Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 414718, 26 pages
doi:10.1155/2012/414718

Research Article

A New General System of Generalized Nonlinear Mixed Composite-Type Equilibria and Fixed Point Problems with an Application to Minimization Problems

Pongsakorn Sunthrayuth and Poom Kumam

Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bang Mod, Bangkok 10140, Thailand

Correspondence should be addressed to Poom Kumam, poom.kum@kmutt.ac.th

Received 9 August 2012; Accepted 24 September 2012

Academic Editor: RuDong Chen

We introduce a new general system of generalized nonlinear mixed composite-type equilibria and propose a new iterative scheme for finding a common element of the set of solutions of a generalized equilibrium problem, the set of solutions of a general system of generalized nonlinear mixed composite-type equilibria, and the set of fixed points of a countable family of strict pseudocontraction mappings. Furthermore, we prove the strong convergence theorem of the purposed iterative scheme in a real Hilbert space. As applications, we apply our results to solve a certain minimization problem related to a strongly positive bounded linear operator. Finally, we also give a numerical example which supports our results. The results obtained in this paper extend the recent ones announced by many others.

1. Introduction

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed and convex subset of H. Let $\phi : C \to \mathbb{R}$ be a real-valued function, where $\mathbb{R}$ is the set of real numbers. Let $G, B : C \to H$ be two nonlinear mappings and $\Theta : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function, that is, $\Theta(w, u, v) = \Theta(w, v, u) = 0$ for all $(w, u, v) \in H \times C \times C$. We consider the following *new generalized equilibrium problem*: find $x^* \in C$ such that

$$\Theta(Gx^*, x^*, y) + \phi(y) - \phi(x^*) + \langle Bx^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of the problem (1.1) is denoted by $\mathrm{GEP}(C, G, \Theta, \phi, B)$. As special cases of the problem (1.1), we have the following results.

(1) If $B = 0$, then problem (1.1) reduces to the following *generalized equilibrium problem*: find $x^* \in C$ such that

$$\Theta(Gx^*, x^*, y) + \phi(y) - \phi(x^*) \geq 0, \quad \forall y \in C, \tag{1.2}$$

which was considered by Cho et al. [1] for more details. The set of solutions of the problem (1.1) is denoted by $\mathrm{GEP}(C, G, \Theta, \phi)$.

(2) If $B = 0$ and $\Theta(w, u, v) = F(u, v)$, where $F : C \times C \to \mathbb{R}$ is an equilibrium bifunction, then problem (1.1) reduces to the following *mixed equilibrium problem*: find $x^* \in C$ such that

$$F(x^*, y) + \phi(y) - \phi(x^*) \geq 0, \quad \forall y \in C, \tag{1.3}$$

which was considered by Ceng and Yao [2] for more details. The set of solutions of the problem (1.3) is denoted by $\mathrm{MEP}(F, \phi)$.

(3) If $B = 0$, $\phi = 0$ and $\Theta(w, u, v) = F(u, v)$ where $F : C \times C \to \mathbb{R}$ is an equilibrium bifunction, then problem (1.1) reduces to the following *equilibrium problem*: find $x^* \in C$ such that

$$F(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of problem (1.4) is denoted by $\mathrm{EP}(F)$.

(4) If $\Theta = 0$, $\phi = 0$, then problem (1.1) reduces to the following *classical variational inequality problem*: find $x^* \in C$ such that

$$\langle Bx^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of the problem (1.5) is denoted by $\mathrm{VI}(C, B)$.

In brief, for an appropriate choice of the mapping G, the function ϕ, and the convex set C, one can obtain a number of the various classes of equilibrium problems as special cases. In particular, the equilibrium problems (1.4) which were introduced by Blum and Oettli [3] and Noor and Oettli [4] in 1994 have had a great impact and influence on the development of several branches of pure and applied sciences. It has been shown that the equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity, and optimization. In [3, 4], it has been shown that equilibrium problems include variational inequalities, fixed point, minimax problems, Nash equilibrium problems in noncooperative games, and others as special cases. This means that the equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity, and optimization. Hence collectively, equilibrium problems cover a vast range of applications. Related to the equilibrium problems, we also have the problems of finding the fixed points of the nonlinear mappings, which is the subject of current interest in functional analysis. It is natural to construct a unified approach for these problems. In this direction, several authors have introduced some iterative schemes for finding a common element of the set of solutions of the equilibrium problems and the set of fixed points of nonlinear mappings (e.g., see [5–18] and the references therein).

Recall the following definitions.

Definition 1.1. The mapping $S : C \to C$ is said to be

(1) nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C, \tag{1.6}$$

(2) *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Sx - Sy\| \leq L\|x - y\|, \quad \forall x, y \in C, \tag{1.7}$$

(3) *k-strict pseudocontraction* [19] if there exists a constant $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C, \tag{1.8}$$

(4) *pseudocontractive* if

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{1.9}$$

Clearly, the class of strict pseudocontractions falls into the one between classes of nonexpansive mappings and pseudocontractions. It is easy to see that (1.8) is equivalent to

$$\langle Sx - Sy, x - y\rangle \leq \|x - y\|^2 - \frac{1-k}{2}\|(I - S)x - (I - S)y\|^2, \quad \forall x,\ y \in C, \tag{1.10}$$

that is, $I - S$ is $(1-k)/2$-inverse-strongly monotone. From [19], we know that if S is a k-strictly pseudocontractive mapping, then S is Lipschitz continuous with constant $(3-k)/(1-k)$, that is, $\|Sx - Sy\| \leq (3-k)/(1-k)\|x - y\|$, for all $x, y \in C$.

In this paper, we use $\mathrm{Fix}(S) = \{x \in C : Sx = x\}$ to denote the set of fixed points of S.

Definition 1.2. A countable family of mapping $\{T_n\}_{n=1}^{\infty} : C \to C$ is called a *family of k-strict pseudocontraction mappings* if there exists a constant $k \in [0, 1)$ such that

$$\|T_n x - T_n y\|^2 \leq \|x - y\|^2 + k\|(I - T_n)x - (I - T_n)y\|^2, \quad \forall x, y \in C, n \geq 1. \tag{1.11}$$

On the other hand, let C be a nonempty closed and convex subset of a real Hilbert space H. Let $F_1, F_2, F_3 : C \times C \to \mathbb{R}$ be three bifunctions and let $\Psi_1, \Psi_2, \Psi_3, \Phi_1, \Phi_2, \Phi_3 : C \to H$ be six nonlinear mappings and let $\varphi_1, \varphi_2, \varphi_3 : C \to \mathbb{R}$ be three functions. We consider the following problem of finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{aligned}
&\mu_1 F_1(x^*, x) + \langle \mu_1(\Psi_1 + \Phi_1)y^* + x^* - y^*, x - x^* \rangle \geq \mu_1\varphi_1(x^*) - \mu_1\varphi_1(x), \quad \forall x \in C, \\
&\mu_2 F_2(y^*, y) + \langle \mu_2(\Psi_2 + \Phi_2)z^* + y^* - z^*, y - y^* \rangle \geq \mu_2\varphi_2(y^*) - \mu_2\varphi_2(y), \quad \forall y \in C, \\
&\mu_3 F_3(z^*, z) + \langle \mu_3(\Psi_3 + \Phi_3)x^* + z^* - x^*, z - z^* \rangle \geq \mu_3\varphi_3(z^*) - \mu_3\varphi_3(z), \quad \forall z \in C,
\end{aligned} \tag{1.12}$$

which is called a *new general system of generalized nonlinear mixed composite-type equilibria*, where $\mu_i > 0$ for all $i = 1, 2, 3$. Next, we present some special cases of problem (1.12) as follows.

(1) If $\Psi_i = \Psi$, $\Phi_i = \Phi$, $F_i = F$, and $\varphi_i = \varphi$ for all $i = 1, 2, 3$, then the problem (1.12) reduces to the following *new general system of generalized nonlinear mixed composite-type equilibria*: find $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{aligned}
&\mu_1 F(x^*, x) + \langle \mu_1(\Psi + \Phi)y^* + x^* - y^*, x - x^* \rangle \geq \mu_1\varphi(x^*) - \mu_1\varphi(x), \quad \forall x \in C, \\
&\mu_2 F(y^*, y) + \langle \mu_2(\Psi + \Phi)z^* + y^* - z^*, y - y^* \rangle \geq \mu_2\varphi(y^*) - \mu_2\varphi(y), \quad \forall y \in C, \\
&\mu_3 F(z^*, z) + \langle \mu_3(\Psi + \Phi)x^* + z^* - x^*, z - z^* \rangle \geq \mu_3\varphi(z^*) - \mu_3\varphi(z), \quad \forall z \in C,
\end{aligned} \tag{1.13}$$

where $\mu_i > 0$ for all $i = 1, 2$.

(2) If $F_3 = 0$, $\Psi_3 = \Phi_3 = 0$, $\mu_3 = 0$, and $z^* = x^*$, then the problem (1.12) reduces to the following *general system of generalized nonlinear mixed composite-type equilibria*: find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned}
&\mu_1 F_1(x^*, x) + \langle \mu_1(\Psi_1 + \Phi_1)y^* + x^* - y^*, x - x^* \rangle \geq \mu_1\varphi_1(x^*) - \mu_1\varphi_1(x), \quad \forall x \in C, \\
&\mu_2 F_2(y^*, y) + \langle \mu_2(\Psi_2 + \Phi_2)x^* + y^* - x^*, y - y^* \rangle \geq \mu_2\varphi_2(y^*) - \mu_2\varphi_2(y), \quad \forall y \in C,
\end{aligned} \tag{1.14}$$

which was introduced and considered by Ceng et al. [20], where $\mu_i > 0$ for all $i = 1, 2$.

(3) If $F_3 = 0$, $\Psi_3 = \Phi_3 = 0$, $\mu_3 = 0$, and $\varphi_i = 0$ for all $i = 1, 2, 3$ and $z^* = x^*$, then the problem (1.12) reduces to the following a *general system of generalized equilibria*: find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned}
&F_1(x^*, x) + \langle \Psi_1 y^*, x - x^* \rangle + \frac{1}{\mu_1}\langle x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\
&F_2(y^*, y) + \langle \Psi_2 x^*, y - y^* \rangle + \frac{1}{\mu_2}\langle y^* - x^*, y - y^* \rangle \geq 0, \quad \forall y \in C,
\end{aligned} \tag{1.15}$$

which was introduced and considered by Ceng and Yao [21], where $\mu_i > 0$ for all $i = 1, 2$.

