \int_T^t x^{\Delta}(s) = \int_T^t \frac{\left(r(s)(x^{\Delta}(s))^{\gamma}\right)^{1/\gamma}}{r^{1/\gamma}(s)} \Delta s \\ &> \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{1/\gamma} \int_T^t \frac{1}{r^{1/\gamma}(s)} \Delta s = R_T(t) r^{1/\gamma}(t) x^{\Delta}(t), \end{aligned} \tag{2.15}$$

we have

$$\begin{aligned} \frac{x^{\sigma}(t)}{x(t)} &= \frac{x(t) + \mu(t) x^{\Delta}(t)}{x(t)} = 1 + \mu(t) \frac{x^{\Delta}(t)}{x(t)} \\ &< 1 + \mu(t) \frac{1}{R_T(t) r^{1/\gamma}(t)} = \frac{R_T(t) r^{1/\gamma}(t) + \mu(t)}{R_T(t) r^{1/\gamma}(t)}. \end{aligned} \tag{2.16}$$

Since $\delta_i(t) \geq t$ and $x^{\Delta}(t) > 0$, we get

$$\frac{x^{\delta_i}(t)}{x^{\sigma}(t)} = \frac{x^{\delta_i}(t)}{x(t)} \cdot \frac{x(t)}{x^{\sigma}(t)} \geq \frac{x(t)}{x^{\sigma}(t)} > \frac{R_T(t) r^{1/\gamma}(t)}{R_T(t) r^{1/\gamma}(t) + \mu(t)}, \quad i = 1, 2. \tag{2.17}$$

The proof is complete. □

Remark 2.5. By $x(t) \geq y(t)$ on $[T,\infty)_{\mathbb{T}}$, $x^{\Delta} > 0$, (1.3), (2.8) and (h_1)–(h_3), we get

$$\begin{aligned} 0 &\geq \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + q_1(t)\left[y(\delta_1(t))\right]^{\alpha} + q_2(t)\left[y(\delta_2(t))\right]^{\beta} \\ &= \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + q_1(t)\left[x(\delta_1(t)) - p(\delta_1(t)) y(\tau(\delta_1(t)))\right]^{\alpha} \\ &\quad + q_2(t)\left[x(\delta_2(t)) - p(\delta_2(t)) y(\tau(\delta_2(t)))\right]^{\beta} \\ &\geq \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + q_1(t)\left[x(\delta_1(t)) - p(\delta_1(t)) x(\tau(\delta_1(t)))\right]^{\alpha} \\ &\quad + q_2(t)\left[x(\delta_2(t)) - p(\delta_2(t)) x(\tau(\delta_2(t)))\right]^{\beta} \\ &\geq \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + q_1(t)\left[1 - p(\delta_1(t))\right]^{\alpha} (x(\delta_1(t)))^{\alpha} \\ &\quad + q_2(t)\left[1 - p(\delta_2(t))\right]^{\beta} (x(\delta_2(t)))^{\beta}. \end{aligned} \tag{2.18}$$

Lemma 2.6 (see [3]). *Let $g(u) = Bu - Au^{(\gamma+1)/\gamma}$, where $A > 0$ and B are constants, γ is a quotient of odd positive integers. Then g attains its maximum value on $\mathbb{R}$ at $u^* = (B\gamma/(A(\gamma+1)))^\gamma$, and*

$$\max_{u\in\mathbb{R}} g = g(u^*) = \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}. \tag{2.19}$$

Lemma 2.7 (see [11]). *x and z are delta-differentiable on $\mathbb{T}$. For $x \neq 0$ and any $t \in \mathbb{T}$, one has*

$$x^\Delta(t)\left(\frac{z^2(t)}{x(t)}\right)^\Delta = \left(z^\Delta(t)\right)^2 - x(t)x^\sigma(t)\left[\left(\frac{z(t)}{x(t)}\right)^\Delta\right]^2. \tag{2.20}$$

3. Main Results

In this section, by employing the Riccati transformation technique we will establish oscillation criteria for (1.3) in two cases: $\gamma \geq 1$ and $0 < \gamma < 1$. Set

$$\begin{gathered}
Q(s) = \left(q_1(s)(1-p(\delta_1(s)))^\alpha\right)^{(\beta-\gamma)/(\beta-\alpha)} \cdot \left(q_2(s)(1-p(\delta_2(s)))^\beta\right)^{(\gamma-\alpha)/(\beta-\alpha)}, \\
Q_1(s) = \left(\frac{R_T(s)r^{1/\gamma}(s)}{R_T(s)r^{1/\gamma}(s)+\mu(s)}\right)^\gamma z(s)Q(s), \\
Q_2(s) = \left(\frac{R_T(s)r^{1/\gamma}(s)}{R_T(s)r^{1/\gamma}(s)+\mu(s)}\right)^\gamma \left(z^2(s)\right)^\sigma Q(s), \\
C(t,s) = H_s^\Delta(t,s) + H(t,s)\frac{z^\Delta(s)}{z(\sigma(s))}, \qquad A(s) = \frac{z^\Delta(s)}{z(\sigma(s))} + \frac{H_s^\Delta(t,s)}{H(t,s)},
\end{gathered} \tag{3.1}$$

(I) $\gamma \geq 1$.

Theorem 3.1. *Assume that (h_1)–(h_3) hold and $\gamma \geq 1$. Furthermore, assume that there exists a positive* rd-continuous Δ-differentiable function $z(t)$ such that for all sufficiently large $T \in \mathbb{T}$,

$$\limsup_{t\to\infty} \int_T^t \left[Q_1(s) - \frac{1}{(\gamma+1)^{\gamma+1}} \frac{r(s)(z^\Delta(s))^{\gamma+1}}{z^\gamma(s)}\right]\Delta s = \infty, \tag{3.2}$$

then (1.3) is oscillatory.

Proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of (1.3). Without loss of generality, we may assume that $y(t)$ is eventually positive (note that in the case when $y(t)$ is eventually negative, the proof is similar, since the substitution $Y(t) = -y(t)$ transforms (1.3) into the same form). Then, by (h_1)–(h_3) there exists $T \geq t_0$ sufficiently large such that $y(t) > 0$,

$y(\tau(t)) > 0$, $y(\delta_{1,2}(t)) > 0$, and Lemma 2.4 holds for $t \geq T$, where $x(t)$ is defined by (2.8). Define the function $w(t)$ by the Riccati substitution

$$w(t) := \frac{z(t)r(t)\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}(t)}, \quad \text{for } t \geq T, \tag{3.3}$$

then $w(t) > 0$ and

$$\begin{aligned} w^{\Delta}(t) &= \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}\left(\frac{z(t)}{x^{\gamma}(t)}\right) + \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\left(\frac{z(t)}{x^{\gamma}(t)}\right)^{\Delta} \\ &= \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}\left(\frac{z(t)}{x^{\gamma}(t)}\right) + \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\left[\frac{z^{\Delta}(t)x^{\gamma}(t) - z(t)(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}}\right]. \end{aligned} \tag{3.4}$$

By (1.3), $x^{\Delta}(t) > 0$, and (2.18), we obtain

$$\begin{aligned} \frac{\left(r(t)(x^{\Delta}(t))^{\gamma}\right)^{\Delta}}{x^{\gamma}(t)} \leq &-\frac{q_1(t)\left[1 - p(\delta_1(t))\right]^{\alpha}\left(x^{\delta_1}(t)\right)^{\alpha}}{(x^{\sigma}(t))^{\gamma}} \\ &-\frac{q_2(t)\left[1 - p(\delta_2(t))\right]^{\beta}\left(x^{\delta_2}(t)\right)^{\beta}}{(x^{\sigma}(t))^{\gamma}}. \end{aligned} \tag{3.5}$$

Noting that $0 < \alpha < \gamma < \beta$, we have

$$\frac{\beta - \gamma}{\beta - \alpha} < 1, \qquad \frac{\gamma - \alpha}{\beta - \alpha} < 1. \tag{3.6}$$

By Young's inequality

$$a^{\chi}b^{1-\chi} \leq \chi a + (1 - \chi)b, \quad 0 < \chi < 1, \tag{3.7}$$

with

$$\begin{gathered} \chi = \frac{\beta - \gamma}{\beta - \alpha}, \qquad a = \frac{q_1(t)\left[1 - p(\delta_1(t))\right]^{\alpha}\left(x^{\delta_1}(t)\right)^{\alpha}}{(x^{\sigma}(t))^{\gamma}}, \\ b = \frac{q_2(t)\left[1 - p(\delta_2(t))\right]^{\beta}\left(x^{\delta_2}(t)\right)^{\beta}}{(x^{\sigma}(t))^{\gamma}}, \end{gathered} \tag{3.8}$$

we have

$$\begin{aligned}
&\frac{q_1(t)\left[1-p(\delta_1(t))\right]^{\alpha}\left(x^{\delta_1}(t)\right)^{\alpha}}{(x^{\sigma}(t))^{\gamma}}+\frac{q_2(t)\left[1-p(\delta_2(t))\right]^{\beta}\left(x^{\delta_2}(t)\right)^{\beta}}{(x^{\sigma}(t))^{\gamma}}\\
&\quad\geq \frac{\beta-\gamma}{\beta-\alpha}\frac{q_1(t)\left[1-p(\delta_1(t))\right]^{\alpha}\left(x^{\delta_1}(t)\right)^{\alpha}}{(x^{\sigma}(t))^{\gamma}}+\frac{\gamma-\alpha}{\beta-\alpha}\frac{q_2(t)\left[1-p(\delta_2(t))\right]^{\beta}\left(x^{\delta_2}(t)\right)^{\beta}}{(x^{\sigma}(t))^{\gamma}}\\
&\quad\geq \left(\frac{q_1(t)\left[1-p(\delta_1(t))\right]^{\alpha}\left(x^{\delta_1}(t)\right)^{\alpha}}{(x^{\sigma}(t))^{\gamma}}\right)^{(\beta-\gamma)/(\beta-\alpha)}\cdot\left(\frac{q_2(t)\left[1-p(\delta_2(t))\right]^{\beta}\left(x^{\delta_2}(t)\right)^{\beta}}{(x^{\sigma}(t))^{\gamma}}\right)^{(\gamma-\alpha)/(\beta-\alpha)}\\
&\quad= \left(q_1(t)\left[1-p(\delta_1(t))\right]^{\alpha}\right)^{(\beta-\gamma)/(\beta-\alpha)}\cdot\left(q_2(t)\left[1-p(\delta_2(t))\right]^{\beta}\right)^{(\gamma-\alpha)/(\beta-\alpha)}\\
&\qquad\cdot\frac{\left(x^{\delta_1}(t)\right)^{(\alpha\beta-\alpha\gamma)/(\beta-\alpha)}\cdot\left(x^{\delta_2}(t)\right)^{(\beta\gamma-\beta\alpha)/(\beta-\alpha)}}{(x^{\sigma}(t))^{\gamma}}.
\end{aligned}\tag{3.9}$$

By $\gamma=((\alpha\beta-\alpha\gamma)/(\beta-\alpha))+((\beta\gamma-\beta\alpha)/(\beta-\alpha))$ and Lemma 2.4, we get

$$\begin{aligned}
&\frac{q_1(t)\left[1-p(\delta_1(t))\right]^{\alpha}\left(x^{\delta_1}(t)\right)^{\alpha}}{(x^{\sigma}(t))^{\gamma}}+\frac{q_2(t)\left[1-p(\delta_2(t))\right]^{\beta}\left(x^{\delta_2}(t)\right)^{\beta}}{(x^{\sigma}(t))^{\gamma}}\\
&\quad\geq \left(q_1(t)\left[1-p(\delta_1(t))\right]^{\alpha}\right)^{(\beta-\gamma)/(\beta-\alpha)}\cdot\left(q_2(t)\left[1-p(\delta_2(t))\right]^{\beta}\right)^{(\gamma-\alpha)/(\beta-\alpha)}\\
&\qquad\cdot\left(\frac{x^{\delta_1}(t)}{x^{\sigma}(t)}\right)^{(\alpha\beta-\alpha\gamma)/(\beta-\alpha)}\cdot\left(\frac{x^{\delta_2}(t)}{x^{\sigma}(t)}\right)^{(\beta\gamma-\beta\alpha)/(\beta-\alpha)}\\
&\quad> \left(\frac{R_T(t)r^{1/\gamma}(t)}{R_T(t)r^{1/\gamma}(t)+\mu(t)}\right)^{\gamma}Q(t).
\end{aligned}\tag{3.10}$$

In view of $x^{\Delta}(t)>0$ and (3.5)–(3.10), for all $t\geq T$, we obtain

$$\begin{aligned}
w^{\Delta}(t)&<-z(t)\left(\frac{R_T(t)r^{1/\gamma}(t)}{R_T(t)r^{1/\gamma}(t)+\mu(t)}\right)^{\gamma}Q(t)+w^{\sigma}(t)\frac{z^{\Delta}(t)}{z^{\sigma}(t)}\\
&\quad-\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\frac{z(t)(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}}\\
&=-Q_1(t)+w^{\sigma}(t)\frac{z^{\Delta}(t)}{z^{\sigma}(t)}-\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\frac{z(t)(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}}.
\end{aligned}\tag{3.11}$$

