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*Research Article*

# Kink Solutions for a Class of Generalized Dissipative Equations

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We study, in this paper, a generalized viscoelastic equation which includes several interesting models considered in some recent papers. Many physically important nonlinear PDEs can be reduced to nonlinear ODEs by means of reduction techniques. So it is significant and very interesting to study, among all the closed-form solutions admitted by the model, the corresponding kink waves. A plot of the obtained solution is performed.

## 1. Introduction

Nonlinear wave phenomena of dissipation, dispersion, diffusion, reaction, and convection appear in a wide variety of scientific applications and are very important in nonlinear sciences. In this paper, we report an interesting integrable equation that recently (see, e.g., [1, 2]) we have studied from the point of view of symmetries.

The integrable equation we study is

$$u_{tt} = \left[f(u)u_x\right]_x + \left[\lambda(u)u_t\right]_{xx}, \tag{1.1}$$

where $f$ and $\lambda$ are smooth functions, $u(t,x)$ is the dependent variable, and subscripts denote partial derivative with respect to the independent variables $t$ and $x$.

The behavior of some partial differential equations, in the case that $f$ and $\lambda$ could be discontinuous, is investigated in [3, 4] where some regularity results of the solutions are also considered.

When $\lambda(u) \equiv 0$, (1.1) includes the nonlinear homogeneous vibrating string equation:

$$u_{tt} = \left[f(u)u_x\right]_x \tag{1.2}$$

which was classified by Ames et al. [5] and gives rise to numerous publications on symmetry analysis of nonlinear wave phenomena. While when $\lambda(u) = \lambda_0 = \varepsilon << 1$ an approximate study can be found in a recent paper [6].

In the framework of nonlinear viscoelasticity, some recent results can be found in the paper of Pucci and Saccomandi [7] (see bibliography therein for a review).

Our aim, in this work, is to construct wave solutions of nonlinear evolution equation (1.1).

## 2. Problem Formulation

Let us consider a homogeneous viscoelastic bar of uniform cross-section and assume that the material is a nonlinear Kelvin solid. This model is described by a stress-strain relation of the following form [8]:

$$\tau = \sigma(w_x) + \lambda(w_x)w_{xt}, \tag{2.1}$$

where $\tau$ is the stress, $x$ the position of a cross-section in the homogeneous rest configuration of the bar, $w(t, x)$ the displacement at time $t$ of the section from the rest position, $\sigma(w_x)$ is the elastic part of the stress, while $\lambda(w_x)w_{tx}$ is the dissipative part.

The equation of linear momentum $w_{tt} = \tau_x$, in the absence of body forces, after setting $w_x = u$ and introducing the function $f(u)$ such that

$$\sigma(u) = \int^u f(s)\, ds \tag{2.2}$$

can be reduced to (1.1).

Many physically important nonlinear PDEs can be reduced to nonlinear ODEs by means of reduction techniques. So it is significant and very interesting to study the exact solutions of the reduced equation of (1.1). Among all the solutions admitted by our equation we seek for travelling wave solutions.

Travelling waves are very interesting from the point of view of applications. These types of waves will not change their shapes during propagation and are thus easy to detect. Of particular interest are three types of travelling waves: the solitary waves, which are localized travelling waves, asymptotically zero at large distances, the periodic waves, and the kink waves, which rise or descend from one asymptotic state to another.

## 3. Solutions in Viscoelastic Medium

Motivated by a number of physical problems discussed in [9, 10], Ruggieri and Valenti, in [2], found travelling wave solutions for (1.1) in the case of *ideally hard material*, the main feature of which is that the Lagrangian speed of sound increases monotonically without bound.