(4) If $F_i = F$, $\Psi_i = \Psi$, and $\Phi_i = \Phi$, $\varphi_i = \varphi$, for all $i = 1, 2, 3$, then the problem (1.12) reduces to the following *generalized mixed equilibrium problem with perturbed mapping*: find $x^* \in C$ such that

$$F(x^*, y) + \varphi(y) - \varphi(x^*) + \langle (\Psi + \Phi)x^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \tag{1.16}$$

which was introduced and considered by Hu and Ceng [22].

(5) If $F_i = 0$, $\Phi_i = 0$, and $\varphi_i = 0$ for all $i = 1, 2, 3$, then the problem (1.12) reduces to the following *general system of variational inequalities*: find $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{aligned} &\langle \mu_1 \Psi_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ &\langle \mu_2 \Psi_2 z^* + y^* - z^*, y - y^* \rangle \geq 0, \quad \forall y \in C, \\ &\langle \mu_3 \Psi_3 x^* + z^* - x^*, z - z^* \rangle \geq 0, \quad \forall z \in C, \end{aligned} \tag{1.17}$$

which was introduced and considered by Kumam et al. [23], where $\mu_i > 0$ for all $i = 1, 2, 3$.

(6) If $F_i = 0$, $\Phi_i = 0$, $\varphi_i = 0$ for all $i = 1, 2, 3$, $\Psi_3 = 0$ and $z^* = x^*$, then the problem (1.12) reduces to the following *general system of variational inequalities*: find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \mu_1 \Psi_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ &\langle \mu_2 \Psi_2 x^* + y^* - x^*, y - y^* \rangle \geq 0, \quad \forall y \in C, \end{aligned} \tag{1.18}$$

which was introduced and considered by Ceng et al. [24], where $\mu_i > 0$ for all $i = 1, 2$.

In 2010, Cho et al. [1] introduced an iterative method for finding a common element of the set of solutions of generalized equilibrium problems (1.2), the set of solutions for a systems of nonlinear variational inequalities problems (1.18), and the set of fixed points of nonexpansive mappings in Hilbert spaces. Ceng and Yao [21] introduced and considered a relaxed extragradient-like method for finding a common element of the set of solutions of a system of generalized equilibria, the set of fixed points of a strictly pseudocontractive mapping, and the set of solutions of a equilibrium problem in a real Hilbert space and obtained a strong convergence theorem. The result of Ceng and Yao [21] included, as special cases, the corresponding ones of S. Takahashi and W. Takahashi [10], Ceng et al. [24], Peng and Yao [25], and Yao et al. [26].

Motivated and inspired by the works in the literature, we introduce a new general system of generalized nonlinear mixed composite-type equilibria (1.12) and propose a new iterative scheme for finding a common element of the set of solutions of a generalized equilibrium problem, a general system of generalized nonlinear mixed composite-type equilibria, and the set of fixed points of a countable family of strict pseudocontraction mappings. Furthermore, we prove the strong convergence theorem of the purposed iterative scheme in a real Hilbert space. As applications, we apply our results to solve a certain minimization problem related to a strongly positive bounded linear operator. The results presented in this paper extend the recent results of Cho et al. [1], Ceng and Yao [21], Ceng et al.[20], and many authors.

2. Preliminaries

A bounded linear operator A is said to be *strongly positive*, if there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{2.1}$$

Recall that, a mapping $f : C \to C$ is said to be *contractive* if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in C. \tag{2.2}$$

A mapping $A : C \to H$ is called *α-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{2.3}$$

Let C be a nonempty closed convex subset of a real Hilbert space H. For every point $x \in H$ there exists a unique nearest point in C denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.4}$$

P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive (see [27]) and for $x \in H$,

$$z = P_C x \Longleftrightarrow \langle x - z, y - z\rangle \leq 0, \quad \forall y \in C. \tag{2.5}$$

Let $\phi : C \to \mathbb{R}$ be a real-valued function, $G : C \to H$ be a mapping and $\Theta : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function. Let r be a positive real number. For all $x \in C$, we consider the following problem. Find $y \in C$ such that

$$\Theta(Gx, y, z) + \phi(z) - \phi(y) + \frac{1}{r}\langle y - x, z - y\rangle \geq 0, \quad \forall z \in C, \tag{2.6}$$

which is known as *the auxiliary generalized equilibrium problem*.

Let $T^{(r)} : H \to C$ be the mapping such that, for all $x \in H$, $T^{(r)}$ is the solution set of the auxiliary problem (2.6), that is,

$$T^{(r)}(x) = \left\{y \in C : \Theta(Gx, y, z) + \phi(z) - \phi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall z \in C\right\}. \tag{2.7}$$

Then, we will assume the *Condition* (Δ) [28] as follows:

(a) $T^{(r)}$ is single-valued;

(b) $T^{(r)}$ is nonexpansive;

(c) $\mathrm{Fix}(T^{(r)}) = \mathrm{GEP}(C, G, \Theta, \phi)$.

Notice that the examples of showing the sufficient conditions for the existence of the condition (Δ) can be found in [6].

Throughout this paper, we assume that a bifunction $F : C \times C \to \mathbb{R}$ and $\varphi : C \to \mathbb{R}$ is a lower semicontinuous and convex function satisfy the following conditions:

(H1) $F(x, x) = 0, \forall x \in C$;

(H2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0, \forall x, y \in C$;

(H3) for all $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous;

(H4) for all $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous;

(A1) for all $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subset C$ and $y_x \in C$ such that for all $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \tag{2.8}$$

(A2) C is a bounded set.

In order to prove our main results in the next section, we need the following lemmas.

Lemma 2.1 (see [29]). *Let C be a nonempty closed and convex subset of a real Hilbert sapce H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying condition $(H1)$–$(H4)$ and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function. For $r > 0$ and $x \in H$ define a mapping $T_r^{(F,\varphi)} : H \to C$ follows*

$$T_r^{(F,\varphi)}(x) = \left\{ y \in C : F(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall z \in C \right\}. \tag{2.9}$$

Assume that either $(A1)$ or $(A2)$ holds, then the following statements hold

(i) $T_r^{(F,\varphi)} \neq \emptyset$ *for all $x \in H$ and $T_r^{(F,\varphi)}$ is single-valued;*

(ii) $T_r^{(F,\varphi)}$ *is firmly nonexpansive, that is, for all $x, y \in H$,*

$$\left\| T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y \right\|^2 \leq \left\langle T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y, x - y \right\rangle; \tag{2.10}$$

(iii) $\mathrm{Fix}(T_r^{(F,\varphi)}) = \mathrm{MEP}(F, \varphi)$*;*

(iv) $\mathrm{MEP}(F, \varphi)$ *is closed and convex.*

Remark 2.2. If $\varphi = 0$, then $T_r^{(F,\varphi)}$ is rewritten as T_r^F (see [21, Lemma 2.1] for more details).

Lemma 2.3 (see [30]). *Let $\{x_n\}$ and $\{l_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|l_n - x_n\| = 0$.*

Lemma 2.4 (see [31]). *Let H be a real Hilbert space. Then the following inequalities hold.*

(i) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$, $\forall x, y \in H$ *and* $\lambda \in [0,1]$.

(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$, $\forall x, y \in H$.

Definition 2.5 (see [32]). Let $\{T_n\}$ be a sequence of mappings from a subset C of a real Hilbert space H into itself. We say that $\{T_n\}$ satisfies the PT-condition if

$$\lim_{k,l\to\infty}\rho_l^k = 0, \tag{2.11}$$

where $\rho_l^k = \sup_{z\in C}\{\|T_k z - T_l z\|\} < \infty$, for all $k, l \in \mathbb{N}$.

Lemma 2.6 (see [32]). *Suppose that $\{T_n\}$ satisfies the PT-condition such that*

(i) *for each $x \in C$, $\{T_n\}$ is converse strongly to some point in C*

(ii) *let the mapping $T : C \to C$ defined by $Tx = \lim_{n\to\infty}T_n x$ for all $x \in C$.*

Then, $\lim_{n\to\infty}\sup_{\omega\in C}\|T\omega - T_n\omega\| = 0$.

Lemma 2.7 (see [33]). *Let C be a closed and convex subset of a strictly convex Banach space X. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose $\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$ is nonempty. Let $\{\gamma_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty}\gamma_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=1}^{\infty}\gamma_n T_n x$ for all $x \in C$ is well defined, nonexpansive, and $\mathrm{Fix}(S) = \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$ holds.*

Lemma 2.8 (see [19]). *Let $T : C \to H$ be a k-strict pseudocontraction. Define $S : C \to H$ by $Sx = \delta x + (1-\delta)Tx$ for each $x \in C$. Then, as $\delta \in [k,1)$, S is nonexpansive such that $\mathrm{Fix}(S) = \mathrm{Fix}(T)$.*

Lemma 2.9 (see [34]). *Let C be a closed and convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping. then, the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and $(I-S)x_n \to y$, then $(I-S)z = y$.*

Lemma 2.10 (see [35]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\sigma_n)a_n + \delta_n, \tag{2.12}$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty}\sigma_n = \infty$;

(ii) $\limsup_{n\to\infty}(\delta_n/\sigma_n) \leq 0$ *or* $\sum_{n=0}^{\infty}|\delta_n| < \infty$.

then, $\lim_{n\to\infty}a_n = 0$.

Lemma 2.11. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let mappings $\Psi, \Phi : C \to H$ be $\widetilde{\beta}$-inverse-strongly monotone and $\widetilde{\gamma}$-inverse-strongly monotone, respectively. Then, we have*

$$\begin{aligned}\|(I-\mu(\Psi+\Phi))x-(I-\mu(\Psi+\Phi))y\|^2 \le \|x-y\|^2 &+ 2\mu\left(\mu-\widetilde{\beta}\right)\|\Psi x-\Psi y\|^2 \\ &+ 2\mu(\mu-\widetilde{\gamma})\|\Phi x-\Phi y\|^2,\end{aligned} \tag{2.13}$$

where $\mu > 0$. In particular, if $\mu \in (0, \min\{\widetilde{\beta}, \widetilde{\gamma}\})$, then $I - \mu(\Psi + \Phi)$ is nonexpansive.