Using $\gamma \geq 1$, Lemma 2.4 and the Keller's chain rule, we get

$$\begin{aligned}
(x^{\gamma}(t))^{\Delta} &= \gamma\left[\int_0^1 \left(x(t) + h\mu(t)x^{\Delta}(t)\right)^{\gamma-1} dh\right] x^{\Delta}(t) \\
&= \gamma x^{\Delta}(t) \int_0^1 ((1-h)x(t) + hx^{\sigma}(t))^{\gamma-1} dh \\
&\geq \gamma x^{\Delta}(t) \int_0^1 ((1-h)x(t) + hx(t))^{\gamma-1} dh = \gamma x^{\gamma-1}(t)x^{\Delta}(t).
\end{aligned} \tag{3.12}$$

Also from Lemma 2.4 and $\sigma(t) \geq t$, we have

$$r(t)\left(x^{\Delta}(t)\right)^{\gamma} \geq r(\sigma(t))\left(x^{\Delta}(\sigma(t))\right)^{\gamma}. \tag{3.13}$$

By (3.11)–(3.13), we get

$$\begin{aligned}
w^{\Delta}(t) &< -Q_1(t) + w^{\sigma}(t)\frac{z^{\Delta}(t)}{z^{\sigma}(t)} - \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma} \frac{z(t)\gamma x^{\gamma-1}(t)x^{\Delta}(t)}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}} \\
&\leq -Q_1(t) + w^{\sigma}(t)\frac{z^{\Delta}(t)}{z^{\sigma}(t)} - \frac{z(t)\gamma}{r^{1/\gamma}(t)} \frac{(r^{\sigma}(t))^{(\gamma+1)/\gamma}\left(x^{\Delta}(\sigma(t))\right)^{\gamma+1}}{(x^{\sigma}(t))^{\gamma+1}} \\
&= -Q_1(t) + w^{\sigma}(t)\frac{z^{\Delta}(t)}{z^{\sigma}(t)} - \frac{z(t)\gamma}{r^{1/\gamma}(t)(z^{\sigma}(t))^{(\gamma+1)/\gamma}} (w^{\sigma}(t))^{(\gamma+1)/\gamma}.
\end{aligned} \tag{3.14}$$

Setting

$$B = \frac{z^{\Delta}(t)}{z^{\sigma}(t)}, \qquad A = \frac{z(t)\gamma}{r^{1/\gamma}(t)(z^{\sigma}(t))^{(\gamma+1)/\gamma}}, \qquad u = w^{\sigma}(t), \tag{3.15}$$

then by Lemma 2.6, from (3.14) we obtain that for all $t \geq T$,

$$w^{\Delta}(t) < -Q_1(t) + \frac{1}{(\gamma+1)^{\gamma+1}} \frac{r(t)\left(z^{\Delta}(t)\right)^{\gamma+1}}{z^{\gamma}(t)}. \tag{3.16}$$

Integrating the above inequality from T to $t(\geq T)$, we get

$$\int_T^t \left[Q_1(s) - \frac{1}{(\gamma+1)^{\gamma+1}} \frac{r(s)\left(z^{\Delta}(s)\right)^{\gamma+1}}{z^{\gamma}(s)}\right] \Delta s < w(T) - w(t) < w(T). \tag{3.17}$$

Taking lim sup on both sides of the above inequality as $t \to \infty$, we obtain a contradiction to condition (3.2). The proof is complete. $\square$

The following theorem gives new oscillation criteria for (1.3) which can be considered as the extension of Philos-type oscillation criterion. Define $D = \{(t,s) \in \mathbb{T}^2 : t \ge s \ge 0\}$ and

$$\mathscr{H}_* = \left\{ H(t,s) \in C^1(D,\mathbb{R}_+) : H(t,t) = 0, H(t,s) > 0,\ H_s^{\Delta}(t,s) \ge 0,\ \text{for } t > s \ge 0 \right\}. \tag{3.18}$$

Theorem 3.2. *Assume that* (h_1)–(h_3) *hold and* $\gamma \ge 1$. *Furthermore, assume that there exist a positive* rd-continuous Δ-differentiable function $z(t)$ and a function $H \in \mathscr{H}_*$ such that for all sufficiently large $T \in \mathbb{T}$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)Q_1(s) - \frac{C^{\gamma+1}(t,s)r(s)(z(\sigma(s)))^{\gamma+1}}{H^{\gamma}(t,s)(\gamma+1)^{\gamma+1} z^{\gamma}(s)} \right] \Delta s = \infty, \tag{3.19}$$

then (1.3) is oscillatory.

Proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of (1.3). Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, by (h_1)–(h_3) there exists $T \ge t_0$ sufficiently large such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\delta_{1,2}(t)) > 0$, and Lemma 2.4 holds for $t \ge T$, where $x(t)$ is defined by (2.8). Define $w(t)$ as in (3.3). Proceeding as in the proof of Theorem 3.1, we can get (3.14). From (3.14), for function $H \in \mathscr{H}_*$ and all $t \ge T$ we have

$$\begin{aligned} \int_T^t H(t,s)Q_1(s)\Delta s < &-\int_T^t H(t,s)w^{\Delta}(s)\Delta s + \int_T^t H(t,s)w^{\sigma}(s)\frac{z^{\Delta}(s)}{z^{\sigma}(s)}\Delta s \\ &- \int_T^t H(t,s)\frac{z(s)\gamma}{r^{1/\gamma}(s)(z^{\sigma}(s))^{(\gamma+1)/\gamma}}(w^{\sigma}(s))^{(\gamma+1)/\gamma}\Delta s. \end{aligned} \tag{3.20}$$

Using the integration by parts formula (2.7), we obtain

$$\begin{aligned} -\int_T^t H(t,s)w^{\Delta}(s)\Delta s &= -H(t,s)w(s)|_T^t + \int_T^t H_s^{\Delta}(t,s)w^{\sigma}(s)\Delta s \\ &= H(t,T)w(T) + \int_T^t H_s^{\Delta}(t,s)w^{\sigma}(s)\Delta s. \end{aligned} \tag{3.21}$$

