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Research Article

Approximate Analytical Solutions Using Hyperbolic Functions for the Generalized Blasius Problem

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We propose simple forms of approximate analytical solutions for the generalized Blasius problem based on the given boundary conditions and some known properties of the solution. The efficiency of the proposed solutions is shown for various cases. As a result, one can see that the solutions are uniformly accurate over the whole region.

1. Introduction

We consider the following generalized Blasius problem corresponding to two-dimensional laminar viscous flow over a thin plate:

$$Nf(x) := f'''(x) + \alpha f(x) f''(x) = 0, \quad 0 \le x < \infty \tag{1.1}$$

for $\alpha > 0$, subject to the boundary conditions

$$f(0) = f'(0) = 0, \qquad f'(\infty) = \beta, \tag{1.2}$$

where $\beta > 0$. We call the solution $f(x)$ the Blasius function. Up to now many analytical methods, for example, Adomian decomposition method [1–3], variational iteration method [4–11], and homotopy analysis method [12–15] have been proposed. In addition, numerical solutions were given in [16–18].

For the special case of $\alpha = 1/2$ and $\beta = 1$, the Blasius problem was completely reviewed by Boyd [19] with several known properties of the Blasius function $f(x)$. In the recent work

[20], the author proposed simple approximate analytical solutions which result in good uniform approximations to the exact solution $f(x)$.

In this paper, we extend the method developed in [20] to the generalized Blasius problem (1.1). Based on the given boundary conditions and known properties of the Blasius function $f(x)$, we propose three approximate analytical solutions which consist of the hyperbolic cosine and tangent functions. From the results of the numerical experiments, we can observe that for every cases of α and β the presented approximate solutions are efficient and available over the whole region. In particular, the proposed three-term approximate solution results in the relative errors less than 0.033% and 0.065% for approximation to the exact solution and its derivative, respectively. In addition, using the known properties of the Blasius function, we apply the proposed approach to the homotopy perturbation method [6–8]. Numerical results show the validity of the obtained approximate solution.

2. Uniformly Accurate Analytical Solutions

We can see that for arbitrary α and β, the Blasius function $f(x)$ satisfies the following properties [21]:

(i) $\kappa(\alpha;\beta) := \ f''(0) \ = \ \sqrt{\alpha\beta^3}\,\kappa_0$,

(ii) $B(\alpha;\beta) := \ \lim_{x\to\infty}\{f(x)-\beta x\} \ = \ \sqrt{\beta/\alpha}\,B_0$,

where κ_0 and B_0 are the constants which correspond to the case of $\alpha = 1$ and $\beta = 1$ as follows [19, 22]:

$$\kappa_0 \ = \ 0.4695999883\cdots, \quad B_0 \ = \ -1.2167806216\cdots. \tag{2.1}$$

For the special case of $\alpha = 1/2$ and $\beta = 1$, in the recent work [20], the author introduced the single term approximate solution:

$$h(x) \ = \ \frac{1}{b}\ \log[\cosh(bx)] \tag{2.2}$$

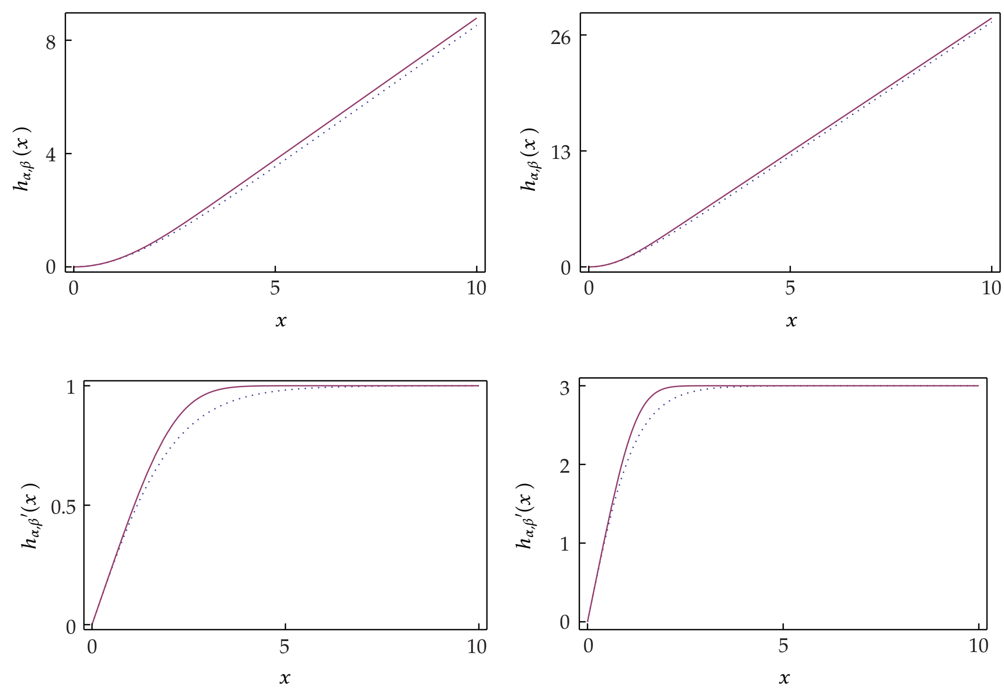
and the two term approximate solution

$$g(x) \ = \ \frac{1}{b}\log[\cosh(bx)] + c\tanh^4(rx), \quad r > 0, \tag{2.3}$$

where the constants b and c are determined from the known properties of the Blasius function as follows:

$$g''(0) = \frac{\kappa_0}{\sqrt{2}}, \qquad \lim_{x\to\infty}\{g(x)-\beta x\} = \sqrt{2}B_0. \tag{2.4}$$

The parameter r is chosen by minimizing the L_2-norm of the residual function $Ng(x)$ over the whole region $[0,\infty)$.



(a) $\alpha = \beta = 1$ (b) $\alpha = 1, \beta = 3$

Figure 1: Graphs of the presented solution $h_{\alpha,\beta}(x)$ and its derivative (dotted lines) compared with those of the numerical solution $f_{\alpha,\beta}(x)$ (solid lines).

In this paper, we extend the aforementioned idea to the generalized Blasius problem given in (1.1) and (1.2). First, referring to the single term approximate solution $h(x)$ in (2.2), we modify it as

$$h_{\alpha,\beta}(x) = \frac{\beta}{b} \log[\cosh(bx)]. \tag{2.5}$$

It is straightforward to see that the function $h_{\alpha,\beta}(x)$ has the first and second derivatives,

$$h'_{\alpha,\beta}(x) = \beta \tanh(bx), \qquad h''_{\alpha,\beta}(x) = b\beta \operatorname{sech}^2(bx) \tag{2.6}$$

and all of the boundary conditions in (1.2) are satisfied. Moreover, taking $h''_{\alpha,\beta}(0) = \kappa(\alpha;\beta)$, from the property (i) we have

$$b = \frac{\kappa(\alpha;\beta)}{\beta} = \sqrt{\alpha\beta}\, \kappa_0. \tag{2.7}$$

For the cases of $\alpha = \beta = 1$ and $\alpha = 1$, $\beta = 3$, Figure 1 shows graphs of the proposed approximate solution $h_{\alpha,\beta}(x)$ and its first derivative $h'_{\alpha,\beta}(x)$ compared with those of

Table 1: Maximum relative % errors of the presented approximate solution $h_{\alpha,\beta}(x)$ and its derivative for the sample points $x_j = (0.2)j$, $j = 1, 2, \ldots, 50$.

α	β	$M^{[0]}h_{\alpha,\beta}$	$M^{[1]}h_{\alpha,\beta}$
1/2	1/2	8.24	10.2
	1	8.24	10.2
	2	8.24	10.2
	3	8.23	10.2
1	1/2	8.24	10.2
	1	8.24	10.2
	2	8.24	10.2
	3	8.24	10.1

the numerical solution $f_{\alpha,\beta}(x)$ obtained by using the software *Mathematica*. We, in this work, regard $f_{\alpha,\beta}(x)$ as the exact solution.