Proof. From Lemma 2.4(i), for all $x, y \in C$, we have

$$\begin{aligned}
&\|(I-\mu(\Psi+\Phi))x-(I-\mu(\Psi+\Phi))y\|^2 \\
&\quad = \|(x-y)-\mu((\Psi+\Phi)x-(\Psi+\Phi)y)\|^2 \\
&\quad = \left\|\frac{1}{2}((x-y)-2\mu(\Psi x-\Psi y))+\frac{1}{2}((x-y)-2\mu(\Phi x-\Phi y))\right\|^2 \\
&\quad \le \frac{1}{2}\|((x-y)-2\mu(\Psi x-\Psi y))\|^2+\frac{1}{2}\|((x-y)-2\mu(\Phi x-\Phi y))\|^2 \\
&\quad = \frac{1}{2}\left(\|x-y\|^2-4\mu\langle x-y,\Psi x-\Psi y\rangle+4\mu^2\|\Psi x-\Psi y\|^2\right) \\
&\qquad + \frac{1}{2}\left(\|x-y\|^2-4\mu\langle x-y,\Phi x-\Phi y\rangle+4\mu^2\|\Phi x-\Phi y\|^2\right) \\
&\quad \le \frac{1}{2}\left(\|x-y\|^2+4\mu\left(\mu-\widetilde{\beta}\right)\|\Psi x-\Psi y\|^2\right) \\
&\qquad + \frac{1}{2}\left(\|x-y\|^2+4\mu(\mu-\widetilde{\gamma})\|\Phi x-\Phi y\|^2\right) \\
&\quad = \|x-y\|^2+2\mu\left(\mu-\widetilde{\beta}\right)\|\Psi x-\Psi y\|^2+2\mu(\mu-\widetilde{\gamma})\|\Phi x-\Phi y\|^2.
\end{aligned} \tag{2.14}$$

It is clear that, if $0 < \mu \le \min\{\widetilde{\beta}, \widetilde{\gamma}\}$, then $I - \mu(\Psi + \Phi)$ is nonexpansive. This completes the proof. □

Lemma 2.12. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let mappings $\Psi_i, \Phi_i : C \to H (i = 1, 2, 3)$ be $\widetilde{\beta}_i$-inverse-strongly monotone and $\widetilde{\gamma}_i$-inverse-strongly monotone, respectively. Let $Q : C \to C$ be the mapping defined by*

$$\begin{aligned}
Qx = T_{\mu_1}^{(F_1,\varphi_1)}&\left[T_{\mu_2}^{(F_2,\varphi_2)}\left[T_{\mu_3}^{(F_3,\varphi_3)}(x-\mu_3(\Psi_3+\Phi_3)x)-\mu_2(\Psi_2+\Phi_2)T_{\mu_3}^{(F_3,\varphi_3)}(x-\mu_3(\Psi_3+\Phi_3)x)\right]\right. \\
&-\mu_1(\Psi_1+\Phi_1)T_{\mu_2}^{(F_2,\varphi_2)}\left[T_{\mu_3}^{(F_3,\varphi_3)}(x-\mu_3(\Psi_3+\Phi_3)x)\right. \\
&\left.\left.-\mu_2(\Psi_2+\Phi_2)T_{\mu_3}^{(F_3,\varphi_3)}(x-\mu_3(\Psi_3+\Phi_3)x)\right]\right], \quad \forall x \in C.
\end{aligned} \tag{2.15}$$

If $\mu_i \in (0, \min\{\widetilde{\beta}_i, \widetilde{\gamma}_i\})$ $(i = 1, 2, 3)$, then Q is nonexpansive.

Proof. From Lemma 2.11, for all $x, y \in C$, we have

$$
\begin{aligned}
&\|Qx - Qy\| = \\
&\quad\Big\| T_{\mu_1}^{(F_1,\varphi_1)} \Big[T_{\mu_2}^{(F_2,\varphi_2)} \Big[T_{\mu_3}^{(F_3,\varphi_3)} \big(x - \mu_3(\Psi_3 + \Phi_3)x\big) - \mu_2(\Psi_2 + \Phi_2) T_{\mu_3}^{(F_3,\varphi_3)} \big(x - \mu_3(\Psi_3 + \Phi_3)x\big) \Big] \\
&\qquad\qquad - \mu_1(\Psi_1 + \Phi_1) T_{\mu_2}^{(F_2,\varphi_2)} \Big[T_{\mu_3}^{(F_3,\varphi_3)} \big(x - \mu_3(\Psi_3 + \Phi_3)x\big) \\
&\qquad\qquad\qquad - \mu_2(\Psi_2 + \Phi_2) T_{\mu_3}^{(F_3,\varphi_3)} \big(x - \mu_3(\Psi_3 + \Phi_3)x\big) \Big] \Big] \\
&\quad - T_{\mu_1}^{(F_1,\varphi_1)} \Big[T_{\mu_2}^{(F_2,\varphi_2)} \Big[T_{\mu_3}^{(F_3,\varphi_3)} \big(y - \mu_3(\Psi_3 + \Phi_3)y\big) - \mu_2(\Psi_2 + \Phi_2) T_{\mu_3}^{(F_3,\varphi_3)} \big(y - \mu_3(\Psi_3 + \Phi_3)y\big) \Big] \\
&\qquad\qquad - \mu_1(\Psi_1 + \Phi_1) T_{\mu_2}^{(F_2,\varphi_2)} \Big[T_{\mu_3}^{(F_3,\varphi_3)} \big(y - \mu_3(\Psi_3 + \Phi_3)y\big) \\
&\qquad\qquad\qquad - \mu_2(\Psi_2 + \Phi_2) T_{\mu_3}^{(F_3,\varphi_3)} \big(y - \mu_3(\Psi_3 + \Phi_3)y\big) \Big] \Big] \Big\| \\
&\le \Big\| T_{\mu_2}^{(F_2,\varphi_2)} \Big[T_{\mu_3}^{(F_3,\varphi_3)} \big(x - \mu_3(\Psi_3 + \Phi_3)x\big) - \mu_2(\Psi_2 + \Phi_2) T_{\mu_3}^{(F_3,\varphi_3)} \big(x - \mu_3(\Psi_3 + \Phi_3)x\big) \Big] \\
&\quad - \mu_1(\Psi_1 + \Phi_1) T_{\mu_2}^{(F_2,\varphi_2)} \Big[T_{\mu_3}^{(F_3,\varphi_3)} \big(x - \mu_3(\Psi_3 + \Phi_3)x\big) \\
&\qquad\qquad - \mu_2(\Psi_2 + \Phi_2) T_{\mu_3}^{(F_3,\varphi_3)} \big(x - \mu_3(\Psi_3 + \Phi_3)x\big) \Big] \\
&\quad - \Big[T_{\mu_2}^{(F_2,\varphi_2)} \Big[T_{\mu_3}^{(F_3,\varphi_3)} \big(y - \mu_3(\Psi_3 + \Phi_3)y\big) - \mu_2(\Psi_2 + \Phi_2) T_{\mu_3}^{(F_3,\varphi_3)} \big(y - \mu_3(\Psi_3 + \Phi_3)y\big) \Big] \\
&\quad - \mu_1(\Psi_1 + \Phi_1) T_{\mu_2}^{(F_2,\varphi_2)} \Big[T_{\mu_3}^{(F_3,\varphi_3)} \big(y - \mu_3(\Psi_3 + \Phi_3)y\big) \\
&\qquad\qquad - \mu_2(\Psi_2 + \Phi_2) T_{\mu_3}^{(F_3,\varphi_3)} \big(y - \mu_3(\Psi_3 + \Phi_3)y\big) \Big] \Big] \Big\| \\
&= \Big\| T_{\mu_2}^{(F_2,\varphi_2)} \big(I - \mu_2(\Psi_2 + \Phi_2)\big) T_{\mu_3}^{(F_3,\varphi_3)} \big(I - \mu_3(\Psi_3 + \Phi_3)\big) \big(I - \mu_1(\Psi_1 + \Phi_1)\big) x \\
&\quad - T_{\mu_2}^{(F_2,\varphi_2)} \big(I - \mu_2(\Psi_2 + \Phi_2)\big) T_{\mu_3}^{(F_3,\varphi_3)} \big(I - \mu_3(\Psi_3 + \Phi_3)\big) \big(I - \mu_1(\Psi_1 + \Phi_1)\big) y \Big\| \\
&\le \Big\| \big(I - \mu_2(\Psi_2 + \Phi_2)\big) T_{\mu_3}^{(F_3,\varphi_3)} \big(I - \mu_3(\Psi_3 + \Phi_3)\big) \big(I - \mu_1(\Psi_1 + \Phi_1)\big) x \\
&\quad - \big(I - \mu_2(\Psi_2 + \Phi_2)\big) T_{\mu_3}^{(F_3,\varphi_3)} \big(I - \mu_3(\Psi_3 + \Phi_3)\big) \big(I - \mu_1(\Psi_1 + \Phi_1)\big) y \Big\| \\
&\le \Big\| T_{\mu_3}^{(F_3,\varphi_3)} \big(I - \mu_3(\Psi_3 + \Phi_3)\big) \big(I - \mu_1(\Psi_1 + \Phi_1)\big) x \\
&\quad - T_{\mu_3}^{(F_3,\varphi_3)} \big(I - \mu_3(\Psi_3 + \Phi_3)\big) \big(I - \mu_1(\Psi_1 + \Phi_1)\big) y \Big\| \\
&\le \big\| \big(I - \mu_3(\Psi_3 + \Phi_3)\big) \big(I - \mu_1(\Psi_1 + \Phi_1)\big) x - \big(I - \mu_3(\Psi_3 + \Phi_3)\big) \big(I - \mu_1(\Psi_1 + \Phi_1)\big) y \big\| \\
&\le \big\| \big(I - \mu_1(\Psi_1 + \Phi_1)\big) x - \big(I - \mu_1(\Psi_1 + \Phi_1)\big) y \big\| \\
&\le \|x - y\|,
\end{aligned}
\tag{2.16}
$$

which implies that $Q : C \to C$ is nonexpansive. This completes the proof. □

Lemma 2.13. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $F_i : C \times C \to \mathbb{R}$ $(i = 1,2,3)$ be a bifunction satisfying conditions $(H1)$–$(H4)$ and let $\Psi_i, \Phi_i : C \to H (i = 1,2,3)$ be a nonlinear mapping. Suppose that μ_i $(i = 1,2,3)$ be a real positive number. Let $\varphi_i : C \to \mathbb{R}$ $(i = 1,2,3)$ be a lower semicontinuous and convex function. Assume that either condition $(A1)$ or $(A2)$ holds. Then, for $(x^*, y^*, z^*) \in C \times C \times C$ is a solution of the problem* (1.12) *if and only if $x^* \in \mathrm{Fix}(Q)$, $y^* = T_{\mu_2}^{(F_2,\varphi_2)}(z^* - \mu_2(\Psi_2 + \Phi_2)z^*)$ and $z^* = T_{\mu_3}^{(F_3,\varphi_3)}(x^* - \mu_3(\Psi_3 + \Phi_3)x^*)$, where Q is the mapping defined as in Lemma 2.12.*

