Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 683021, 14 pages
doi:10.1155/2012/683021

Research Article

The Maximal Subspace for Generation of (a, k)-Regularized Families

Edgardo Alvarez-Pardo[1] and Carlos Lizama[2]

[1] *Facultad de Ciencias Básicas, Universidad Tecnológica de Bolívar, Cartagena, Colombia*
[2] *Departamento de Matemática y Ciencia de la Computación, Facultad de Ciencia, Universidad de Santiago de Chile, Casilla 307, Correo 2, Santiago, Chile*

Correspondence should be addressed to Carlos Lizama, carlos.lizama@usach.cl

Received 25 May 2012; Revised 7 September 2012; Accepted 12 September 2012

Academic Editor: Patricia J. Y. Wong

Let A be a linear operator in a Banach space X. We define a subspace of X and a norm such that the part of A in such subspace generates an (a, k)-regularized resolvent family. This space is maximal-unique in a suitable sense and nontrivial, under certain conditions on the kernels a and k.

1. Introduction

Inspired by the well-known Hille-Yosida theorem, Kantorovitz defined in 1988 a linear subspace and a norm such that the restriction of A to this subspace generates a strongly continuous semigroup of contractions (see [1]). This so-called Hille- Yosida space is maximal unique in a suitable sense. The same problem has been considered in the context of strongly continuous operator families of contractions by Cioranescu in [2]. In this case, the generation theorem of Sova and Fattorini was fundamental for her work. Later, Lizama in [3] used the generation theorem for resolvent families due to Da Prato and Iannelli (see [4]) as basis to generalize the results of Kantorovitz and Cioranescu to the context of resolvent families of bounded and linear operators. In this paper, also some applications to Volterra equations were given.

It is remarkable that resolvent families do not include α-times integrated semigroups, α-times integrated cosine functions, K-convoluted semigroups, K-convoluted cosine families, and integrated Volterra equations. For a historical account of these classes of operators, see ([5], page 234). Actually, these types of families are (a, k)-regularized. The concept of (a, k)-regularized resolvent families was introduced in [6]. The systematic treatment based on techniques of Laplace transforms was developed in several papers (see, e.g., [7–13]). The theory of (a, k)-regularized families has been developed in many directions and we refer to the recent monograph of Kostić [14] for further information. In this context, the problem to find maximal subspaces for generation of (a, k)-regularized families remained open in case $k(t) \neq 1$. In this paper, we are able to close this gap generalizing, in particular, [1, 2, 6].

In this work, we will use the generation theorem for (a, k)-regularized resolvent family (see [6]) to show that there exists a linear subspace $Z_{a,k}$ in X and a norm $|\cdot|_{a,k}$ majorizing the given norm, such that $(Z_{a,k}, |\cdot|_{a,k})$ is a Banach space, and the part of A in $Z_{a,k}$ generates a (a, k)-regularized resolvent family of contractions in $Z_{a,k}$. Moreover, the space $(Z_{a,k}, |\cdot|_{a,k})$ is a maximal-unique in a sense to be defined below. Concerning the non-triviality of $Z_{a,k}$, we prove that it contains the eigenvectors corresponding to non-positive eigenvalues of A. We close this paper with illustrative examples concerning the cases $a(t) = t^{\alpha-1}$ and $k(t) = t^{\beta}$ in some region $\alpha > 0$ and $\beta > 0$.

This paper is organized as follows. In the first section, we recall the definition as well as basic results about (a, k)-regularized families.

In Section 2, we show the existence of the maximal subspace such that the part of A in this subspace generates an (a, k)-regularized family. We prove that such subspace is a Banach space with the norm defined below. The maximality is also proved and we show how this is used to obtain a relation with the Hille-Yosida space corresponding to the semigroup case.

In Section 3, we present some applications of the theory developed in the preceding section. Here we show the particular cases of generation corresponding to resolvent families, cosine operator families, semigroups, α-times semigroups and α-times cosine operator families. After that, we give concrete conditions on a given operator A to obtain the non-triviality of the maximal spaces and hence the well posedness on these spaces, for the abstract Cauchy problems of first and second order.

2. Preliminaries

In this section, we recall some useful results in the literature about (a, k)-regularized resolvent families. Let us fix some notations. From now on, we take X to be a Banach space with the norm $\|\cdot\|$. We denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X endowed with the operator norm, which again is denoted by $\|\cdot\|$. The identity operator on X is denoted by $I \in \mathcal{B}(X)$, and $\mathbb{R}_+$ denotes the interval $[0, \infty)$. For a closed operator A, we denote by $\sigma(A)$, $\sigma_p(A)$, $\rho(A)$ the spectrum, the point spectrum, and resolvent of A, respectively.