We can see that the function $h_{\alpha,\beta}(x)$ has rather a similar behavior with the numerical solution. Table 1 includes maximum values of the relative percentage errors of the presented approximate solution $h_{\alpha,\beta}(x)$ and its derivative. Therein, for a function u, $M^{[m]}u$ is defined as

$$M^{[m]}u = \operatorname*{Max}_{j}\left\{\left|E^{[m]}u(x_j)\right|\right\}, \quad m = 0, 1, \tag{2.8}$$

where x_j are sample points selected as $x_j = (0.2)j$, $j = 1, 2, \ldots, 50$, and

$$E^{[m]}u(x_j) = \frac{u^{(m)}(x_j) - f_{\alpha,\beta}^{(m)}(x_j)}{f_{\alpha,\beta}^{(m)}(x_j)} \times 100\ (\%). \tag{2.9}$$

The table shows that $h_{\alpha,\beta}(x)$ uniformly approximates $f_{\alpha,\beta}(x)$ with the maximum relative errors less than 8.3% and 10.3% for approximation to $f_{\alpha,\beta}(x)$ and its derivative, respectively. In addition, this tendency of the proposed solution $h_{\alpha,\beta}(x)$ seems to be independent of the selected values of α $(= 1/2, 1)$ and β $(= 1/2, 1, 2, 3)$.

In order to improve the error of $h_{\alpha,\beta}(x)$, we employ the two term approximate solution

$$g_{\alpha,\beta}(x) = \frac{\beta}{b}\log[\cosh(bx)] + c\tanh^4(rx), \tag{2.10}$$

where b is given in (2.7) and c and r are unknowns to be determined later. We can see that $g_{\alpha,\beta}(x)$ satisfies the boundary conditions in (1.2) and the properties (i) as well, that is,

$$\begin{gathered} g_{\alpha,\beta}(0) = g'_{\alpha,\beta}(0) = 0, \qquad g'_{\alpha,\beta}(\infty) = \beta, \\ g''_{\alpha,\beta}(0) = \sqrt{\alpha\beta^3}\,\kappa_0. \end{gathered} \tag{2.11}$$

Table 2: Numerical optimal values of the parameter r and maximum relative % errors of the two-term approximate solution $g_{\alpha,\beta}(x)$ for each value of α and β.

α	β	r^*	$M^{[0]}g_{\alpha,\beta}$	$M^{[1]}g_{\alpha,\beta}$
1/2	1/2	0.21475516	0.434	0.965
	1	0.30370966	0.434	0.962
	2	0.42951032	0.434	0.965
	3	0.52604056	0.433	0.955
1	1/2	0.30370966	0.434	0.962
	1	0.42951032	0.434	0.965
	2	0.60741932	0.434	0.962
	3	0.74393370	0.433	0.933

To find the appropriate value of the constant c, we note that from (2.7) and (2.10)

$$\lim_{x\to\infty}\{g_{\alpha,\beta}(x)-\beta x\} = -\frac{\sqrt{\beta}}{\sqrt{\alpha}\kappa_0}\log 2 + c. \tag{2.12}$$

Then, taking a constraint $\lim_{x\to\infty}\{g_{\alpha,\beta}(x)-\beta x\} = B(\alpha,\beta)$, from the property (ii) we can determine the value of c as follows:

$$c = B(\alpha,\beta) + \frac{\sqrt{\beta}}{\sqrt{\alpha}\kappa_0}\log 2 = \sqrt{\frac{\beta}{\alpha}}\left\{B_0 + \frac{\log 2}{\kappa_0}\right\}. \tag{2.13}$$

For determination of the optimal value of the parameter r in the approximate solution $g_{\alpha,\beta}(x)$, we consider minimization of the residual function

$$Ng_{\alpha,\beta}(x) = g'''_{\alpha,\beta}(x) + \alpha\, g_{\alpha,\beta}(x)g''_{\alpha,\beta}(x) \tag{2.14}$$

over the interval $0 < x < \infty$ in the L_2-norm sense, that is, minimization of

$$\|Ng_{\alpha,\beta}\|_2^2 = \int_0^\infty \{Ng_{\alpha,\beta}(x)\}^2 dx \tag{2.15}$$

with respect to the parameter r included therein. In practice, using the software *Mathematica*, we can obtain numerical optimal values of r in $g_{\alpha,\beta}(x)$, denoted by $r^* = r^*(\alpha,\beta)$, for each given α and β. Numerical results for the values of r^* and the maximum relative percentage errors of $g_{\alpha,\beta}(x)$ with r^* are given in Table 2 for each $\alpha = 1/2, 1$ and $\beta = 1/2, 1, 2, 3$. The table shows that $g_{\alpha,\beta}(x)$ uniformly approximates the numerical solution $f_{\alpha,\beta}(x)$ with the maximum relative errors less than 0.44% and 0.97% for approximation to $f_{\alpha,\beta}(x)$ and its derivative, respectively. As a result, one can see that the two term approximate solution $g_{\alpha,\beta}(x)$ with $r = r^*$ well improves the single term approximation $h_{\alpha,\beta}(x)$.