Proof. Let $(x^*, y^*, z^*) \in C \times C \times C$ be a solution of the problem (1.12). Then, we have

$$\begin{aligned} &\mu_1 F_1(x^*, x) + \langle \mu_1(\Psi_1 + \Phi_1)y^* + x^* - y^*, x - x^* \rangle \geq \mu_1\varphi_1(x^*) - \mu_1\varphi_1(x), \quad \forall x \in C,\\ &\mu_2 F_2(y^*, y) + \langle \mu_2(\Psi_2 + \Phi_2)z^* + y^* - z^*, y - y^* \rangle \geq \mu_2\varphi_2(y^*) - \mu_2\varphi_2(y), \quad \forall y \in C,\\ &\mu_3 F_3(z^*, z) + \langle \mu_3(\Psi_3 + \Phi_3)x^* + z^* - x^*, z - z^* \rangle \geq \mu_3\varphi_3(z^*) - \mu_3\varphi_3(z), \quad \forall z \in C, \end{aligned} \tag{2.17}$$

$\Leftrightarrow$

$$\begin{aligned} x^* &= T_{\mu_1}^{(F_1,\varphi_1)}(y^* - \mu_1(\Psi_1 + \Phi_1)y^*),\\ y^* &= T_{\mu_2}^{(F_2,\varphi_2)}(z^* - \mu_2(\Psi_2 + \Phi_2)z^*),\\ z^* &= T_{\mu_3}^{(F_3,\varphi_3)}(x^* - \mu_3(\Psi_3 + \Phi_3)x^*), \end{aligned} \tag{2.18}$$

$\Leftrightarrow$

$$\begin{aligned} x^* = T_{\mu_1}^{(F_1,\varphi_1)} &\Big[T_{\mu_2}^{(F_2,\varphi_2)} \Big[T_{\mu_3}^{(F_3,\varphi_3)}(x^* - \mu_3(\Psi_3 + \Phi_3)x^*) - \mu_2(\Psi_2 + \Phi_2) T_{\mu_3}^{(F_3,\varphi_3)}(x^* - \mu_3(\Psi_3 + \Phi_3)x^*) \Big]\\ &\quad - \mu_1(\Psi_1 + \Phi_1) T_{\mu_2}^{(F_2,\varphi_2)} \Big[T_{\mu_3}^{(F_3,\varphi_3)}(x^* - \mu_3(\Psi_3 + \Phi_3)x^*)\\ &\qquad - \mu_2(\Psi_2 + \Phi_2) T_{\mu_3}^{(F_3,\varphi_3)}(x^* - \mu_3(\Psi_3 + \Phi_3)x^*) \Big] \Big]. \end{aligned} \tag{2.19}$$

This completes the proof. □

Corollary 2.14 (see [20]). *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $F_i : C \times C \to \mathbb{R}$ $(i = 1,2)$ be a bifunction satisfying conditions $(H1)$–$(H4)$ and let $\Psi_i, \Phi_i : C \to H (i = 1,2)$ be a nonlinear mapping. Suppose that μ_i $(i = 1,2)$ be a real positive number. Let $\varphi_i : C \to \mathbb{R}$ $(i = 1,2)$ be a lower semicontinuous and convex function. Assume that either condition $(A1)$ or $(A2)$ holds. Then, for $(x^*, y^*) \in C \times C$ is a solution of the problem* (1.14) *if and only if $x^* \in \mathrm{Fix}(Q)$, $y^* = T_{\mu_2}^{(F_2,\varphi_2)}(x^* - \mu_2(\Psi_2 + \Phi_2)x^*)$, where G is the mapping defined by*

$$Qx = T_{\mu_1}^{(F_1,\varphi_1)} \Big[T_{\mu_2}^{(F_2,\varphi_2)}(x - \mu_2(\Psi_2 + \Phi_2)x) - \mu_1(\Psi_1 + \Phi_1) T_{\mu_2}^{(F_2,\varphi_2)}(x - \mu_2(\Psi_2 + \Phi_2)x) \Big], \quad \forall x \in C. \tag{2.20}$$

Corollary 2.15 (see [21]). *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $F_i : C \times C \to \mathbb{R}$ $(i = 1,2)$ be a bifunction satisfying conditions $(H1)$–$(H4)$ and let $\Psi_i : C \to$*

H $(i = 1, 2)$ be a nonlinear mapping. Suppose that μ_i $(i = 1, 2)$ is a real positive number. Assume that either condition $(A1)$ or $(A2)$ holds. Then, for $(x^*, y^*) \in C \times C$ is a solution of the problem (1.15) *if and only if $x^* \in \mathrm{Fix}(Q)$, $y^* = T_{\mu_2}^{F_2}(x^* - \mu_2\Psi_2 x^*)$, where Q is the mapping defined by*

$$Qx = T_{\mu_1}^{F_1}\left[T_{\mu_2}^{F_2}(x - \mu_2\Psi_2 x) - \mu_1\Psi_1 T_{\mu_2}^{F_2}(x - \mu_2\Psi_2 x)\right], \quad \forall x \in C. \tag{2.21}$$

Corollary 2.16 (see [23]). *Let C be a nonempty closed and convex subset of a real Hilbert space H. For given $(x^*, y^*, z^*) \in C \times C \times C$ is a solution of the problem* (1.17) *if and only if $x^* \in \mathrm{Fix}(Q)$, $y^* = P_C(z^* - \mu_2\Psi_2 z^*)$ and $z^* = P_C(x^* - \mu_3\Psi_3 x^*)$, where Q is the mapping defined by*

$$\begin{aligned} Qx = P_C[\,&P_C[P_C(x - \mu_3\Psi_3 x) - \mu_2\Psi_2 P_C(x - \mu_3\Psi x)] \\ &-\mu_1\Psi_1 P_C[P_C(x - \mu_3\Psi_3 x) - \mu_2\Psi_2 P_C(x - \mu_3\Psi_3 x)]], \quad \forall x \in C. \end{aligned} \tag{2.22}$$

Corollary 2.17 (see [24]). *Let C be a nonempty closed and convex subset of a real Hilbert space H. For given $(x^*, y^*) \in C \times C$ is a solution of the problem* (1.18) *if and only if $x^* \in \mathrm{Fix}(Q)$, $y^* = P_C(x^* - \mu_2\Psi_2 x^*)$, where Q is the mapping defined by*

$$Qx = P_C[P_C(x - \mu_2\Psi_2 x) - \mu_1\Psi_1 P_C(x - \mu_2\Psi_2 x)], \quad \forall x \in C. \tag{2.23}$$

3. Main Results

We are now in a position to prove the main result of this paper.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H such that $C \pm C \subset C$. Let $\phi, \varphi_i : C \to \mathbb{R}$ $(i = 1, 2, 3)$ be lower semicontinuous and convex functionals, $\Theta : H \times C \times C \to \mathbb{R}$ be an equilibrium-like function, $G : C \to H$ be a mapping, and $F_i : C \times C \to \mathbb{R}$ $(i = 1, 2, 3)$ be a bifunction satisfying conditions $(H1)$–$(H4)$. Assume that either condition $(A1)$ or $(A2)$ holds. Let $B : C \to H$ be β-inverse-strongly monotone, $\Psi_i, \Phi_i : C \to H$ $(i = 1, 2, 3)$ be $\widetilde{\beta}_i$-inverse-strongly monotone and $\widetilde{\gamma}_i$-inverse-strongly monotone, respectively. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a family of k-strict pseudocontraction mappings. Define a mapping $S_n x := \delta x + (1 - \delta)T_n x$, for all $x \in C$, $\delta \in [k, 1)$ and $n \geq 1$. Assume that the condition (Δ) is satisfied and $\Omega := \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{Fix}(Q) \cap \mathrm{GEP}(C, G, \Theta, \phi) \neq \emptyset$, where Q is defined as in Lemma 2.13. Let $\mu > 0$, $\gamma > 0$, and $r > 0$ be three constants. Let $f : C \to C$ be a contraction mapping with a coefficient $\alpha \in (0, 1)$ and let A be a strongly positive bounded linear operator on C with a coefficient $\overline{\gamma} \in (0, 1]$ such that $0 < \gamma < ((1 + \mu)\overline{\gamma})/\alpha$. For $x_1 \in C$, let the sequence $\{x_n\}$ defined by*

$$\begin{aligned} \Theta(Gx_n, u_n, y) + \phi(y) - \phi(u_n) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rBx_n)\rangle \geq 0, \quad \forall y \in C, \\ z_n = T_{\mu_3}^{(F_3,\varphi_3)}(x_n - \mu_3(\Psi_3 + \Phi_3)x_n), \\ y_n = T_{\mu_2}^{(F_2,\varphi_2)}(z_n - \mu_2(\Psi_2 + \Phi_2)z_n), \\ v_n = T_{\mu_1}^{(F_1,\varphi_1)}(y_n - \mu_1(\Psi_1 + \Phi_1)y_n), \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n(I + \mu A))[\gamma_1 S_n x_n + \gamma_2 u_n + \gamma_3 v_n], \quad \forall n \geq 1, \end{aligned} \tag{3.1}$$

where $\gamma_1, \gamma_2, \gamma_3 \in [0,1]$ *such that* $\gamma_1 + \gamma_2 + \gamma_3 = 1$, $\mu_1 \in (0, \min\{\widetilde{\beta}_1, \widetilde{\gamma}_1\})$, $\mu_2 \in (0, \min\{\widetilde{\beta}_2, \widetilde{\gamma}_2\})$, $\mu_3 \in (0, \min\{\widetilde{\beta}_3, \widetilde{\gamma}_3\})$, $r \in (0, 2\beta)$ *and* $\{\alpha_n\}, \{\beta_n\}$ *are two sequences in* $[0,1]$. *Suppose that* $\{T_n\}$ *satisfies the* PT-*condition. Let* $T : C \to C$ *be the mapping defined by* $Tx = \lim_{n\to\infty} T_n x$ *for all* $x \in C$ *and suppose that* $\mathrm{Fix}(T) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. *Assume the following conditions are satisfied:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$.

Then the sequence $\{x_n\}$ *defined by* (3.1) *converges strongly to* $\widehat{x} \in \Omega$, *where* $\widehat{x}$ *is the unique solution of the variational inequality*

$$\langle (\gamma f - (I + \mu A))\widehat{x}, v - \widehat{x} \rangle \le 0, \quad \forall v \in \Omega, \tag{3.2}$$

or equivalently, $\widehat{x} = P_\Omega(\gamma f - \mu A)\widehat{x}$, *where* P_Ω *is a metric projection mapping from* C *onto* Ω, *and* $(\widehat{x}, \widehat{y}, \widehat{z})$ *is a solution of the problem* (1.12), *where* $\widehat{y} = T_{\mu_2}^{(F_2,\varphi_2)}(\widehat{z} - \mu_2(\Psi_2 + \Phi_2)\widehat{z})$ *and* $\widehat{z} = T_{\mu_3}^{(F_3,\varphi_3)}(\widehat{x} - \mu_3(\Psi_3 + \Phi_3)\widehat{x})$.