Definition 2.1. Let $k \in C(\mathbb{R}_+)$, $k \neq 0$, and $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ be given. Assume that A is a linear operator with domain $D(A)$. A strongly continuous family $\{R(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called (a, k)-regularized family on X having A as a generator if the following hold:

(a) $R(0) = k(0)I$;

(b) $R(t)x \in D(A)$ and $R(t)Ax = AR(t)x$ for all $x \in D(A)$ and $t \geq 0$;

(c) $R(t)x = k(t)x + \int_0^t a(t-s)AR(s)x\,ds$ for all $t \geq 0$ and $x \in D(A)$.

In the case where $k(t) \equiv 1$, this definition corresponds to the resolvent family for the Volterra equation of convolution type in [6]. Moreover, if, in addition, $a(t) \equiv 1$ then this family is a C_0-semigroup on X or if $a(t) \equiv t$ is a cosine family on X.

We note that the study of (a, k)-regularized families is associated to a wide class of linear evolution equation, including, for example, fractional abstract differential equations (see [15]).

Definition 2.2. We say that $(R(t))_{t\geq 0}$ is of type (M,ω) if there exist constants $M \geq 0$ and $\omega \in \mathbb{R}$ such that

$$\|R(t)\| \leq Me^{\omega t} \tag{2.1}$$

for all $t \geq 0$.

We will require the following theorem on generation of (a,k)-regularized families (see [6]).

Theorem 2.3. *Let A be a closed and densely defined operator on a Banach space X. Then $\{R(t)\}_{t\geq 0}$ is an (a,k)-regularized family of type (M,ω) if and only if the following hold:*

(a) $\widehat{a}(\lambda) \neq 0$ *and* $1/\widehat{a}(\lambda) \in \rho(A)$ *for all* $\lambda > \omega$;

(b) $H(\lambda) := \widehat{k}(\lambda)(I - \widehat{a}(\lambda)A)^{-1}$ *satisfies the estimates*

$$\left\|H^{(n)}(\lambda)\right\| \leq \frac{Mn!}{(\lambda-\omega)^{n+1}}, \quad \lambda > \omega,\ n \in \mathbb{N}_0. \tag{2.2}$$

In the case where $k(t) \equiv 1$, Theorem 2.3 is well known. In fact, if $a(t) \equiv 1$ then it is just the Hille-Yosida theorem; if $a(t) \equiv t$, then it is the generation theorem due essentially to Da Prato and Iannelli in [4]. In the case where $k(t) = t^n/n!$ and $a(t) \equiv 1$, it is the generation theorem for n-times integrated semigroups [16]; if $k(t) = t^n/n!$ and $a(t)$ is arbitrary, it corresponds to the generation theorem for integrated solutions of Volterra equations due to Arendt and Kellerman [17].

In order to give applications to our results we recall the following concepts of fractional calculus. The Mittag-Leffler function (see, e.g., [18–20]) is defined as follows:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} = \frac{1}{2\pi i}\int_{H_a} e^{\mu}\frac{\mu^{\alpha-\beta}}{\mu^{\alpha}-z}d\mu, \quad \alpha,\beta > 0,\ z \in \mathbb{C}, \tag{2.3}$$

where H_a is a Hankel path, that is, a contour which starts and ends at $-\infty$ and encircles the disc $|\mu| \leq |z|^{1/\alpha}$ counter clockwise. The function $E_{\alpha,\beta}$ is an entire function which provides a generalization of several usual functions. For a recent review, we refer to the monograph [21].

An interesting property related with the Laplace transform of the Mittag-Leffler function is the following (cf. [18], (A.27) page 267):

$$\mathcal{L}\left(t^{\beta-1}E_{\alpha,\beta}(-\rho^{\alpha}t^{\alpha})\right)(\lambda) = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha}+\rho^{\alpha}}, \quad \operatorname{Re}\lambda > |\rho|^{1/\alpha};\ \alpha > 0,\ \beta > 0,\ \rho \in \mathbb{R}. \tag{2.4}$$