It follows that

$$\begin{aligned} \int_T^t H(t,s)Q_1(s)\Delta s &< H(t,T)w(T) + \int_T^t \left[H_s^{\Delta}(t,s) + H(t,s)\frac{z^{\Delta}(s)}{z^{\sigma}(s)} \right] w^{\sigma}(s)\Delta s \\ &\quad - \int_T^t H(t,s)\frac{z(s)\gamma}{r^{1/\gamma}(s)(z^{\sigma}(s))^{(\gamma+1)/\gamma}}(w^{\sigma}(s))^{(\gamma+1)/\gamma}\Delta s \\ &= H(t,T)w(T) + \int_T^t C(t,s)w^{\sigma}(s)\Delta s \\ &\quad - \int_T^t \frac{H(t,s)z(s)\gamma}{r^{1/\gamma}(s)(z^{\sigma}(s))^{(\gamma+1)/\gamma}}(w^{\sigma}(s))^{(\gamma+1)/\gamma}\Delta s. \end{aligned} \tag{3.22}$$

Setting

$$B = C(t,s), \qquad A = \frac{H(t,s)z(s)\gamma}{r^{1/\gamma}(s)(z^{\sigma}(s))^{(\gamma+1)/\gamma}}, \qquad u = w^{\sigma}(s), \tag{3.23}$$

by Lemma 2.6 we obtain that for all $t \geq T$,

$$\int_T^t H(t,s)Q_1(s)\Delta s < H(t,T)w(T) + \int_T^t \frac{[C(t,s)]^{\gamma+1}(z^{\sigma}(s))^{\gamma+1}r(s)}{H^{\gamma}(t,s)(\gamma+1)^{\gamma+1}z^{\gamma}(s)}\Delta s. \tag{3.24}$$

That is,

$$\frac{1}{H(t,T)}\int_T^t \left[H(t,s)Q_1(s) - \frac{[C(t,s)]^{\gamma+1}r(s)(z^{\sigma}(s))^{\gamma+1}}{H^{\gamma}(t,s)(\gamma+1)^{\gamma+1}z^{\gamma}(s)}\right]\Delta s < w(T). \tag{3.25}$$

Taking lim sup on both sides of the above inequality as $t \to \infty$, we obtain a contradiction to condition (3.19). The proof is complete. □

Theorem 3.3. *Assume that* (h_1)–(h_3) *hold and* $\gamma \geq 1$. *Then* (1.3) *is oscillatory if for all sufficiently large* $T \in \mathbb{T}$,

$$\limsup_{t\to\infty}\int_T^t Q(s)R_T^{\gamma}(s)\Delta s = \infty. \tag{3.26}$$

Proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of (1.3). Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, by (h_1)–(h_3) there exists $T \geq t_0$ sufficiently large such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\delta_{1,2}(t)) > 0$, and Lemma 2.4 holds for $t \geq T$, where $x(t)$ is defined by (2.8). Set $\phi(t) = r^{1/\gamma}(t)x^{\Delta}(t)$. By Lemma 2.4, we get $\phi > 0$, $(\phi^{\gamma})^{\Delta} < 0$. Using $\gamma \geq 1$ and the keller's chain rule, we get

$$\begin{aligned}(\phi^{\gamma}(t))^{\Delta} &= \gamma\left[\int_0^1 \left(\phi(t) + h\mu(t)\phi^{\Delta}(t)\right)^{\gamma-1}dh\right]\phi^{\Delta}(t) \\ &= \gamma\left[\int_0^1 ((1-h)\phi(t) + h\phi^{\sigma}(t))^{\gamma-1}dh\right]\phi^{\Delta}(t) < 0.\end{aligned} \tag{3.27}$$

So we have $\phi^{\Delta}(t) < 0$ and there is a constant $L > 0$ such that $\phi(t) \leq L$ for $t \geq T$. Then (1.3) becomes $(\phi^{\gamma})^{\Delta}(t) + f_1(t, y(\delta_1(t))) + f_2(t, y(\delta_2(t))) = 0$. By (2.18), we have

$$0 \geq \frac{(\phi^{\gamma}(t))^{\Delta}}{(\phi^{\sigma}(t))^{\gamma}} + q_1(t)(1-p(\delta_1(t)))^{\alpha}\frac{(x^{\delta_1}(t))^{\alpha}}{(\phi^{\sigma}(t))^{\gamma}} + q_2(t)(1-p(\delta_2(t)))^{\beta}\frac{(x^{\delta_2}(t))^{\beta}}{(\phi^{\sigma}(t))^{\gamma}}. \tag{3.28}$$

Similar to the proof of (3.10), we get

$$0 \geq \frac{(\phi^{\gamma}(t))^{\Delta}}{(\phi^{\sigma}(t))^{\gamma}} + Q(t)\left(\frac{x^{\delta_1}(t)}{\phi^{\sigma}(t)}\right)^{(\alpha\beta-\alpha\gamma)/(\beta-\alpha)} \cdot \left(\frac{x^{\delta_2}(t)}{\phi^{\sigma}(t)}\right)^{(\beta\gamma-\beta\alpha)/(\beta-\alpha)}. \tag{3.29}$$

Using the Keller's chain rule and $\phi^{\Delta}(t) < 0$, we get $\phi^{\sigma} \leq \phi$ and

$$(\phi^{\gamma}(t))^{\Delta} \geq \gamma(\phi^{\sigma}(t))^{\gamma-1}\phi^{\Delta}(t). \tag{3.30}$$

From $\delta_{1,2}(t) \geq t$ and $x^{\Delta}(t) > 0$, it follows that

$$\begin{aligned} 0 &\geq \frac{\gamma\phi^{\Delta}(t)}{\phi^{\sigma}(t)} + Q(t)\left(\frac{x^{\delta_1}(t)}{\phi^{\sigma}(t)}\right)^{(\alpha\beta-\alpha\gamma)/(\beta-\alpha)} \cdot \left(\frac{x^{\delta_2}(t)}{\phi^{\sigma}(t)}\right)^{(\beta\gamma-\beta\alpha)/(\beta-\alpha)} \geq \frac{\gamma\phi^{\Delta}(t)}{L} + Q(t)\left(\frac{x(t)}{\phi^{\sigma}(t)}\right)^{\gamma} \\ &\geq \frac{\gamma\phi^{\Delta}(t)}{L} + Q(t)\left(\frac{x(t)}{\phi(t)}\right)^{\gamma}. \end{aligned} \tag{3.31}$$

By Lemma 2.4, we get

$$\frac{x(t)}{\phi(t)} = \frac{x(t)}{r^{1/\gamma}(t)x^{\Delta}(t)} > R_T(t). \tag{3.32}$$