Proof. Note that from the conditions $(C1)$ and $(C2)$, we may assume, without loss of generality, that $\alpha_n \le (1-\beta_n)(1+\mu\|A\|)^{-1}$ for all $n \in \mathbb{N}$. Since A is a linear bounded self-adjoint operator on C, by (2.2), we have

$$\|A\| = \sup\{|\langle Au, u\rangle| : u \in C, \|u\| = 1\}. \tag{3.3}$$

Observe that

$$\begin{aligned} \langle ((1-\beta_n)I - \alpha_n(I + \mu A))u, u \rangle &= 1 - \beta_n - \alpha_n - \alpha_n\mu\langle Au, u\rangle \\ &\ge 1 - \beta_n - \alpha_n - \alpha_n\mu\|A\| \\ &\ge 0. \end{aligned} \tag{3.4}$$

This show that $(1-\beta_n)I - \alpha_n(I + \mu A)$ is positive. It follows that

$$\begin{aligned} \|(1-\beta_n)I - \alpha_n(I+\mu A)\| &= \sup\{|\langle ((1-\beta_n)I - \alpha_n(I+\mu A))u, u\rangle| : u \in C, \|u\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n - \alpha_n\mu\langle Au, u\rangle : u \in C, \|u\| = 1\} \\ &\le 1 - \beta_n - \alpha_n(1 + \mu\overline{\gamma}) \\ &\le 1 - \beta_n - \alpha_n(1+\mu)\overline{\gamma}. \end{aligned} \tag{3.5}$$

First, we show that $\{x_n\}$ is bounded. Taking $x^* \in \Omega$, it follows from Lemma 2.13 that

$$\begin{aligned} x^* = T_{\mu_1}^{(F_1,\varphi_1)}\Big[& T_{\mu_2}^{(F_2,\varphi_2)}\Big[T_{\mu_3}^{(F_3,\varphi_3)}(x^* - \mu_3(\Psi_3+\Phi_3)x^*) - \mu_2(\Psi_2+\Phi_2)T_{\mu_3}^{(F_3,\varphi_3)}(x^* - \mu_3(\Psi_3+\Phi_3)x^*)\Big] \\ & - \mu_1(\Psi_1+\Phi_1)T_{\mu_2}^{(F_2,\varphi_2)}\Big[T_{\mu_3}^{(F_3,\varphi_3)}(x^* - \mu_3(\Psi_3+\Phi_3)x^*) \\ & -\mu_2(\Psi_2+\Phi_2)T_{\mu_3}^{(F_3,\varphi_3)}(x^* - \mu_3(\Psi_3+\Phi_3)x^*)\Big]\Big]. \end{aligned} \tag{3.6}$$

Putting $y^* = T_{\mu_2}^{(F_2,\varphi_2)}(z^* - \mu_2(\Psi_2 + \Phi_2)z^*)$ and $z^* = T_{\mu_3}^{(F_3,\varphi_3)}(x^* - \mu_3(\Psi_3 + \Phi_3)x^*)$, we obtain $x^* = T_{\mu_1}^{(F_1,\varphi_1)}(y^* - \mu_1(\Psi_1 + \Phi_1)y^*)$. Notice that $u_n = T^{(r)}(I - rB)x_n$. Since $T^{(r)}$ is nonexpansive and B is β-inverse-strongly monotone, we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \left\|T^{(r)}(I - rB)x_n - T^{(r)}(I - rB)x^*\right\|^2 \\
&\le \|(I - rB)x_n - (I - rB)x^*\|^2 \\
&= \|(x_n - x^*) - r(Bx_n - Bx^*)\|^2 \\
&= \|x_n - x^*\|^2 - 2r\langle x_n - x^*, Bx_n - Bx^*\rangle + r^2\|Bx_n - Bx^*\|^2 \\
&\le \|x_n - x^*\|^2 + r(r - 2\beta)\|Bx_n - Bx^*\|^2 \\
&\le \|x_n - x^*\|^2,
\end{aligned} \tag{3.7}$$

and hence

$$\|u_n - x^*\| \le \|x_n - x^*\|. \tag{3.8}$$

We observe that

$$\begin{aligned}
\|v_n - x^*\| &= \|Qx_n - Qx^*\| \\
&\le \|x_n - x^*\|.
\end{aligned} \tag{3.9}$$

Setting $\theta_n := \gamma_1 S_n x_n + \gamma_2 u_n + \gamma_3 v_n$. By Lemma 2.8, we have S_n is a nonexpansive mapping such that $\mathrm{Fix}(S_n) = \mathrm{Fix}(T_n)$ for all $n \ge 1$. Then, we have

$$\begin{aligned}
\|\theta_n - x^*\| &= \left\|\gamma_1(S_n x_n - x^*) + \gamma_2(u_n - x^*) + \gamma_3(v_n - x^*)\right\| \\
&\le \gamma_1\|S_n x_n - x^*\| + \gamma_2\|u_n - x^*\| + \gamma_3\|v_n - x^*\| \\
&\le \gamma_1\|x_n - x^*\| + \gamma_2\|x_n - x^*\| + \gamma_3\|x_n - x^*\| \\
&= \|x_n - x^*\|.
\end{aligned} \tag{3.10}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \left\|\alpha_n(\gamma f(x_n) - (I + \mu A)x^*) + \beta_n(x_n - x^*) + ((1 - \beta_n)I - \alpha_n(I + \mu A))(\theta_n - x^*)\right\| \\
&\le \alpha_n\left\|\gamma f(x_n) - \alpha_n(I + \mu A)x^*\right\| + \beta_n\|x_n - x^*\| + (1 - \beta_n - \alpha_n(1 + \mu\overline{\gamma}))\|\theta_n - x^*\| \\
&\le \alpha_n\gamma\left\|f(x_n) - f(x^*)\right\| + \alpha_n\left\|\gamma f(x^*) - (I + \mu A)x^*\right\| + (1 - \alpha_n(1 + \mu\overline{\gamma}))\|x_n - x^*\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\gamma\alpha\|x_n - x^*\| + \alpha_n\left\|\gamma f(x^*) - (I+\mu A)x^*\right\| + (1-\alpha_n(1+\mu\overline{\gamma}))\|x_n - x^*\| \\
&\leq (1-((1+\mu)\overline{\gamma}-\gamma\alpha)\alpha_n)\|x_n - x^*\| + \alpha_n\left\|\gamma f(x^*) - (I+\mu A)x^*\right\| \\
&= (1-((1+\mu)\overline{\gamma}-\gamma\alpha)\alpha_n)\|x_n - x^*\| + ((1+\mu)\overline{\gamma}-\gamma\alpha)\alpha_n\frac{\left\|\gamma f(x^*) - (I+\mu A)x^*\right\|}{(1+\mu)\overline{\gamma}-\gamma\alpha}.
\end{aligned}
\tag{3.11}
$$

By induction, we have

$$
\|x_n - x^*\| \leq \max\left\{\|x_1 - x^*\|, \frac{\left\|\gamma f(x^*) - (I+\mu A)x^*\right\|}{(1+\mu)\overline{\gamma}-\gamma\alpha}\right\}, \quad \forall n \geq 1. \tag{3.12}
$$

Hence, $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{v_n\}$, $\{y_n\}$, and $\{z_n\}$. From definition of S_n and for all $k, l \in \mathbb{N}$, it follows that

$$
\sup_{\omega\in C}\|S_k\omega - S_l\omega\| = \gamma_1\sup_{\omega\in C}\|T_k\omega - T_l\omega\|. \tag{3.13}
$$

By our assumption, $\{T_n\}$ satisfies the PT-condition, we obtain that

$$
\lim_{n\to\infty}\sup_{\omega\in C}\|S_k\omega - S_l\omega\| = 0, \tag{3.14}
$$

that is $\{S_n\}$ satisfies the PT-condition.

Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since $u_n = T^{(r)}(I - rB)x_n$, we have

$$
\begin{aligned}
\|u_{n+1} - u_n\| &= \left\|T^{(r)}(x_{n+1} - rBx_{n+1}) - T^{(r)}(x_n - rBx_n)\right\| \\
&\leq \|(x_{n+1} - rBx_{n+1}) - (x_n - rBx_n)\| \\
&\leq \|x_{n+1} - x_n\|, \\
\|v_{n+1} - v_n\| &= \|Qx_{n+1} - Qx_n\| \\
&\leq \|x_{n+1} - x_n\|.
\end{aligned}
\tag{3.15}
$$

It follows from (3.15) that

$$
\begin{aligned}
\|\theta_{n+1} - \theta_n\| &= \left\|(\gamma_1 S_{n+1}x_{n+1} + \gamma_2 u_{n+1} + \gamma_3 v_{n+1}) - (\gamma_1 S_n x_n + \gamma_2 u_n + \gamma_3 v_n)\right\| \\
&= \left\|\gamma_1(S_{n+1}x_{n+1} - S_n x_n) + \gamma(u_{n+1} - u_n) + \gamma_3(v_{n+1} - v_n)\right\| \\
&\leq \gamma_1\|S_{n+1}x_{n+1} - S_n x_n\| + \gamma_2\|u_{n+1} - u_n\| + \gamma_3\|v_{n+1} - v_n\| \\
&\leq \gamma_1\|S_{n+1}x_{n+1} - S_{n+1}x_n\| + \gamma_1\|S_{n+1}x_n - S_n x_n\| + \gamma_2\|u_{n+1} - u_n\| + \gamma_3\|v_{n+1} - v_n\| \\
&\leq \gamma_1\|x_{n+1} - x_n\| + \gamma_2\|x_{n+1} - x_n\| + \gamma_3\|x_{n+1} - x_n\| + \gamma_1\|S_{n+1}x_n - S_n x_n\| \\
&= \|x_{n+1} - x_n\| + \gamma_1\|S_{n+1}x_n - S_n x_n\|.
\end{aligned}
\tag{3.16}
$$

Let $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$ for all $n \geq 1$. Then, we have

$$\begin{aligned}
l_{n+1} - l_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + \left((1-\beta_{n+1})I - \alpha_{n+1}(I+\mu A)\right)\theta_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n \gamma f(x_n) + \left((1-\beta_n)I - \alpha_n(I+\mu A)\right)\theta_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1}) + \frac{(1-\beta_{n+1})I - \alpha_{n+1}(I+\mu A)\theta_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n}{1-\beta_n}\gamma f(x_n) - \frac{(1-\beta_n)I - \alpha_n(I+\mu A)\theta_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\gamma f(x_{n+1}) - (I+\mu A)\theta_{n+1}\right) - \frac{\alpha_n}{1-\beta_n}\left(\gamma f(x_n) - (I+\mu A)\theta_n\right) + \theta_{n+1} - \theta_n.
\end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17), we have

$$\begin{aligned}
\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \leq{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left\|\gamma f(x_{n+1}) - (I+\mu A)\theta_{n+1}\right\| + \frac{\alpha_n}{1-\beta_n}\left\|\gamma f(x_n) - (I+\mu A)\theta_n\right\| \\
&+ \gamma_1\|S_{n+1}x_n - S_n x_n\|.
\end{aligned} \tag{3.18}$$