Remark 2.4 (see [22]). If $0 < \alpha < 2$ and $\beta > 0$, then

$$\begin{aligned} E_{\alpha,\beta}(z) &= \frac{1}{\alpha}z^{(1-\beta)/\alpha}\exp\left(z^{1/\alpha}\right) + \varepsilon_{\alpha,\beta}(z), \qquad |\arg(z)| \leq \frac{1}{2}\alpha\pi; \\ E_{\alpha,\beta}(z) &= \varepsilon_{\alpha,\beta}(z), \qquad |\arg(-z)| < \left(1-\frac{1}{2}\alpha\right)\pi, \end{aligned} \tag{2.5}$$

where

$$\varepsilon_{\alpha,\beta}(z) := -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta-\alpha n)} + \mathcal{O}\left(|z|^{-N}\right), \tag{2.6}$$

as $z \to \infty$ with $2 \leq N \in \mathbb{N}$. This implies that for each $\pi(\alpha/2) < \omega < \min\{\pi, \pi\alpha\}$, there is a constant $C := C(\omega) > 0$ such that

$$\left|E_{\alpha,\beta}(z)\right| \leq \frac{C}{1+|z|}, \quad \omega \leq \left|\arg(z)\right| \leq \pi. \tag{2.7}$$

3. The Maximal Subspace

In this section, X is a Banach space with norm $\|\cdot\|$. Let A be a linear operator and $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $k \in C(\mathbb{R}_+)$, $k \neq 0$. Assume that

$$\widehat{a}(\lambda) = \int_0^\infty e^{-\lambda t} a(t)dt \neq 0, \quad \forall \lambda > 0, \tag{3.1}$$

and $1/\widehat{a}(\lambda) \in \rho(A)$ for all $\lambda > 0$. Observe that we are implicitly assuming that the inequality (2.2) holds with $\omega = 0$. Let $H_{a,k}(\lambda) := (\widehat{k}(\lambda)/\widehat{a}(\lambda))((1/\widehat{a}(\lambda)) - A)^{-1}$ for $\lambda > 0$, and define

$$Y_{a,k} := \left\{ x \in X : |x|_{a,k} := \sup_{\lambda_j > 0,\, n_j, l \in \mathbb{N}_0} \left\| \prod_{j=1}^{l} \left(\frac{1}{n_j!}\right) \lambda_j^{n_j+1} H_{a,k}^{(n_j)}(\lambda_j) x \right\| < +\infty \right\}, \tag{3.2}$$

where for $l = 0$ the product is defined as x. It is clear that $|\cdot|_{a,k}$ is a norm on $Y_{a,k}$.

Proposition 3.1. $(Y_{a,k}, |\cdot|_{a,k})$ *is a Banach space.*

Proof. Let $\{x_i\} \subset Y_{a,k}$ be a Cauchy sequence. We observe that $\|x\| \leq |x|_{a,k}$ if $x \in X$. Then $\{x_i\}$ is a Cauchy sequence on X. Let $x := \lim_{i\to\infty} x_i$.

First, we show that $x \in Y_{a,k}$. Indeed, let $\lambda_j > 0$, $n_j, l \in \mathbb{N}_0$ be fixed. Then

$$\begin{aligned} \left\| \prod_{j=1}^{l} \left(\frac{1}{n_j!}\right) \lambda_j^{n_j+1} H_{a,k}^{(n_j)}(\lambda_j) x \right\| &= \lim_{i\to\infty} \left\| \prod_{j=1}^{l} \left(\frac{1}{n_j!}\right) \lambda_j^{n_j+1} H_{a,k}^{(n_j)}(\lambda_j) x_i \right\| \\ &\leq \limsup_{i\to\infty} |x_i|_{a,k} < +\infty. \end{aligned} \tag{3.3}$$

Second, we prove that $\{x_i\}$ converges to $x \in Y_{a,k}$. Let $\varepsilon > 0$. There exists $N := N(\varepsilon) \in \mathbb{N}$ such that $|x_i - x_m|_{a,k} < \varepsilon/2$ for $i, m > N$. Since $x = \lim_{i\to\infty} x_i$ in the norm of X, we also have $\|x_m - x\| < \varepsilon/2M$ where $M > 0$ is the constant given in (2.2). Hence for every $\lambda_j > 0$, $n_j, l \in \mathbb{N}_0$, we have by inequality (2.2)