Table 3: Numerical optimal values of the parameter r and maximum relative % errors of the three-term approximate solution $k_{\alpha,\beta}(x)$ for each value of α and β.

α	β	r^*	$M^{[0]}k_{\alpha,\beta}$	$M^{[1]}k_{\alpha,\beta}$
1/2	1/2	0.25243528	0.0321	0.0649
	1	0.35699739	0.0314	0.0647
	2	0.50487056	0.0321	0.0646
	3	0.61833763	0.0320	0.0647
1	1/2	0.35699740	0.0314	0.0647
	1	0.50487056	0.0321	0.0646
	2	0.71399479	0.0311	0.0647
	3	0.87446146	0.0310	0.0649

For further improvement of the approximate solutions given above, we propose another approximate analytical solution as

$$k_{\alpha,\beta}(x) = \frac{\beta}{b}\log[\cosh(bx)] + \frac{c}{2}\left\{\tanh^4(rx) + \tanh^8(rx)\right\}, \tag{2.16}$$

where b and c are as given in (2.7) and (2.13). Similarly to the case of $g_{\alpha,\beta}(x)$, the optimal value r^* of the parameter r in (2.16) should be determined by the minimization of $\|Nk_{\alpha,\beta}\|_2^2$. Table 3 includes numerical results for the values of r^* and the maximum relative percentage errors for each $\alpha = 1/2, 1$ and $\beta = 1/2, 1, 2, 3$. One can see that $k_{\alpha,\beta}(x)$ uniformly approximates $f_{\alpha,\beta}(x)$ with the maximum relative errors less than 0.033% and 0.065% for approximation to $f_{\alpha,\beta}(x)$ and its derivative, respectively. Therefore, the three-term approximate solution $k_{\alpha,\beta}(x)$ with $r = r^*$ highly improves the previous approximate solutions $h_{\alpha,\beta}(x)$ and $g_{\alpha,\beta}(x)$.

Figure 2 includes graphs of the relative percentage errors of the proposed approximate analytical solutions $g_{\alpha,\beta}(x)$ and $k_{\alpha,\beta}(x)$ and those of their derivatives $g'_{\alpha,\beta}(x)$ and $k'_{\alpha,\beta}(x)$, where the optimal values $r = r^*$ given in Tables 2 and 3 are used.

3. Application to the Homotopy Perturbation Method

In this section, we consider application of the presented approach to the homotopy perturbation method [6] which is composed of coupling iteration method and perturbation method.

First, we take an iteration formula for the original equation (1.1) as

$$\phi'''_{n+1}(x) + \alpha\phi_n(x)\phi''_{n+1}(x) = 0, \quad n \geq 0. \tag{3.1}$$