Since $\{S_n\}$ satisfies the PT-condition, we can define a mapping $S : C \to C$ by

$$Sx = \lim_{n\to\infty} S_n x = \lim_{n\to\infty}[\delta x + (1-\delta)T_n x] = \delta x + (1-\delta)Tx, \quad \forall x \in C. \tag{3.19}$$

We observe that

$$\begin{aligned}
\|S_{n+1}x_n - S_n x_n\| &\leq \|S_{n+1}x_n - Sx_n\| + \|Sx_n - S_n x_n\| \\
&\leq \sup_{\omega\in C}\|S_{n+1}\omega - S\omega\| + \sup_{\omega\in C}\|S\omega - S_n\omega\|.
\end{aligned} \tag{3.20}$$

Be Lemma 2.6, we have

$$\lim_{n\to\infty}\|S_{n+1}x_n - S_n x_n\| = 0. \tag{3.21}$$

Consequently, it follows from the conditions $(C1)$, $(C2)$, and (3.18) that

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{3.22}$$

Hence, by Lemma 2.3, we obtain that

$$\lim_{n\to\infty}\|l_n - x_n\| = 0. \tag{3.23}$$

Consequently, we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|l_n - x_n\| = 0. \tag{3.24}$$

On the other hand, we observe that

$$x_{n+1} - x_n = \alpha_n(\gamma f(x_n) - (I+\mu A)x_n) + ((1-\beta_n)I - \alpha_n(I+\mu A))(\theta_n - x_n). \tag{3.25}$$

It follows that

$$(1-\beta_n - \alpha_n(1+\mu)\overline{\gamma})\|\theta_n - x_n\| \le \|x_{n+1} - x_n\| + \alpha_n\left\|\gamma f(x_n) - (I+\mu A)x_n\right\|. \tag{3.26}$$

From the conditions $(C1)$, $(C2)$, and (3.24), we obtain that

$$\lim_{n\to\infty}\|\theta_n - x_n\| = 0. \tag{3.27}$$

Next, we show that

$$\limsup_{n\to\infty}\langle\gamma f(\hat{x}) - (I+\mu A)\hat{x}, x_n - \hat{x}\rangle \le 0. \tag{3.28}$$

To show this, we take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle\gamma f(\hat{x}) - (I+\mu A)\hat{x}, x_n - \hat{x}\rangle = \lim_{j\to\infty}\left\langle\gamma f(\hat{x}) - (I+\mu A)\hat{x}, x_{n_j} - \hat{x}\right\rangle. \tag{3.29}$$

Since $\{x_{n_j}\}$ is bounded, without loss of generality, we can assume that $x_{n_j} \rightharpoonup v \in C$. So, we get

$$\begin{aligned}\limsup_{n\to\infty}\langle\gamma f(\hat{x}) - (I+\mu A)\hat{x}, x_n - \hat{x}\rangle &= \lim_{j\to\infty}\left\langle\gamma f(\hat{x}) - (I+\mu A)\hat{x}, x_{n_j} - \hat{x}\right\rangle \\ &= \langle\gamma f(\hat{x}) - (I+\mu A)\hat{x}, v - \hat{x}\rangle \\ &\le 0.\end{aligned} \tag{3.30}$$

Next, we show that $v \in \Omega := \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n) \cap \mathrm{Fix}(Q) \cap \mathrm{GEP}(C, G, \Theta, \phi)$. Define a mapping $K_n : C \to C$ by

$$K_n x = \gamma_1 S_n x + \gamma_2 Q x + \gamma_3 T^{(r)}(I - rB)x, \quad \forall x \in C. \tag{3.31}$$

From (3.27), we have

$$\lim_{n\to\infty}\|K_n x_n - K x_n\| = 0. \tag{3.32}$$

For all $k, l \in \mathbb{N}$, it follows that

$$\sup_{\omega\in C}\|K_k\omega - K_l\omega\| = \gamma_1 \sup_{\omega\in C}\|S_k\omega - S_l\omega\|. \tag{3.33}$$

Since $\{S_n\}$ satisfies the PT-condition, we obtain that

$$\lim_{k,l\to\infty}\sup_{\omega\in C}\|K_k\omega - K_l\omega\| = 0, \tag{3.34}$$

that is $\{K_n\}$ satisfies the PT-condition. Define a mapping $K : C \to C$ by

$$\begin{aligned} Kx &= \lim_{n\to\infty} K_n x = \lim_{n\to\infty}\left[\gamma_1 S_n x + \gamma_2 Q x + \gamma_3 T^{(r)}(I - rB)x\right] \\ &= \gamma_1 S x + \gamma_2 Q x + \gamma_3 T^{(r)}(I - rB)x, \quad \forall x \in C. \end{aligned} \tag{3.35}$$

By Lemma 2.6, we obtain that

$$\lim_{n\to\infty}\sup_{\omega\in C}\|K_n\omega - K\omega\| = 0. \tag{3.36}$$

From Lemma 2.7, we see that K is nonexpansive and

$$\begin{aligned} \operatorname{Fix}(K) &= \operatorname{Fix}(S) \cap \operatorname{Fix}(Q) \cap \operatorname{Fix}\left(T^{(r)}(I - rB)\right) \\ &= \bigcap_{n=1}^{\infty} \operatorname{Fix}(S_n) \cap \operatorname{Fix}(Q) \cap \operatorname{Fix}\left(T^{(r)}(I - rB)\right) \\ &= \bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n) \cap \operatorname{Fix}(Q) \cap \operatorname{GEP}(C, G, \Theta, \varphi) \\ &= \bigcap_{n=1}^{\infty} \operatorname{Fix}(K_n). \end{aligned} \tag{3.37}$$

Notice that

$$\begin{aligned} \|x_n - Kx_n\| &\le \|x_n - K_n x_n\| + \|K_n x_n - K x_n\| \\ &\le \|x_n - K_n x_n\| + \sup_{\omega\in C}\|K_n\omega - K\omega\|. \end{aligned} \tag{3.38}$$

From (3.32) and (3.39), we obtain that

$$\lim_{n\to\infty}\|x_n - Kx_n\| = 0. \tag{3.39}$$

Thus, by Lemma 2.9, we obtain that $v \in \mathrm{Fix}(K) = \Omega$.

Finally, we show that $x_n \to \widehat{x}$ as $n \to \infty$. From Lemma 2.4, we compute

$$\begin{aligned}
\|x_{n+1} - \widehat{x}\|^2 &= \left\|\alpha_n(\gamma f(x_n) - (I+\mu A)\widehat{x}) + \beta_n(x_n - \widehat{x}) + ((1-\beta_n)I - \alpha_n(I+\mu A))(\theta_n - \widehat{x})\right\|^2 \\
&\le \left\|((1-\beta_n)I - \alpha_n(I+\mu A))(\theta_n - \widehat{x}) + \beta_n(x_n - \widehat{x})\right\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - (I+\mu A)\widehat{x}, x_{n+1} - \widehat{x}\rangle \\
&\le \left[(1-\beta_n - \alpha_n(1+\mu\overline{\gamma}))\|\theta_n - \widehat{x}\| + \beta_n\|x_n - \widehat{x}\|\right]^2 \\
&\quad + 2\alpha_n\gamma\langle f(x_n) - f(\widehat{x}), x_{n+1} - \widehat{x}\rangle + 2\alpha_n\langle \gamma f(\widehat{x}) - (I+\mu A)\widehat{x}, x_{n+1} - \widehat{x}\rangle \\
&\le \left[(1-\beta_n - \alpha_n(1+\mu)\overline{\gamma})\|x_n - \widehat{x}\| + \beta_n\|x_n - \widehat{x}\|\right]^2 + 2\alpha_n\gamma\alpha\|x_n - \widehat{x}\|\|x_{n+1} - \widehat{x}\| \\
&\quad + 2\alpha_n\langle \gamma f(\widehat{x}) - (I+\mu A)\widehat{x}, x_{n+1} - \widehat{x}\rangle \\
&\le \left[(1-\beta_n - \alpha_n(1+\mu)\overline{\gamma})\|x_n - \widehat{x}\| + \beta_n\|x_n - \widehat{x}\|\right]^2 + 2\alpha_n\gamma\alpha\|x_n - \widehat{x}\|\|x_{n+1} - \widehat{x}\| \\
&\quad + 2\alpha_n\langle \gamma f(\widehat{x}) - (I+\mu A)\widehat{x}, x_{n+1} - \widehat{x}\rangle \\
&\le (1-\alpha_n(1+\mu)\overline{\gamma})^2\|x_n - \widehat{x}\|^2 + \alpha_n\gamma\alpha\left(\|x_n - \widehat{x}\|^2 + \|x_{n+1} - \widehat{x}\|^2\right) \\
&\quad + 2\alpha_n\langle \gamma f(\widehat{x}) - (I+\mu A)\widehat{x}, x_{n+1} - \widehat{x}\rangle \\
&= \left(1 - 2\alpha_n(1+\mu)\overline{\gamma} + \alpha_n^2\left[(1+\mu)\overline{\gamma}\right]^2 + \alpha_n\gamma\alpha\right)\|x_n - \widehat{x}\|^2 + \alpha_n\gamma\alpha\|x_{n+1} - \widehat{x}\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(\widehat{x}) - (I+\mu A)\widehat{x}, x_{n+1} - \widehat{x}\rangle.
\end{aligned} \tag{3.40}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - \widehat{x}\|^2 &\le \frac{1 - 2\alpha_n(1+\mu)\overline{\gamma} + \alpha_n^2\left[(1+\mu)\overline{\gamma}\right]^2 + \alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n - \widehat{x}\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle \gamma f(\widehat{x}) - (I+\mu A)\widehat{x}, x_{n+1} - \widehat{x}\rangle \\
&= \frac{1 - 2\alpha_n(1+\mu)\overline{\gamma} + \alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n - \widehat{x}\|^2 + \frac{\alpha_n^2\left[(1+\mu)\overline{\gamma}\right]^2}{1-\alpha_n\gamma\alpha}\|x_n - \widehat{x}\|^2 \\
&\quad + \frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle \gamma f(\widehat{x}) - (I+\mu A)\widehat{x}, x_{n+1} - \widehat{x}\rangle
\end{aligned}$$

$$\leq \left[1-\frac{2((1+\mu)\overline{\gamma}-\gamma\alpha)\alpha_n}{1-\alpha_n\gamma\alpha}\right]\|x_n-\widehat{x}\|^2+\frac{\alpha_n^2\left[(1+\mu)\overline{\gamma}\right]^2}{1-\alpha_n\gamma\alpha}M$$
$$+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(\widehat{x})-(I+\mu A)\widehat{x},x_{n+1}-\widehat{x}\rangle, \tag{3.41}$$

where $M=\sup_{n\geq 1}\{\|x_n-\widehat{x}\|^2\}$. Put $\sigma_n=(2((1+\mu)\overline{\gamma}-\gamma\alpha)\alpha_n)/(1-\alpha_n\gamma\alpha)$ and

$$\delta_n=\frac{\alpha_n^2\left[(1+\mu)\overline{\gamma}\right]^2}{1-\alpha_n\gamma\alpha}M+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(\widehat{x})-(I+\mu A)\widehat{x},x_{n+1}-\widehat{x}\rangle. \tag{3.42}$$