Then, in order to seek for other solutions of physical interest, we apply the Lie method analysis and we use the notion of symmetry to generate solutions. A key notion in Lie's method is that of an infinitesimal generator for a symmetry group; then, we look for the one-parameter Lie group of infinitesimal transformations in $(t, x, u)$-space given by

$$\begin{aligned}
\hat{t} &= t + a\ \xi^1(t, x, u) + \mathcal{O}\left(a^2\right), \\
\hat{x} &= x + a\ \xi^2(t, x, u) + \mathcal{O}\left(a^2\right), \\
\hat{u} &= u + a\ \eta(t, x, u) + \mathcal{O}\left(a^2\right),
\end{aligned} \tag{3.1}$$

where $a$ is the group parameter and the associated Lie algebra $\mathcal{L}$ is the set of vector fields of the form

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u}. \tag{3.2}$$

We then require that the transformation (3.1) leaves invariant the set of solutions of (1.1); in others words, we require that the transformed equation has the same form as the original one.

Following the well-known monographs on this argument (see, e.g., [11, 12]), we introduce the third prolongation of the operator X which has the form

$$X^{(3)} = X + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{11} \frac{\partial}{\partial u_{tt}} + \zeta_{22} \frac{\partial}{\partial u_{xx}} + \zeta_{221} \frac{\partial}{\partial u_{xxt}}, \tag{3.3}$$

where we have set

$$\begin{aligned}
\zeta_1 &= D_t(\eta) - u_t D_t\left(\xi^1\right) - u_x D_t\left(\xi^2\right), \\
\zeta_2 &= D_x(\eta) - u_t D_x\left(\xi^1\right) - u_x D_x\left(\xi^2\right), \\
\zeta_{11} &= D_t(\zeta_1) - u_{tt} D_t\left(\xi^1\right) - u_{tx} D_t\left(\xi^2\right), \\
\zeta_{22} &= D_x(\zeta_2) - u_{tx} D_x\left(\xi^1\right) - u_{xx} D_x\left(\xi^2\right), \\
\zeta_{221} &= D_t(\zeta_{22}) - u_{xxt} D_t\left(\xi^1\right) - u_{xxx} D_t\left(\xi^2\right),
\end{aligned} \tag{3.4}$$

where the operators $D_t$ and $D_x$ denote total derivatives with respect to $t$ and $x$.

In order to better clarify the technique used to obtain reduced equation of (1.1), we quickly recall the results obtained in [2].

The *determining system* of (1.1) arises from the following invariance condition:

$$X^{(3)}\left(u_{tt} - \left[f(u)u_x\right]_x - \left[\lambda(u)u_t\right]_{xx}\right) = 0, \tag{3.5}$$

under the constraint that the variable $u_{tt}$ has to satisfy (1.1). This latter allows us to find the infinitesimal generator of the symmetry transformations and, at the same time, gives the

**Table 1:** Group classification of (1.1). $f_0$, $\lambda_0$, $p$, $q$, $r$, and $s$ are constitutive constants with $f_0$, $\lambda_0 > 0$, and $p$, $s \neq 0$.

| Case | $f(u)$ and $\lambda(u)$ | Extensions of $\mathcal{L}_{\mathcal{P}}$ |
|---|---|---|
| I | $f = f_0 e^{u/p}$, $\lambda = \lambda_0 e^{u/s}$ | $X_3 = 2(s-p)t\dfrac{\partial}{\partial t} + (s-2p)x\dfrac{\partial}{\partial x} - 2ps\dfrac{\partial}{\partial u}$ |
| II | $f = f_0(u+q)^{1/p}$, $\lambda = \lambda_0(u+q)^{(1+r)/p}$ | $X_3 = 2rt\dfrac{\partial}{\partial t} + (1+2r)x\dfrac{\partial}{\partial x} + 2p(u+q)\dfrac{\partial}{\partial u}$ |
| III | $f = f_0\ (u+q)^{-4/3}$, $\lambda = \lambda_0\ (u+q)^{-4/3}$ | $X_3 = x\dfrac{\partial}{\partial x} - \dfrac{3}{2}(u+q)\dfrac{\partial}{\partial u}$ <br> $X_4 = x^2\dfrac{\partial}{\partial x} - 3x(u+q)\dfrac{\partial}{\partial u}$ |

functional dependence of the constitutive functions $f$ and $\lambda$ for which the equation does admit symmetries. From (3.5) we obtain the following relations:

$$\begin{gathered}
\xi^1 = a_8 t + a_1, \\
\xi^2 = a_9 x^2 + a_5 x + a_2, \\
\eta = (-3a_9 x_2 + a_6 - a_5)u - 3a_3 a_9 x_2 + a_7, \\
a_9\left[3(u+a_3)f' + 4f\right] = 0, \\
a_9\left[3(u+a_3)\lambda' + 4\lambda\right] = 0, \\
[(a_6 - a_5)\ u + a_7]f' + 2(a_8 - a_5)f = 0, \\
[(a_6 - a_5)u + a_7]\lambda' + (a_8 - 2a_5)\lambda = 0,
\end{gathered} \tag{3.6}$$

where $a_i$ $(i = 1, 2, \ldots, 9)$ are constants and the prime denotes derivative of a function with respect to the only variable upon which it depends.

We consider for arbitrary $f$ and $\lambda$, the Principal Lie Algebra $\mathcal{L}_{\mathcal{P}}$ of (1.1) that is two-dimensional and is spanned by the operators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}. \tag{3.7}$$

Otherwise the group classification is summarized in Table 1.

So, reducing (1.1) by means of Principal Lie Algebra we obtain that the similarity variable, the similarity solution, and the reduced ODE of (1.1), respectively, are

$$z = x - c_2 t, \qquad u = \phi(z), \tag{3.8}$$

$$c_2^2\ \phi'' - (f\ \phi')' + c_2(\lambda\ \phi')'' = 0, \tag{3.9}$$

with $f$ and $\lambda$ arbitrary functions of $\phi$ and we observe that the third-order partial differential equation (1.1) admits travelling wave solutions for arbitrary $f$ and $\lambda$.

Now, in order to seek new solutions which are of physical interest, let us consider the following form for the tension function $\sigma(u)$:

$$-\sigma(u) = 2\tilde{\gamma}^2 \rho_0 (-u)^{1/2}, \quad \tilde{\gamma} = \text{const} \tag{3.10}$$

which arises, as Bell has shown [13], in polycristalline solids during a dynamic uniaxial compression.

Under this assumption, taking into account (2.2) and choosing for the compatibility of the problem [2] the following expression for the function $\lambda = \lambda_0(-\phi)^{-1/2}$ with $\lambda_0 > 0$, the reduced equation (3.9) becomes

$$c_2^2\, \phi'' - \tilde{\gamma}^2 \rho_0 \left[(-\phi)^{-1/2} \phi'\right]' + c_2 \lambda_0 \left[(-\phi)^{-1/2} \phi'\right]'' = 0. \tag{3.11}$$

An exact solution of (3.11) is

$$\phi = \left[\frac{2\tilde{\gamma}^2 \rho_0}{e^{(-\tilde{\gamma}^2 \rho_0 / c_2 \lambda_0)(z+k_1)} + c_2^2}\right]^2, \tag{3.12}$$

with $k_1$ being an arbitrary constant of integration.

Coming back to the original variables (3.8) and taking (3.12) into account, the solution can be written as

$$u = \left[\frac{2\tilde{\gamma}^2 \rho_0}{e^{(-\tilde{\gamma}^2 \rho_0 / c_2 \lambda_0)\ (x - c_2 t + k_1)} + c_2^2}\right]^2. \tag{3.13}$$