It follows that

$$0 > \frac{\gamma\phi^{\Delta}(t)}{L} + Q(t)R_T^{\gamma}(t). \tag{3.33}$$

Integrating the above inequality from T to $t(\geq T)$, we obtain

$$\int_T^t Q(s)R_T^{\gamma}(s)\Delta s < -\frac{\gamma}{L}\int_T^t \phi^{\Delta}(s)\Delta s = \frac{\gamma}{L}(\phi(T) - \phi(t)) < \frac{\gamma}{L}\phi(T). \tag{3.34}$$

Taking lim sup on both sides of the above inequality as $t \to \infty$, we obtain a contradiction to condition (3.26). The proof is complete. □

Theorem 3.4. *Assume that* (h_1)–(h_3) *hold and* $\gamma \geq 1$*. Furthermore, assume that there exists a positive* rd-continuous Δ-differentiable function $z(t)$ such that for all sufficiently large $T \in \mathbb{T}$,

$$\limsup_{t\to\infty} \int_T^t \left[Q_1(s) - \frac{r^{1/\gamma}(s)\left(z^{\Delta}(s)\right)^2}{4\gamma(R_T(s))^{\gamma-1}z(s)}\right]\Delta s = \infty, \tag{3.35}$$

then (1.3) is oscillatory.

Proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of (1.3). Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, by (h_1)–(h_3) there exists $T \geq t_0$ sufficiently large such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\delta_{1,2}(t)) > 0$, and Lemma 2.4 holds for $t \geq T$, where $x(t)$ is defined by (2.8). Define $w(t)$ as in (3.3). By (2.6), we obtain

$$\begin{aligned} w^{\Delta}(t) &= z^{\Delta}(t)\left[\frac{r(t)\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}(t)}\right]^{\sigma} + z(t)\left[\frac{r(t)\left(x^{\Delta}(t)\right)^{\gamma}}{x^{\gamma}(t)}\right]^{\Delta} \\ &= \frac{z^{\Delta}(t)}{z^{\sigma}(t)}w^{\sigma}(t) + z(t)\left[\frac{\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}x^{\gamma}(t) - r(t)\left(x^{\Delta}(t)\right)^{\gamma}\left(x^{\gamma}(t)\right)^{\Delta}}{x^{\gamma}(t)\left(x^{\sigma}(t)\right)^{\gamma}}\right] \\ &= \frac{z^{\Delta}(t)}{z^{\sigma}(t)}w^{\sigma}(t) + z(t)\left[\frac{\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}\left(x^{\sigma}(t)\right)^{\gamma} - r^{\sigma}(t)\left(\left(x^{\Delta}(t)\right)^{\sigma}\right)^{\gamma}\left(x^{\gamma}(t)\right)^{\Delta}}{x^{\gamma}(t)\left(x^{\sigma}(t)\right)^{\gamma}}\right]. \end{aligned} \tag{3.36}$$

By Lemma 2.4, $\sigma(t) \geq t$, and (3.5)–(3.13), for all $t \geq T$ we obtain

$$\begin{aligned} w^{\Delta}(t) &< -Q_1(t) + w^{\sigma}(t)\frac{z^{\Delta}(t)}{z^{\sigma}(t)} - z(t)\frac{r^{\sigma}(t)\left(x^{\Delta}(\sigma(t))\right)^{\gamma}\gamma x^{\gamma-1}(t)x^{\Delta}(t)}{x^{\gamma}(t)\left(x^{\sigma}(t)\right)^{\gamma}} \\ &= -Q_1(t) + w^{\sigma}(t)\frac{z^{\Delta}(t)}{z^{\sigma}(t)} - \gamma z(t)r^{\sigma}(t)\left[\frac{\left(x^{\Delta}(\sigma(t))\right)^{\gamma}}{\left(x^{\sigma}(t)\right)^{\gamma}}\right]^2\frac{\left(x^{\sigma}(t)\right)^{\gamma}x^{\Delta}(t)}{x(t)\left(x^{\Delta}(\sigma(t))\right)^{\gamma}} \\ &< -Q_1(t) + w^{\sigma}(t)\frac{z^{\Delta}(t)}{z^{\sigma}(t)} - \gamma z(t)\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)}\right)^2\frac{x^{\gamma-1}(t)}{\left(x^{\Delta}(t)\right)^{\gamma-1}r(t)} \\ &< -Q_1(t) + w^{\sigma}(t)\frac{z^{\Delta}(t)}{z^{\sigma}(t)} - \gamma z(t)\frac{1}{r(t)}\left(\frac{w^{\sigma}(t)}{z^{\sigma}(t)}\right)^2\left(R_T(t)r^{1/\gamma}(t)\right)^{\gamma-1}. \end{aligned} \tag{3.37}$$

It follows that

$$w^{\Delta}(t) < -Q_1(t) + \frac{z^{\Delta}(t)}{z^{\sigma}(t)}w^{\sigma}(t) - \frac{\gamma z(t)\left(R_T(t)\right)^{\gamma-1}}{r^{1/\gamma}(t)\left(z^{\sigma}(t)\right)^2}\left(w^{\sigma}(t)\right)^2. \tag{3.38}$$

By completing the square, we have

$$w^{\Delta}(t) < -Q_1(t) + \frac{r^{1/\gamma}(t)\left(z^{\Delta}(t)\right)^2}{4\gamma\left(R_T(t)\right)^{\gamma-1}z(t)}. \tag{3.39}$$

Integrating the above inequality from T to $t(\geq T)$, we get

$$\int_T^t\left[Q_1(s) - \frac{r^{1/\gamma}(s)\left(z^{\Delta}(s)\right)^2}{4\gamma\left(R_T(s)\right)^{\gamma-1}z(s)}\right]\Delta s < w(T) - w(t) < w(T). \tag{3.40}$$

Taking lim sup on both sides of the above inequality as $t \to \infty$, we obtain a contradiction to condition (3.35). The proof is complete. □

Theorem 3.5. *Assume that* (h_1)–(h_3) *and* $\gamma \geq 1$ *hold. Furthermore, assume that there exist a positive* rd-continuous Δ-differentiable function $z(t)$ and a function $H \in \mathscr{H}_*$ such that for all sufficiently large $T \in \mathbb{T}$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)Q_1(s) - \frac{H(t,s)A^2(s)r^{1/\gamma}(s)(z^\sigma)^2(s)}{4\gamma(R_T(s))^{\gamma-1}z(s)} \right] \Delta s = \infty, \tag{3.41}$$

then (1.3) is oscillatory.