Then, the (3.41) reduces to the formula

$$\|x_{n+1}-\widehat{x}\|^2\leq(1-\sigma_n)\|x_n-\widehat{x}\|^2+\delta_n. \tag{3.43}$$

It is easily seen that $\sum_{n=1}^{\infty}\sigma_n=\infty$ and (using (3.30)), we get

$$\limsup_{n\to\infty}\frac{\delta_n}{\sigma_n}=\limsup_{n\to\infty}\frac{1}{2((1+\mu)\overline{\gamma}-\gamma\alpha)}\Big[\alpha_n\left[(1+\mu)\overline{\gamma}\right]^2M$$
$$+2\langle\gamma f(\widehat{x})-(I+\mu A)\widehat{x},x_{n+1}-\widehat{x}\rangle\Big]\leq 0. \tag{3.44}$$

Hence, by Lemma 2.10, we conclude that $x_n\to\widehat{x}$ as $n\to\infty$. This completes the proof. □

Remark 3.2. Theorem 3.1 improves and generalizes [1, Theorem 2.1] in the following ways.

(i) From a one nonexpansive mapping to a countable family of strict pseudocontraction mappings.

(ii) From a general system of variational inequalities to a general system of generalized nonlinear mixed composite-type equilibria.

(iii) Theorem 3.1 for finding an element $\widehat{x}\in\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)\cap\mathrm{Fix}(Q)\cap\mathrm{GEP}(C,G,\Theta,\Phi)$ (Q is defined as in Lemma 2.13) is more general the one of finding elements of $\mathrm{Fix}(S)\cap\mathrm{Fix}(D)\cap\mathrm{GEP}(C,G,\Theta,\Phi)$ (D is defined as in Lemma 2.17) in [1, Theorem 2.1].

Furthermore, our method of the proof is very different from that in [1, Theorem 2.1] because (3.1) involves the countable family of strict pseudocontraction mappings and strongly positive bounded linear operator.

4. Application to Minimization Problems

Let C be a nonempty closed and convex subset of a real Hilbert space H and $A : H \to H$ be a strongly positive linear bounded operator with a constant $\overline{\gamma} > 0$. In this section, we will utilize the results presented in Section 3 to study the following minimization problem:

$$\min_{x\in\Omega}\frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x\|^2 - h(x), \tag{4.1}$$

where Ω is a nonempty closed and convex subsets of C, $\mu \geq 0$ is some constant and $h : C \to \mathbb{R}$ is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for all $x \in C$), where $f : C \to C$ is a contraction mapping with a constant $\alpha \in (0,1)$. Note that this kind of minimization problems has been studied extensively by many authors (e.g., see [18, 36–39]). We can apply Theorem 3.1 to solve the above minimization problem in the framework of Hilbert spaces as follows.

Theorem 4.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H such that $C \pm C \subset C$. Let $S : C \to C$ be a nonexpansive mappings such that $\mathrm{Fix}(S) \neq \emptyset$. Let $\mu > 0$ and $\gamma > 0$ be two constants. Let $f : C \to C$ be a contraction mapping with a coefficient $\alpha \in (0,1)$ and A be a strongly positive bounded linear operator on C with a coefficient $\overline{\gamma} \in (0,1)$ such that $0 < \gamma < ((1+\mu)\overline{\gamma})/\alpha$. Let $\{x_n\}$ be a sequence defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \alpha_n(I+\mu A)\right)Sx_n, \quad \forall n \geq 1, \tag{4.2}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $[0,1]$. Assume the following conditions are satisfied:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Suppose that $\mathrm{Fix}(S)$ is a compact subset of C. Then the sequence $\{x_n\}$ defined by (4.2) converges strongly to $\widehat{x} \in \mathrm{Fix}(S)$ which solves the minimization problem (4.1).

Proof. Taking $\gamma_1 = 1$ and $\gamma_2 = \gamma_3 = 0$ in Theorem 3.1. Hence, from Theorem 3.1, we know that the sequence $\{x_n\}$ defined by (4.2) converges strongly to $\widehat{x} \in \mathrm{Fix}(S)$, where $\widehat{x}$ is the unique solution of the variational inequality

$$\left\langle \left(\gamma f - (I+\mu A)\right)\widehat{x}, v - \widehat{x}\right\rangle \leq 0, \quad \forall v \in \mathrm{Fix}(S). \tag{4.3}$$

Since S is nonexpansive, then $\mathrm{Fix}(S)$ is convex. Again by the assumption that $\mathrm{Fix}(S)$ is compact, then it is a compact and convex subset of C, and

$$\frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x\|^2 - h(x) : C \longrightarrow \mathbb{R} \tag{4.4}$$

is a continuous mapping. By virtue of the well-known Weierstrass's theorem, there exists a point $\widetilde{x} \in \mathrm{Fix}(S)$ which is a minimal point of minimization problem (4.1). As is known to all, (4.3) is the optimality necessary condition [18] for the minimization problem (4.1). Therefore we also have

$$\left\langle \left(\gamma f - (I+\mu A)\right)\widetilde{x}, v - \widetilde{x}\right\rangle \leq 0, \quad \forall v \in \mathrm{Fix}(S). \tag{4.5}$$

Since $\widehat{x}$ is the unique solution of (4.3), we have $\widehat{x} = \widetilde{x}$. This completes the proof. □

5. A Numerical Example

In this section, we give a real example in which the conditions satisfy the ones of Theorem 3.1 and some numerical experiment results to explain the main result Theorem 3.1 as follows.

Example 5.1. Let $H = \mathbb{R}$, $C = [0,1]$, $\Theta(x,y,z) = \phi(x) = Bx = 0$, $\forall x,y,z \in C$, $r = 1$, $F_i(x,y) = \varphi_i(x) = \Psi_i x = \Phi_i x = 0$, $\forall x,y \in C$, $\mu_i = 1$ $(i = 1,2,3)$, $A = I$, $f(x) = (1/2)x$, and $\forall x \in C$, with a constant $\alpha = 1/2$, $\alpha_n = 1/n$, $\beta_n = (n+1)/2n$, $\forall n \in \mathbb{N}$, $\gamma_i = (1/3)$ $(i = 1,2,3)$, $\gamma = 1$, and $\mu = 1$. For all $n \in \mathbb{N}$, let $T_n : C \to C$ define by $T_n x = nx^2/(2n+1)$, $\forall x \in C$, we see that, $\{T_n\}$ is a family of 0-strictly pseudocontractive with $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) = \{0\}$. Then, $\{x_n\}$ is the sequence defined by

$$x_{n+1} = \left(\frac{5}{6} - \frac{2}{3n}\right)x_n + \frac{1}{6}\left(\frac{n-5}{2n+1}\right)x_n^2, \tag{5.1}$$

and $x_n \to 0$ as $n \to \infty$, where 0 is the unique solution of the minimization problem

$$\min_{x\in C} \frac{3}{4}x^2 + q. \tag{5.2}$$

Proof. Step 1. We show that

$$T^{(r)}(x) = P_C x, \quad \forall x \in H, \tag{5.3}$$

where

$$P_C x = \begin{cases} \dfrac{x}{|x|}, & x \in H \setminus C, \\ x, & x \in C. \end{cases} \tag{5.4}$$

Since $\Theta(x,y,z) = \phi(x) = Bx = 0$, due to the definition of $T^{(r)}(x)$, $\forall x \in H$ in (2.7), we have

$$T^{(r)}(x) = \left\{y \in C : \left\langle y - z, z - x\right\rangle \geq 0,\ \forall z \in C\right\}. \tag{5.5}$$

Also by the equivalent property (2.5) of the nearest projection P_C from H to C, we obtain this conclusion. When we take $x \in C$, then $T^{(r)}(x) = P_C x = x$. By the condition $(\Delta)(c)$, we have $\mathrm{GEP}(C,G,\Theta,\phi) = C$. In a similar way, for all $i = 1,2,3$, we can get

$$T_{\mu_i}^{(F_i,\varphi_i)}(x) = P_C x = x, \quad \forall x \in C, \tag{5.6}$$

and $\mathrm{Fix}(Q) = C$. Hence

$$\mathrm{GEP}\left(C,G,\Theta,\phi\right) \cap \mathrm{Fix}(Q) = C. \tag{5.7}$$

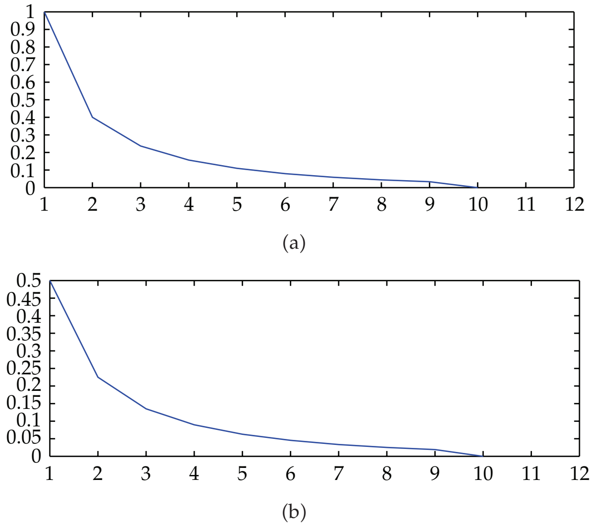

Figure 1: These figures show the iteration comparison chart of different initial values (a) $x_1 = 1$ and (b) $x_1 = 0.5$, respectively.