$$\begin{aligned}
\left\|\prod_{j=1}^{l}\left(\frac{1}{n_j!}\right)\lambda_j^{n_j+1}H_{a,k}^{(n_j)}(\lambda_j)(x_i - x)\right\| &\le \left\|\prod_{j=1}^{l}\left(\frac{1}{n_j!}\right)\lambda_j^{n_j+1}H_{a,k}^{(n_j)}(\lambda_j)(x_i - x_m)\right\| \\
&\quad + \left\|\prod_{j=1}^{l}\left(\frac{1}{n_j!}\right)\lambda_j^{n_j+1}H_{a,k}^{(n_j)}(\lambda_j)(x_m - x)\right\| \\
&\le \left\|\prod_{j=1}^{l}\left(\frac{1}{n_j!}\right)\lambda_j^{n_j+1}H_{a,k}^{(n_j)}(\lambda_j)(x_i - x_m)\right\| \\
&\quad + \prod_{j=1}^{l}\left(\frac{1}{n_j!}\right)\lambda_j^{n_j+1}\left\|H_{a,k}^{(n_j)}(\lambda_j)(x_m - x)\right\| \\
&\le \left\|\prod_{j=1}^{l}\left(\frac{1}{n_j!}\right)\lambda_j^{n_j+1}H_{a,k}^{(n_j)}(\lambda_j)(x_i - x_m)\right\| \\
&\quad + M\|x_m - x\|, \quad i, m > N.
\end{aligned} \tag{3.4}$$

Taking supremum over all $\lambda_j > 0$, $n_j, l \in \mathbb{N}_0$, we obtain

$$|x_i - x|_{a,k} \le |x_i - x_m|_{a,k} + M\|x_m - x\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad i > N. \tag{3.5}$$

Therefore the sequence $\{x_i\}$ converges to $x \in Y_{a,k}$ in the norm $|\cdot|_{a,k}$. Consequently $(Y_{a,k}, |\cdot|_{a,k})$ is a Banach space. □

Definition 3.2. Let $A_{Y_{a,k}} : D(A_{Y_{a,k}}) \subseteq Y_{a,k} \to Y_{a,k}$ be defined by $A_{Y_{a,k}}x := Ax$, where

$$D(A_{Y_{a,k}}) := \{x \in D(A) : x,\ Ax \in Y_{a,k}\}. \tag{3.6}$$

This operator is sometimes called the part of A in $Y_{a,k}$.
We denote

$$Z_{a,k} := \overline{D(A_{Y_{a,k}})}, \tag{3.7}$$

where the closure is taken in the norm $|\cdot|_{a,k}$.

Lemma 3.3. *With the preceding definitions and hypothesis, we have*

(a) $A_{Y_{a,k}}$ *is a closed linear operator on* $Y_{a,k}$*,*

(b) $(1/\hat{a}(\lambda)) - A_{Y_{a,k}}$ *is invertible on* $Y_{a,k}$ *for each* $\lambda > 0$*,*

(c) $\|H_{a,y}^{(n)}(\lambda)\|_{B(Y_{a,k})} \le n!/\lambda^{n+1}$ *for each* $\lambda > 0$ *and* $n \in \mathbb{N}_0$*, where* $H_{a,y}(\lambda) := (\hat{k}(\lambda)/\hat{a}(\lambda))((1/\hat{a}(\lambda)) - A_{Y_{a,k}})^{-1}$.

Proof. Let $\lambda > 0$ be fixed. Since $1/\widehat{a}(\lambda) \in \rho(A)$, then

$$\begin{aligned}
&\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - A\right)H_{a,k}(\lambda)x = x, \quad \text{for each } x \in X \\
&H_{a,k}(\lambda)\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - A\right)x = x, \quad \text{for each } x \in D(A).
\end{aligned} \tag{3.8}$$

Note that $A_{Y_{a,k}}$ is closed because $Y_{a,k}$ is a Banach space and A is closed. So the first part is done. Now, let $y \in Y_{a,k}$ be fixed. Since $H_{a,k}(\lambda) : X \to D(A)$, then $H_{a,k}(\lambda)y \in D(A)$. Moreover,

$$\begin{aligned}
\left|H_{a,k}(\lambda)y\right|_{a,k} &:= \sup\left\|\prod_{j=1}^{l}\left(\frac{1}{n_j!}\right)\lambda_j^{n_j+1}H_{a,k}^{n_j}(\lambda_j)H_{a,k}(\lambda)y\right\| \\
&\leq \|H_{a,k}(\lambda)\| \sup\left\|\prod_{j=1}^{l}\left(\frac{1}{n_j!}\right)\lambda_j^{n_j+1}H_{a,k}^{n_j}(\lambda_j)y\right\| \\
&= \|H_{a,k}(\lambda)\|\left|y\right|_{a,k} < \infty,
\end{aligned} \tag{3.9}$$

therefore $H_{a,k}(\lambda)y \in Y_{a,k}$. On the other hand, from identities above, we have

$$AH_{a,k}(\lambda)y = \frac{1}{\widehat{a}(\lambda)}H_{a,k}(\lambda)y - \frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}y; \tag{3.10}$$

thus $AH_{a,k}(\lambda)y \in Y_{a,k}$. Hence $H_{a,k}(\lambda)y \in D(A_{Y_{a,k}})$, and we conclude that

$$\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - A_{Y_{a,k}}\right)H_{a,k}(\lambda)y = y, \quad \text{for each } y \in Y_{a,k}. \tag{3.11}$$