Setting an initial approximate solution

$$\phi_0(x) = \frac{a}{\alpha} \tag{3.2}$$

for a constant $a > 0$ and substituting it into (3.1), we have

$$\phi'''_1(x) + a\phi''_1(x) = 0. \tag{3.3}$$

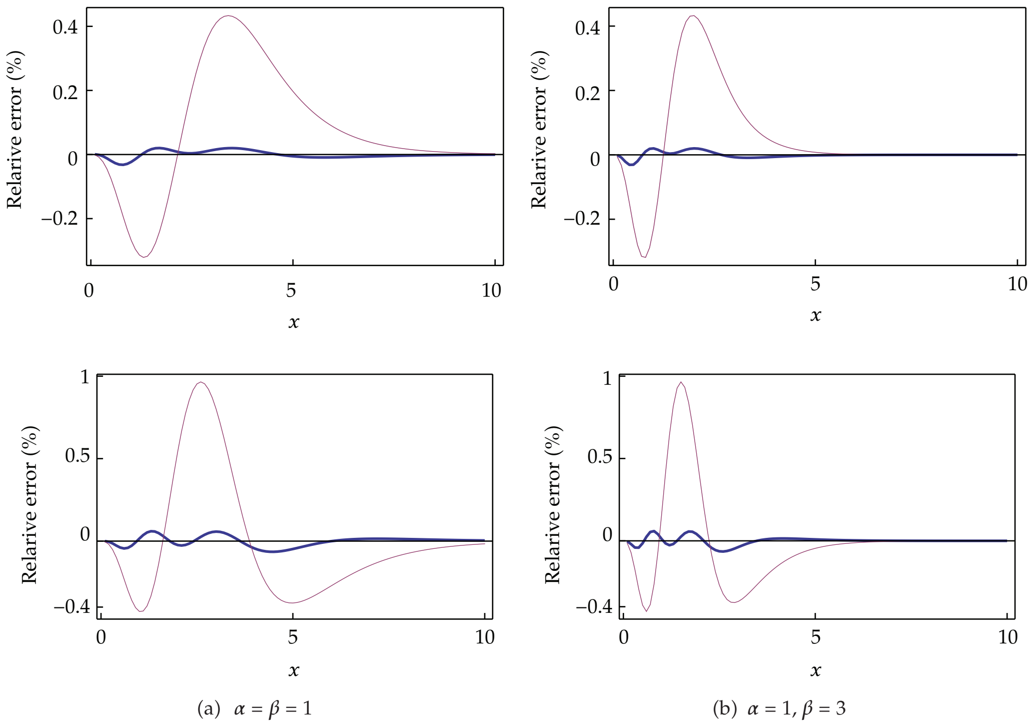


Figure 2: Relative % errors of the presented approximate solutions (in upper row) and those of their derivatives (in lower row). Thin lines indicate the two term approximate solution $g_{\alpha,\beta}(x)$, and thick lines indicate the three-term approximate solution $k_{\alpha,\beta}(x)$.

Referring to the boundary conditions in (1.2), we have a solution

$$\phi_1(x) = \beta x - \frac{\beta}{a}\left(1 - e^{-ax}\right) \tag{3.4}$$

which satisfies $\phi_1(0) = \phi_1'(0) = 0$ and $\phi_1'(\infty) = \beta$.

Substituting $\phi_1(x)$ into (3.1), we obtain

$$\phi_2'''(x) + a\phi_2''(x) + \alpha\left\{\beta x - \frac{\beta}{a}\left(1 - e^{-ax}\right) - \frac{a}{\alpha}\right\}\phi_2''(x) = 0. \tag{3.5}$$

If we embed an artificial parameter ϵ, then it follows that

$$\phi_2'''(x) + a\phi_2''(x) + \epsilon\alpha\left\{\beta x - \frac{\beta}{a}\left(1 - e^{-ax}\right) - \frac{a}{\alpha}\right\}\phi_2''(x) = 0. \tag{3.6}$$

Suppose the solution of this equation can be expressed as

$$\phi_2(x) = \phi_2^{[0]}(x) + \epsilon\phi_2^{[1]}(x), \tag{3.7}$$

then we have the following two equations:

$$\left(\phi_2^{[0]}(x)\right)''' + a\left(\phi_2^{[0]}(x)\right)'' = 0 \tag{3.8}$$

with $\phi_2^{[0]}(0) = (\phi_2^{[0]})'(0) = 0, (\phi_2^{[0]})'(\infty) = \beta$ and

$$\left(\phi_2^{[1]}(x)\right)''' + a\left(\phi_2^{[1]}(x)\right)'' + \alpha\left\{\beta x - \frac{\beta}{a}(1 - e^{-ax}) - \frac{a}{\alpha}\right\}\left(\phi_2^{[0]}(x)\right)'' = 0 \tag{3.9}$$

with $\phi_2^{[1]}(0) = (\phi_2^{[1]})'(0) = 0, (\phi_2^{[1]})'(\infty) = 0$.

One can see that the solution of (3.8) is

$$\phi_2^{[0]}(x) = \beta x - \frac{\beta}{a}(1 - e^{-ax}). \tag{3.10}$$

Substitution of $\phi_2^{[0]}(x)$ into (3.9) results in

$$\left(\phi_2^{[1]}(x)\right)''' + a\left(\phi_2^{[1]}(x)\right)'' = -\alpha\beta\left\{a\beta x - \beta(1 - e^{-ax}) - \frac{a^2}{\alpha}\right\}e^{-ax}. \tag{3.11}$$

Assume that the approximate solution of (3.11) can be expressed as

$$\phi_2^{[1]}(x) = -2Ce^{-ax} + Ce^{-2ax} + C \tag{3.12}$$

for some constant C.