Step 2. We show that $\{T_n\}$ is satisfies the PT-condition. Since $T_n x = (nx^2/(2n+1))$, $\forall x \in C$, and $n \in \mathbb{N}$. For all $k, l \in \mathbb{N}$, we have

$$\begin{aligned}\lim_{k,l\to\infty}\left|\frac{kx^2}{2k+1} - \frac{lx^2}{2l+1}\right| &\le \lim_{k,l\to\infty}\sup_{\omega\in C}\left|\frac{k\omega^2}{2k+1} - \frac{l\omega^2}{2l+1}\right| \\ &= \lim_{k,l\to\infty}\left|\frac{k}{2k+1} - \frac{l}{2l+1}\right|\sup_{\omega\in C}\left\{\omega^2\right\} \\ &= 0,\end{aligned} \tag{5.8}$$

that is $\{T_n\}$ satisfies the PT-condition. □

Step 3. We show that

$$x_{n+1} = \left(\frac{5}{6} - \frac{2}{3n}\right)x_n + \frac{1}{6}\left(\frac{n-5}{2n+1}\right)x_n^2, \quad x_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{5.9}$$

where 0 is the unique solution of the minimization problem:

$$\min_{x\in C}\frac{3}{4}x^2 + q. \tag{5.10}$$

Due to (5.3) and (5.4), we can obtain a special sequence $\{x_n\}$ of (3.1) in Theorem 3.1 as follows:

$$x_{n+1} = \left(\frac{5}{6} - \frac{2}{3n}\right)x_n + \frac{1}{6}\left(\frac{n-5}{2n+1}\right)x_n^2. \tag{5.11}$$

Table 1: This table shows the value of sequence $\{x_n\}$ on each iteration step (initial value $x_1 = 1$).

n	x_n
1	1.000000000000000
2	0.400000000000000
3	0.236825396825397
4	0.156844963280237
5	0.109791474296166
6	0.079448383160412
7	0.058780740964773
8	0.044187178532649
9	0.033618033450111
10	0.000000000000000

Table 2: This table shows the value of sequence $\{x_n\}$ on each iteration step (initial value $x_1 = 0.5$).

n	x_n
1	0.500000000000000
2	0.225000000000000
3	0.135089285714286
4	0.089721577233324
5	0.062805104063327
6	0.045409812093980
7	0.033562589425687
8	0.025205072875411
9	0.019159476038106
10	0.000000000000000

Since $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) = \{0\}$, combining with (5.7), we have

$$\Omega := \mathrm{GEP}(C, G, \Theta, \phi) \cap \mathrm{Fix}(Q) \cap \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) = \{0\}. \tag{5.12}$$

By Lemma 2.10, we obtain that $x_n \to 0$, where 0 is the unique solution of the minimization problem $\min_{x \in C} (3/4)x^2 + q$, where q is a constant number.

5.1. Numerical Experiment Results

Next, we show the numerical experiment results using software MATLAB 7.0 and we obtain the results shown in Tables 1 and 2 and Figure 1, which show that the iteration process of the sequence $\{x_n\}$ as initial points $x_1 = 1$ and $x_1 = 0.5$, respectively.

Acknowledgment

The authors were supported by the Higher Education Research Promotion and the National Research University Project of Thailand, Office of the Higher Education Commission (NRU-CSEC no. 55000613).

References

[1] Y. J. Cho, I. K. Argyros, and N. Petrot, "Approximation methods for common solutions of generalized equilibrium, systems of nonlinear variational inequalities and fixed point problems," *Computers & Mathematics with Applications*, vol. 60, no. 8, pp. 2292–2301, 2010.

[2] L.-C. Ceng and J.-C. Yao, "A hybrid iterative scheme for mixed equilibrium problems and fixed point problems," *Journal of Computational and Applied Mathematics*, vol. 214, no. 1, pp. 186–201, 2008.

[3] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.

[4] M. A. Noor and W. Oettli, "On general nonlinear complementarity problems and quasi-equilibria," *Le Matematiche*, vol. 49, no. 2, pp. 313–331, 1994.

[5] S.-S. Chang, H. W. Joseph Lee, and C. K. Chan, "A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 9, pp. 3307–3319, 2009.

[6] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, "Viscosity approximation methods for generalized equilibrium problems and fixed point problems," *Journal of Global Optimization*, vol. 43, no. 4, pp. 487–502, 2009.

[7] X. Qin, Y. J. Cho, and S. M. Kang, "Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces," *Journal of Computational and Applied Mathematics*, vol. 225, no. 1, pp. 20–30, 2009.

[8] X. Qin, M. Shang, and Y. Su, "Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems," *Mathematical and Computer Modelling*, vol. 48, no. 7-8, pp. 1033–1046, 2008.

[9] A. Tada and W. Takahashi, "Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem," *Journal of Optimization Theory and Applications*, vol. 133, no. 3, pp. 359–370, 2007.

[10] S. Takahashi and W. Takahashi, "Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 3, pp. 1025–1033, 2008.

[11] X. Qin, S. Y. Cho, and S. M. Kang, "Iterative algorithms for variational inequality and equilibrium problems with applications," *Journal of Global Optimization*, vol. 48, no. 3, pp. 423–445, 2010.

[12] X. Qin, S.-S. Chang, and Y. J. Cho, "Iterative methods for generalized equilibrium problems and fixed point problems with applications," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 4, pp. 2963–2972, 2010.

[13] X. Qin, Y. J. Cho, and S. M. Kang, "Viscosity approximation methods for generalized equilibrium problems and fixed point problems with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 1, pp. 99–112, 2010.

[14] X. Qin, S. Y. Cho, and S. M. Kang, "An extragradient-type method for generalized equilibrium problems involving strictly pseudocontractive mappings," *Journal of Global Optimization*, vol. 49, no. 4, pp. 679–693, 2011.

[15] P. Sunthrayuth and P. Kumam, "A system of generalized mixed equilibrium problems, maximal monotone operators, and fixed point problems with application to optimization problems," *Abstract and Applied Analysis*, vol. 2012, Article ID 316276, 39 pages, 2012.

[16] P. Sunthrayuth and P. Kumam, "An iterative method for solving a system of mixed equilibrium problems, system of quasivariational inclusions, and fixed point problems of nonexpansive semigroups with application to optimization problems," *Abstract and Applied Analysis*, vol. 2012, Article ID 979870, 30 pages, 2012.

[17] Y. Yao, M. A. Noor, S. Zainab, and Y.-C. Liou, "Mixed equilibrium problems and optimization problems," *Journal of Mathematical Analysis and Applications*, vol. 354, no. 1, pp. 319–329, 2009.

[18] Y. Yao, S. K. Kang, and Y.-C. Liou, "Algorithms for approximating minimization problems in Hilbert spaces," *Journal of Computational and Applied Mathematics*, vol. 235, no. 12, pp. 3515–3526, 2011.

[19] H. Zhou, "Convergence theorems of fixed points for κ-strict pseudo-contractions in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 2, pp. 456–462, 2008.

[20] L.-C. Ceng, S.-M. Guu, and J.-C. Yao, "Iterative algorithms for a general system of generalized nonlinear mixed composite-type equilibria," *Computers & Mathematics with Applications*, vol. 60, no. 3, pp. 890–905, 2010.

[21] L.-C. Ceng and J.-C. Yao, "A relaxed extragradient-like method for a generalized mixed equilibrium problem, a general system of generalized equilibria and a fixed point problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 3-4, pp. 1922–1937, 2010.

[22] H.-Y. Hu and L.-C. Ceng, "A general system of generalized nonlinear mixed composite-type equilibria in Hilbert spaces," *Taiwanese Journal of Mathematics*, vol. 15, no. 3, pp. 927–959, 2011.

[23] W. Kumam, P. Junlouchai, and P. Kumam, "Generalized systems of variational inequalities and projection methods for inverse-strongly monotone mappings," *Discrete Dynamics in Nature and Society*, vol. 2011, Article ID 976505, 23 pages, 2011.

[24] L.-C. Ceng, C.-Y. Wang, and J.-C. Yao, "Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities," *Mathematical Methods of Operations Research*, vol. 67, no. 3, pp. 375–390, 2008.

[25] J.-W. Peng and J.-C. Yao, "A new hybrid-extragradient method for generalized mixed equilibrium problems, fixed point problems and variational inequality problems," *Taiwanese Journal of Mathematics*, vol. 12, no. 6, pp. 1401–1432, 2008.

[26] Y. Yao, Y.-C. Liou, and J.-C. Yao, "New relaxed hybrid-extragradient method for fixed point problems, a general system of variational inequality problems and generalized mixed equilibrium problems," *Optimization*, vol. 60, no. 3, pp. 395–412, 2011.

[27] K. Goebel and W. A. Kirk, *Topics in Metric Fixed-Point Theory*, Cambridge University Press, Cambridge, UK, 1990.

[28] Y. J. Cho, N. Petrot, and S. Suantai, "Fixed point theorems for nonexpansive mappings with applications to generalized equilibrium and system of nonlinear variational inequalities problems," *Journal of Nonlinear Analysis and Optimization*, vol. 1, no. 1, pp. 45–53, 2010.

[29] R. Wangkeeree and R. Wangkeeree, "A general iterative method for variational inequality problems, mixed equilibrium problems, and fixed point problems of strictly pseudocontractive mappings in Hilbert spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 519065, 32 pages, 2009.

[30] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

[31] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, Japan, 2000.

[32] S. Plubtieng and T. Thammathiwat, "A viscosity approximation method for finding a common fixed pointof nonexpansive and firmly nonexpansive mappings in Hilbert spaces," *Thai Journal of Mathematics*, vol. 6, no. 2, pp. 377–390, 2008.

[33] R. E. Bruck, Jr., "Properties of fixed-point sets of nonexpansive mappings in Banach spaces," *Transactions of the American Mathematical Society*, vol. 179, pp. 251–262, 1973.

[34] H.-K. Xu, "Viscosity method for hierarchical fixed point approach to variational inequalities," *Taiwanese Journal of Mathematics*, vol. 14, no. 2, pp. 463–478, 2010.

[35] H.-K. Xu, "Iterative algorithms for nonlinear operators," *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.

[36] H. H. Bauschke and J. M. Borwein, "On projection algorithms for solving convex feasibility problems," *SIAM Review*, vol. 38, no. 3, pp. 367–426, 1996.

[37] P. L. Combettes, "Hilbertian convex feasibility problem: convergence of projection methods," *Applied Mathematics and Optimization*, vol. 35, no. 3, pp. 311–330, 1997.

[38] F. Deutsch and I. Yamada, "Minimizing certain convex functions over the intersection of the fixed point sets of nonexpansive mappings," *Numerical Functional Analysis and Optimization*, vol. 19, no. 1-2, pp. 33–56, 1998.

[39] H. K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.