Now if $y \in D(A_{Y_{a,k}})$ then, in particular, $y \in D(A)$, and therefore,

$$H_{a,k}(\lambda)\frac{\widehat{a}(\lambda)}{\widehat{k}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - A_{Y_{a,k}}\right)y = y. \tag{3.12}$$

This proves the second assertion. In particular, $\rho(A_{Y_{a,k}}) \neq \emptyset$ and hence

$$H_{a,y}(\lambda) = H_{a,k}(\lambda)|_{Y_{a,k}}, \quad \forall\lambda > 0. \tag{3.13}$$

Finally, let $y \in Y_{a,k}$, $\lambda > 0$, $n \in \mathbb{N}_0$ be fixed. We have

$$\begin{aligned}
\left|\frac{1}{n!}\lambda^{n+1}H_{a,y}(\lambda)y\right|_{a,k} &:= \sup\left\|\prod_{j=1}^{l}\left(\frac{1}{n_j!}\right)\lambda_j^{n_j+1}H_{a,k}^{n_j}(\lambda_j)\left(\frac{1}{n!}\lambda^{n+1}H_{a,y}(\lambda)y\right)\right\| \\
&\leq \sup\left\|\prod_{j=1}^{l+1}\left(\frac{1}{n_j!}\right)\lambda_j^{n_j+1}H_{a,y}^{(n_j)}(\lambda_j)y\right\| \\
&\leq |y|_{a,k},
\end{aligned} \tag{3.14}$$

where $\lambda_j > 0$ are arbitrary for $1 \leq j \leq l$, $\lambda_{l+1} := \lambda$ and $n_{l+1} := n$. This proves the third part of the lemma. □

Lemma 3.4. *Let $A_{a,k} : D(A_{a,k}) \subseteq Z_{a,k} \to Z_{a,k}$ be defined by $A_{a,k}x := A_{Y_{a,k}}x$, where*

$$D(A_{a,k}) := \{x \in D(A_{Y_{a,k}}) : x, A_{Y_{a,k}}x \in Z_{a,k}\}. \tag{3.15}$$

Then $A_{a,k}$ is a closed operator such that $\overline{D(A_{a,k})} = Z_{a,k}$ and

(a) $(1/\widehat{a}(\lambda)) - A_{a,k}$ *is invertible on $Z_{a,k}$ for each $\lambda > 0$,*

(b) $\|H_{a,k}^{(n)}(\lambda)\|_{B(Z_{a,k})} \leq n!/\lambda^{n+1}$ *for each $\lambda > 0$ and $n \in \mathbb{N}_0$ where $H_{a,k}(\lambda) := (\widehat{k}(\lambda)/\widehat{a}(\lambda))((1/\widehat{a}(\lambda)) - A_{a,k})^{-1}$.*

Proof. We observe that $H_{a,k}(\lambda) = H_{a,y}(\lambda)|_{Z_{a,k}}$. Then the result is a direct consequence of ([5], Lemma 3.10.2). □

As a consequence, we obtain the main result of this section on the existence of (a,k)-regularized families.

Theorem 3.5. *Let A be a linear operator defined in a Banach space X and $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $k \in C(\mathbb{R}_+)$, $k \neq 0$. Assume that $1/\widehat{a}(\lambda) \in \rho(A)$ for all $\lambda > 0$. Then there exist a linear subspace $Z_{a,k}$ and a norm $|\cdot|_{a,k}$ such that $(Z_{a,k}, |\cdot|_{a,k})$ is a Banach space and $A_{a,k}$ generates an (a,k)-regularized family of contractions in $Z_{a,k}$.*

Proof. According to our hypothesis, we can apply the generation theorem for (a,k)-regularized family (see [6, Theorem 3.4]) and the result of Lemma 3.4. □

Concerning the non-triviality of $Z_{a,k}$, we will prove that it contains the eigenvectors corresponding to nonpositive eigenvalues of A.