By setting $\epsilon = 1$ in (3.7), from (3.10) and (3.12) we obtain

$$\phi_2(x) = \beta x - \frac{\beta}{a}(1 - e^{-ax}) - 2Ce^{-ax} + Ce^{-2ax} + C \tag{3.13}$$

with $\phi_2(0) = \phi_2'(0) = 0$ and $\phi_2'(\infty) = \beta$. To determine the unknown constants C and a we take the conditions $\lim_{x\to\infty}\{\phi_2(x) - \beta x\} = B(\alpha,\beta)$ and $\phi_2''(0) = \kappa(\alpha;\beta)$ given in (ii) and (i), respectively. Then it follows that

$$\begin{aligned} C &= \frac{\beta}{a} + B(\alpha,\beta), \\ a\beta + 2Ca^2 &= \kappa(\alpha;\beta) \end{aligned} \tag{3.14}$$

which results in

$$a = \frac{1}{4B(\alpha,\beta)}\left\{-3\beta + \sqrt{9\beta^2 + 8\kappa(\alpha;\beta)B(\alpha,\beta)}\right\}. \tag{3.15}$$

Table 4: Maximum relative % errors of the approximate solutions $\phi_2(x)$ and $\tilde{\phi}_2(x)$ based on the homotopy perturbation method.

α	β	$M^{[0]}\phi_2$	$M^{[1]}\phi_2$	$M^{[0]}\tilde{\phi}_2$	$M^{[1]}\tilde{\phi}_2$
1/2	1/2	19.2	21.8	8.08	8.95
	1	19.2	21.8	8.07	8.93
	2	19.2	21.8	8.07	8.95
	3	19.2	21.8	8.06	8.96
1	1/2	19.2	21.8	8.07	8.93
	1	19.2	21.8	8.07	8.95
	2	19.2	21.8	8.06	8.88
	3	19.2	21.7	8.09	8.91

On the other hand, if we take an additional term $\tanh^4(rx)$ such as

$$\tilde{\phi}_2(x) = \beta x - \frac{\beta}{a}\left(1 - e^{-ax}\right) - 2Ce^{-ax} + Ce^{-2ax} + C + C\tanh^4(rx), \tag{3.16}$$

then we have

$$\begin{aligned} C &= \frac{1}{2}\left\{\frac{\beta}{a} + B(\alpha,\beta)\right\}, \\ a &= \frac{1}{B(\alpha,\beta)}\left\{-\beta + \sqrt{\beta^2 + \kappa(\alpha;\beta)B(\alpha,\beta)}\right\}. \end{aligned} \tag{3.17}$$

The value of r is taken by minimization of the L_2-norm of the residual function $N\tilde{\phi}_2(x)$ as defined in (2.14).

Numerical results for the maximum relative percentage errors of the homotopy perturbation method (3.13) and the modified method (3.16) are included in Table 4. The table shows that both the solutions $\phi_2(x)$ and $\tilde{\phi}_2(x)$ uniformly approximate the exact solution $f_{\alpha,\beta}(x)$, and that the maximum relative errors of $\tilde{\phi}_2(x)$ are less than 8.09% and 8.96% for approximation to $f_{\alpha,\beta}(x)$ and its derivative, respectively. However, though the modified solution $\tilde{\phi}_2(x)$ improves $\phi_2(x)$ based on the homotopy perturbation method, its accuracy is not comparable with the three-term approximate solution $k_{\alpha,\beta}(x)$ in (2.16).

4. Conclusions

In this paper, we have presented three forms of approximate analytical solutions for the generalized Blasius problem (1.1) and (1.2). The presented solutions uniformly approximate the exact solution on the whole interval $0 \le x < \infty$, regardless of the values of α and β. Particularly, the three-term approximate solution $k_{\alpha,\beta}(x)$ in (2.16) with the parameter $r = r^*$ given in Table 3 results in the relative error less than 0.033%. However, it should be pointed out that there will be room for further improvement if more properties of the exact solution, like (i) and (ii) in Section 2, are informed. In addition, employing the known properties (i) and (ii) of the generalized Blasius problem, we have explored the homotopy perturbation method for application of the presented approach. From the numerical results one can see that the presented three-term approximate solution $k_{\alpha,\beta}(x)$ gives superior results in accuracy.

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