Proof. By (3.39), the proof is similar to Theorems 3.2 and 3.4, so we omit it. □

Theorem 3.6. *Assume that* (h_1)–(h_3) *hold and* $\gamma \geq 1$. *Furthermore, assume that there exists a* rd-continuous Δ-differentiable function $z(t)$ such that for all sufficiently large $T \in \mathbb{T}$,

$$\limsup_{t\to\infty} \int_T^t \left[Q_2(s) - \frac{1}{\gamma}\left(z^\Delta(s)\right)^2 r^{1/\gamma}(s)(R_T(s))^{1-\gamma} \right] \Delta s = \infty, \tag{3.42}$$

then (1.3) is oscillatory.

Proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of (1.3). Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, by (h_1)–(h_3) there exists $T \geq t_0$ sufficiently large such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\delta_{1,2}(t)) > 0$, and Lemma 2.4 holds for $t \geq T$, where $x(t)$ is defined by (2.8). Define the function $v(t)$ by the Riccati substitution

$$v(t) := \frac{z^2(t)r(t)\left(x^\Delta(t)\right)^\gamma}{x^\gamma(t)}, \quad \text{for } t \geq T, \tag{3.43}$$

then $v(t) > 0$. From (3.5) and (3.10), it follows that for all $t \geq T$

$$\begin{aligned} v^\Delta(t) &= \left[r(t)\left(x^\Delta(t)\right)^\gamma\right]^\Delta \left(\frac{z^2(t)}{x^\gamma(t)}\right)^\sigma + r(t)\left(x^\Delta(t)\right)^\gamma \left(\frac{z^2(t)}{x^\gamma(t)}\right)^\Delta \\ &\leq -Q_2(t) + \frac{r(t)\left(x^\Delta(t)\right)^\gamma}{(x^\gamma(t))^\Delta}(x^\gamma(t))^\Delta \left(\frac{z^2(t)}{x^\gamma(t)}\right)^\Delta. \end{aligned} \tag{3.44}$$

By (3.12) and Lemma 2.7, we obtain

$$\begin{aligned} v^\Delta(t) &\leq -Q_2(t) + \frac{r(t)\left(x^\Delta(t)\right)^\gamma}{(x^\gamma(t))^\Delta} \left[\left(z^\Delta(t)\right)^2 - x^\gamma(t)(x^\sigma(t))^\gamma \left(\left(\frac{z(t)}{x^\gamma(t)}\right)^\Delta \right)^2 \right] \\ &< -Q_2(t) + \frac{r(t)\left(x^\Delta(t)\right)^\gamma \left(z^\Delta(t)\right)^2}{(x^\gamma(t))^\Delta} < -Q_2(t) + \frac{r(t)\left(x^\Delta(t)\right)^\gamma \left(z^\Delta(t)\right)^2}{\gamma x^{\gamma-1}(t)x^\Delta(t)}. \end{aligned} \tag{3.45}$$

It follows from Lemma 2.4 that

$$v^{\Delta}(t) < -Q_2(t) + \frac{1}{\gamma}\left(z^{\Delta}(t)\right)^2 r^{1/\gamma}(t)(R_T(t))^{1-\gamma}. \tag{3.46}$$

Integrating the above inequality from T to t $(\geq T)$, we get

$$\int_T^t \left[Q_2(s) - \frac{1}{\gamma}\left(z^{\Delta}(s)\right)^2 r^{1/\gamma}(s)(R_T(s))^{1-\gamma}\right]\Delta s < v(T) - v(t) < v(T). \tag{3.47}$$

Taking lim sup on both sides of the above inequality as $t \to \infty$, we obtain a contradiction to condition (3.42). The proof is complete. □

From Theorem 3.6, we can establish different sufficient conditions for the oscillation of (1.3) by using different choices of $z(t)$. For instance, if $z(t) = 1$ or $z(t) = \sqrt{t}$, we have the following results.

Corollary 3.7. *Assume that* (h_1)–(h_3) *hold and* $\gamma \geq 1$. *Then* (1.3) *is oscillatory if for all sufficiently large* $T \in \mathbb{T}$,

$$\limsup_{t\to\infty} \int_T^t \left(\frac{R_T(s) r^{1/\gamma}(s)}{R_T(s) r^{1/\gamma}(s) + \mu(s)}\right)^{\gamma} Q(s)\Delta s = \infty. \tag{3.48}$$

Corollary 3.8. *Assume that* (h_1)–(h_3) *hold and* $\gamma \geq 1$. *Then* (1.3) *is oscillatory if for all sufficiently large* $T \in \mathbb{T}$,

$$\limsup_{t\to\infty} \int_T^t \left[Q_3(s) - \left(\frac{1}{\sqrt{s} + \sqrt{\sigma(s)}}\right)^2 r^{1/\gamma}(s)\frac{1}{\gamma}(R_T(s))^{1-\gamma}\right]\Delta s = \infty, \tag{3.49}$$

where $Q_3(s) = (R_T(s) r^{1/\gamma}(s)/(R_T(s) r^{1/\gamma}(s) + \mu(s)))^{\gamma}\sigma(s)Q(s)$.

(II) $0 < \gamma < 1$.

Theorem 3.1′. *Assume that* (h_1)–(h_3) *hold and* $0 < \gamma < 1$. *Furthermore, assume that there exists a positive* rd-continuous Δ*-differentiable function* $z(t)$ *such that for all sufficiently large* $T \in \mathbb{T}$,

$$\limsup_{t\to\infty} \int_T^t \left[Q_1(s) - \frac{1}{(\gamma+1)^{\gamma+1}} \frac{r(s)\left(z^{\Delta}(s)\right)^{\gamma+1}}{z^{\gamma}(s)}\right]\Delta s = \infty, \tag{3.50}$$

then (1.3) *is oscillatory*.

Proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of (1.3). Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, by (h_1)–(h_3) there exists $T \geq t_0$ sufficiently large such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\delta_{1,2}(t)) > 0$ and Lemma 2.4 holds for $t \geq T$, where $x(t)$ is defined by (2.8). Define $w(t)$ as in (3.3). Proceeding as in the proof of Theorem 3.1, we get

$$w^{\Delta}(t) = \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}\left(\frac{z(t)}{x^{\gamma}(t)}\right) + \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\left[\frac{z^{\Delta}(t)x^{\gamma}(t) - z(t)(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}}\right]. \tag{3.51}$$

Using $0 < \gamma < 1$, Lemma 2.4 and the Keller's chain rule, we get

$$\begin{aligned}(x^{\gamma}(t))^{\Delta} &= \gamma\left[\int_0^1 \left(x(t) + h\mu(t)x^{\Delta}(t)\right)^{\gamma-1} dh\right]x^{\Delta}(t)\\ &\geq \gamma x^{\Delta}(t)\int_0^1 ((1-h)x^{\sigma}(t) + hx^{\sigma}(t))^{\gamma-1} dh\\ &= \gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t).\end{aligned} \tag{3.52}$$

By (3.10), (3.51), and (3.52), we get

$$w^{\Delta}(t) < -Q_1(t) + w^{\sigma}(t)\frac{z^{\Delta}(t)}{z^{\sigma}(t)} - \left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\sigma}\frac{z(t)\gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t)}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}}. \tag{3.53}$$

Since

$$\begin{aligned}\frac{-\left(r(t)(x^{\Delta}(t))^{\gamma}\right)^{\sigma}z(t)\gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t)}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}} &= -\frac{(r^{\sigma}(t))^{(\gamma+1)/\gamma}\left((x^{\Delta}(t))^{\sigma}\right)^{\gamma+1}z(t)\gamma x^{\Delta}(t)}{x^{\gamma}(t)x^{\sigma}(t)(r^{\sigma}(t))^{1/\gamma}(x^{\Delta}(t))^{\sigma}}\\ (\text{For } (3.13)) &\leq -\frac{(r^{\sigma}(t))^{(\gamma+1)/\gamma}\left((x^{\Delta}(t))^{\sigma}\right)^{\gamma+1}z(t)\gamma x^{\Delta}(t)}{x^{\gamma}(t)x^{\sigma}(t)r^{1/\gamma}(t)x^{\Delta}(t)}\\ &< -\frac{z(t)\gamma}{r^{1/\gamma}(t)(z^{\sigma}(t))^{(\gamma+1)/\gamma}}(w^{\sigma}(t))^{(\gamma+1)/\gamma},\end{aligned} \tag{3.54}$$

it follows that

$$w^{\Delta}(t) < -Q_1(t) + w^{\sigma}(t)\frac{z^{\Delta}(t)}{z^{\sigma}(t)} - \frac{z(t)\gamma}{r^{(1/\gamma)(t)}(z^{\sigma}(t))^{(\gamma+1)/\gamma}}(w^{\sigma}(t))^{(\gamma+1)/\gamma}. \tag{3.55}$$

It is easy to see (3.55) is of the same form as (3.14). The following is similar to the proof of Theorem 3.1 and hence omitted. □

For $\gamma \in (0,1)$, Theorem 3.2 also holds. Its proof is similar to those of Theorems 3.1′ and 3.2.

Theorem 3.2′. *Assume that* (h_1)–(h_3) *hold and* $0 < \gamma < 1$. *Furthermore, assume that there exist a positive* rd-continuous Δ*-differentiable function* $z(t)$ *and a function* $H \in \mathcal{H}_*$ *such that for all sufficiently large* $T \in \mathbb{T}$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)Q_1(s) - \frac{C^{\gamma+1}(t,s)r(s)(z(\sigma(s)))^{\gamma+1}}{H^\gamma(t,s)(\gamma+1)^{\gamma+1}z^\gamma(s)} \right] \Delta s = \infty, \tag{3.56}$$

then (1.3) *is oscillatory.*

Theorem 3.3′. *Assume that* (h_1)–(h_3) *hold and* $0 < \gamma < 1$. *Then* (1.3) *is oscillatory if for all sufficiently large* $T \in \mathbb{T}$,

$$\limsup_{t\to\infty} \int_T^t Q(s)R_T^\gamma(s)\Delta s = \infty. \tag{3.57}$$

Proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of (1.3). Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, by (h_1)–(h_3) there exists $T \geq t_0$ sufficiently large such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\delta_{1,2}(t)) > 0$, and Lemma 2.4 holds for $t \geq T$, where $x(t)$ is defined by (2.8), ϕ is defined as in Theorem 3.3. Similar to the proof of Theorem 3.3, we get

$$0 \geq \frac{(\phi^\gamma(t))^\Delta}{\phi^\gamma(t)} + Q(t)\left(\frac{x^{\delta_1}(t)}{\phi(t)}\right)^{(\alpha\beta-\alpha\gamma)/(\beta-\alpha)} \cdot \left(\frac{x^{\delta_2}(t)}{\phi(t)}\right)^{(\beta\gamma-\beta\alpha)/(\beta-\alpha)}. \tag{3.58}$$

Using the Keller's chain rule, $0 < \gamma < 1$, and $\phi^\Delta(t) < 0$, we get

$$\begin{aligned}
(\phi^\gamma(t))^\Delta &= \gamma\left[\int_0^1 \left(\phi(t) + h\mu(t)\phi^\Delta(t)\right)^{\gamma-1} dh\right]\phi^\Delta(t) \\
&\geq \gamma\left[\int_0^1 ((1-h)\phi(t) + h\phi(t))^{\gamma-1} dh\right]\phi^\Delta(t) \\
&= \gamma\phi^{\gamma-1}(t)\phi^\Delta(t).
\end{aligned} \tag{3.59}$$

From $\delta_{1,2}(t) \geq t$ and $x^\Delta(t) > 0$, it follows that

$$0 \geq \frac{\gamma\phi^\Delta(t)}{L} + Q(t)\left(\frac{x(t)}{\phi(t)}\right)^\gamma. \tag{3.60}$$

It is easy to see (3.31) is of the same form as (3.60). The following is similar to the proof of Theorem 3.3 and hence omitted. □

Last, we give a theorem which holds for all $\gamma > 0$, a quotient of add positive integers.

Theorem 3.9. *Assume that* (h_1)–(h_3) *hold. Furthermore, assume that there exists a positive* rd-continuous Δ-differentiable function $z(t)$ such that for all sufficiently large $T \in \mathbb{T}$,

$$\limsup_{t\to\infty} \int_T^t \left[Q_1(s) - \frac{z^\Delta}{R_T^\gamma(s)} \right] \Delta s = \infty, \tag{3.61}$$

then (1.3) is oscillatory.

Proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of (1.3). Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, by (h_1)–(h_3) there exists $T \ge t_0$ sufficiently large such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\delta_{1,2}(t)) > 0$, and Lemma 2.4 holds for $t \ge T$, where $x(t)$ is defined by (2.8). Define $w(t)$ as in (3.3). Then, $w(t) > 0$ and because $(1/x^\gamma)^\Delta = -((x^\gamma)^\Delta / x^\gamma (x^\gamma)^\sigma) < 0$ we get

$$\begin{aligned} w^\Delta(t) &= \left[z(t) \cdot r(t) \left(x^\Delta(t) \right)^\gamma \right]^\Delta \frac{1}{x^\gamma(t)} + \left[z(t) \cdot r(t) \left(x^\Delta(t) \right)^\gamma \right]^\sigma \left(\frac{1}{x^\gamma(t)} \right)^\Delta \\ &< \left[z^\Delta(t) \cdot \left(r(t) \left(x^\Delta(t) \right)^\gamma \right)^\sigma + z(t) \left(r(t) \left(x^\Delta(t) \right)^\gamma \right)^\Delta \right] \frac{1}{x^\gamma(t)}. \end{aligned} \tag{3.62}$$

From (3.5), (3.10), and (3.13), we obtain

$$\begin{aligned} w^\Delta(t) &< \left[z^\Delta(t) \cdot r(t) \left(x^\Delta(t) \right)^\gamma + z(t) \left(r(t) \left(x^\Delta(t) \right)^\gamma \right)^\Delta \right] \frac{1}{x^\gamma(t)} \\ &< z^\Delta(t) r(t) \left(\frac{x^\Delta(t)}{x(t)} \right)^\gamma - Q_1(t) < \frac{z^\Delta(t)}{R_T^\gamma(t)} - Q_1(t). \end{aligned} \tag{3.63}$$

Integrating the above inequality from T to $t (\ge T)$, we get

$$\int_T^t \left[Q_1(s) - \frac{z^\Delta(s)}{R_T^\gamma(s)} \right] \Delta s < w(T) - w(t) < w(T). \tag{3.64}$$

Taking lim sup on both sides of the above inequality as $t \to \infty$, we obtain a contradiction to condition (3.61). The proof is complete. □

4. Examples

In this section, we give two examples to illustrate our main results. To obtain the conditions for oscillation we will use the following facts:

$$\int_1^\infty \frac{\Delta s}{s^\gamma} = \infty \quad \text{if } 0 \le \gamma \le 1, \qquad \int_1^\infty \frac{\Delta s}{s^\gamma} < \infty, \quad \text{if } \gamma > 1. \tag{4.1}$$

We first give an example to show Theorems 3.1 and 3.1′.

Example 4.1. Consider the equation

$$\left(\frac{1}{(t+\sigma(t))^{\gamma}}\left(\left(y(t)+p(t)y(\tau(t))\right)^{\Delta}\right)^{\gamma}\right)^{\Delta}+\frac{(\sigma(t))^{2\gamma}}{\left(1-p(\delta_1(t))\right)^{\alpha}t^{2\gamma+1}}y^{\alpha}(\delta_1(t))$$
$$+\frac{(\sigma(t))^{2\gamma}}{\left(1-p(\delta_2(t))\right)^{\beta}t^{2\gamma+1}}y^{\beta}(\delta_2(t))=0,\quad t\in\mathbb{T},\tag{4.2}$$

where $\mathbb{T}=[1,\infty)$ is a time scale, $p(t)$ satisfies (h_2), $\tau(t)$ and $\delta_{1,2}(t)$ satisfy (h_1), $r(t)=1/(t+\sigma(t))^{\gamma}$, and γ is a quotient of odd positive integers.

We choose $q_1(t)=(\sigma(t))^{2\gamma}/((1-p(\delta_1(t)))^{\alpha}t^{2\gamma+1})$, $q_2(t)=(\sigma(t))^{2\gamma}/((1-p(\delta_2(t)))^{\beta}t^{2\gamma+1})$, and $z=1$, then $z^{\Delta}=0$, $\int(s+\sigma(s))\Delta s=t^2+c$, and $\int_1^{\infty}(\Delta t/r^{1/\gamma}(t))=\infty$. For any sufficiently large $T\in\mathbb{T}$ and $s>T$, there exists a constant $k>0$ sufficiently large such that

$$\left(\frac{R_T(s)r^{1/\gamma}(s)}{R_T(s)r^{1/\gamma}(s)+\mu(s)}\right)^{\gamma}=\left(\frac{(s^2-T^2)(1/(s+\sigma(s)))}{(s^2-T^2)(1/(s+\sigma(s)))+\sigma(s)-s}\right)^{\gamma}>\left(\frac{s^2}{k\sigma^2(s)}\right)^{\gamma},$$
$$\limsup_{t\to\infty}\int_T^t\left[Q_1(s)-\frac{1}{(\gamma+1)^{\gamma+1}}\frac{r(s)\left(z^{\Delta}(s)\right)^{\gamma+1}}{z^{\gamma}(s)}\right]\Delta s\geq\limsup_{t\to\infty}k^{-\gamma}\int_T^t\frac{1}{s}\Delta s=\infty.\tag{4.3}$$

Hence, by Theorems 3.1 and 3.1′, (4.2) is oscillatory.

The second example illustrates Corollary 3.7.

Example 4.2. Consider the equation

$$\left(y(t)+\frac{y(\tau(t))}{\delta^{-1}(t)}\right)^{\Delta\Delta}+\frac{\sigma(t)}{t^2}y^{1/3}(\delta(t))+\frac{\sigma(t)}{t^2}y^{5/3}(\delta(t))=0,\quad t\in\mathbb{T},\tag{4.4}$$

where $\mathbb{T}=[1,\infty)$ is a time scale, $\gamma=1,\alpha=1/3,\beta=5/3$, $\delta(t)$, and $\tau(t)$ satisfy (h_1), $\delta(t)$ has an inverse function $\delta^{-1}(t)$, and $p(t)=1/\delta^{-1}(t)$ satisfies (h_2).

We choose $q_1(t)=q_2(t)=\sigma(t)/t^2$ and $z=1$. For any sufficiently large $T\in\mathbb{T}$ and $s>T$, there exists a constant $k>0$ sufficiently large such that

$$\frac{R_T(s)r^{1/\gamma}(s)}{R_T(s)r^{1/\gamma}(s)+\mu(s)}=\frac{s-T}{s-T+\sigma(s)-s}>\frac{s}{k\sigma(s)},$$
$$\limsup_{t\to\infty}\int_T^t\left(\frac{R_T(s)r^{1/\gamma}(s)}{R_T(s)r^{1/\gamma}(s)+\mu(s)}\right)^{\gamma}Q(s)\Delta s\geq\limsup_{t\to\infty}\int_T^t\frac{1}{ks}\left(1-\frac{1}{s}\right)^2\Delta s\tag{4.5}$$
$$=\limsup_{t\to\infty}\frac{1}{k}\int_T^t\left[\frac{1}{s}-\frac{2}{s^2}+\frac{1}{s^3}\right]\Delta s=\infty.$$

Hence, by Corollary 3.7, (4.4) is oscillatory.

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