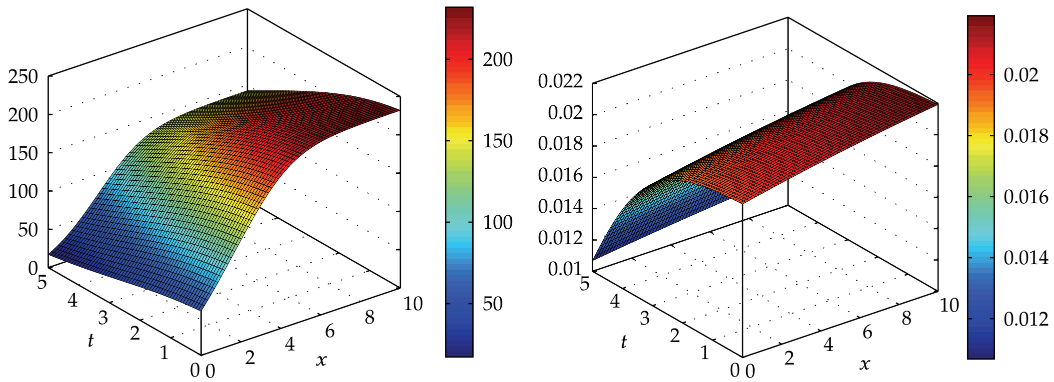
Another solution of physical interest can be obtained when we consider the following form of the tension:

$$\sigma(u) = -\sigma_0 \left(\frac{3T_0}{\rho V_0^2}\right)^3 \left(\frac{3T_0}{\rho V_0^2} + u\right)^{-3} + \sigma_0, \tag{3.14}$$

which models the *ideal soft material* whose main feature is the lagrangian speed of sound which decreases monotonically to zero as $u$ increases without bound.

In this case, taking into account (2.2) and choosing for the compatibility of the problem [2] the following expression for the function $\lambda = \lambda_0(\phi + 3T_0/\rho V_0^2)^{-4}$ with $\lambda_0 > 0$, the reduced equation (3.9) becomes

$$c_2^2 \left(\rho V_0^2\right)^3 \phi'' - (3T_0)^4 \left[\left(\phi + \frac{3T_0}{\rho V_0^2}\right)^{-4} \phi'\right]' + c_2\, \lambda_0 \left(\rho V_0^2\right)^3 \left[\left(\phi + \frac{3T_0}{\rho V_0^2}\right)^{-4} \phi'\right]'' = 0. \tag{3.15}$$




(a) (b)

**Figure 1:** 3D view of the solution (3.13) of (1.1) for increasing values of the wave speed.

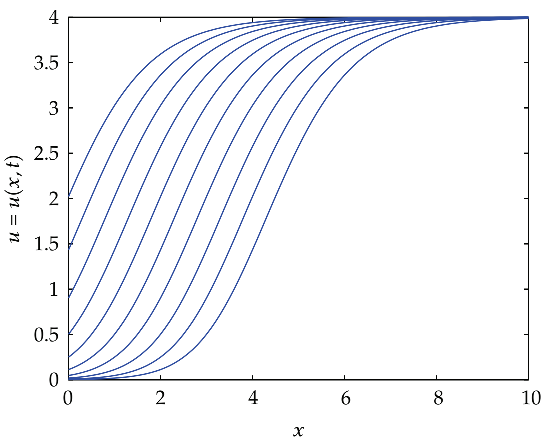



**Figure 2:** Plot 2D of the solution (3.13) of (1.1) at different times, showing $u(x,t)$ versus position.

An exact solution of (3.15) is

$$\phi = 3T_0\left[(\rho V_0^2)^3\left(e^{(108T_0^4/c_2\lambda_0\rho^3V_0^6)(z+k)} - 3c_2^2\right)\right]^{-1/4} - \frac{3T_0}{\rho V_0^2}, \tag{3.16}$$

with $k$ being an arbitrary constant of integration. When we revert to the original variables and take (3.16) into account, the solution can be written as

$$u = 3T_0\left[(\rho V_0^2)^3\left(e^{(108T_0^4/c_2\lambda_0\rho^3V_0^6)(x-c_2t+k)} - 3c_2^2\right)\right]^{-1/4} - \frac{3T_0}{\rho V_0^2}. \tag{3.17}$$

The travelling waves solutions (3.13)–(3.17) have the form of a *kink*, and it is known that *kinks* may propagate in a viscoelastic medium (see [10] and bibliography therein).

In order to show the trend of the obtained solutions (3.13)–(3.17), just as an example some snapshots of the solution (3.13) are shown in Figures 1 and 2.

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