Let $\mu \in \mathbb{C}$ be fixed. Let $r(t,\mu)$ be the unique solution to the scalar equation

$$r(t,\mu) = k(t) + \mu \int_0^t a(t-s) r(s,\mu) ds. \tag{3.16}$$

Thus, provided the kernels $a(t)$ and $k(t)$ are Laplace transformable, we have

$$H_{a,k}(\lambda) = \frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)} \left(\frac{1}{\widehat{a}(\lambda)} - \mu \right)^{-1} = \int_0^\infty e^{-\lambda t} r(t,\mu) dt. \tag{3.17}$$

We define

$$C_{a,k} := \{ \mu \in \sigma(A) : \text{the map } t \longrightarrow r(t,\mu),\ t \geq 0 \text{ is bounded} \}. \tag{3.18}$$

Proposition 3.6. *Let x be an eigenvector of A corresponding to the eigenvalue $\alpha \in C_{a,k}$. Then $x \in Z_{a,k}$.*

Proof. Let x be an eigenvector of A corresponding to the eigenvalue α such that the map $t \to r(t,\alpha)$ is bounded. Let $\lambda > 0$ and $n \in \mathbb{N}$ be fixed. Then

$$\begin{aligned} \left\| \frac{1}{n!} \lambda^{n+1} H_{a,k}^{(n)}(\lambda) x \right\| &= \left\| \frac{1}{n!} \lambda^{n+1} \frac{d^n}{d\lambda^n} \left[\frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)} \left(\frac{1}{\widehat{a}(\lambda)} - \alpha \right)^{-1} x \right] \right\| \\ &= \left\| \frac{1}{n!} \lambda^{n+1} \frac{d^n}{d\lambda^n} \left(\int_0^\infty e^{-\lambda t} r(t,\alpha) dt \right) x \right\| \\ &\leq \frac{1}{n!} \lambda^{n+1} \int_0^\infty t^n e^{-\lambda t} |r(t,\alpha)| dt \|x\| \\ &\leq \sup_{t \geq 0} |r(t,\alpha)| \|x\|. \end{aligned} \tag{3.19}$$

This implies that $|x|_{a,k} \leq \sup_{t\geq 0} |r(t,\alpha)| \|x\|$ and, consequently, $x \in Z_{a,k}$. □

The following result shows us that the spaces $Z_{a,k}$ are maximal-unique in a certain sense.

Theorem 3.7. *Under the same hypothesis of Theorem 3.5, if $(W_{a,k}, \|\cdot\|_{a,k})$ is a Banach space such that $W_{a,k} \subset X$, $\|\cdot\| \leq \|\cdot\|_{a,k}$ and the operator $B_{a,k} = A|_{D(B_{a,k})}$ with $D(B_{a,k}) := \{x \in D(A) : x, Ax \in W_{a,k}\}$ generates an (a,k)-regularized family of contractions in $W_{a,k}$, then $W_{a,k} \subset Z_{a,k}$, $|\cdot|_{a,k} \leq \|\cdot\|_{a,k}$ and $B_{a,k} \subset A_{a,k}$.*

Proof. Suppose that $(W_{a,k}, \|\cdot\|_{a,k})$, $B_{a,k}$ are as in the statement of theorem. Since $H_{a,k}(\lambda)$ is the Laplace transform of the (a,k)-regularized family $S_{a,k}(t)$ $(t \ge 0)$, we have that for $x \in W_{a,k}$, $\lambda > 0$, and $n \in \mathbb{N}_0$

$$\begin{aligned}\left\|\frac{1}{n!}\lambda^{n+1}H_{a,k}^{(n)}(\lambda)x\right\| &= \left\|\frac{1}{n!}\lambda^{n+1}\left(\frac{d}{d\lambda}\right)^n\int_0^\infty e^{-\lambda t}S_{a,k}(t)x\,dt\right\| \\ &\le \frac{1}{n!}\lambda^{n+1}\int_0^\infty e^{-\lambda t}\|S_{a,k}(t)x\|dt \\ &\le \frac{1}{n!}\lambda^{n+1}\int_0^\infty e^{-\lambda t}\|S_{a,k}(t)x\|_{a,k}\,dt \\ &\le \frac{1}{n!}\lambda^{n+1}\int_0^\infty e^{-\lambda t}\|x\|_{a,k}dt \\ &= \|x\|_{a,k}.\end{aligned} \tag{3.20}$$

We conclude that for $x \in W_{a,k}$, $|x|_{a,k} \le \|x\|_{a,k}$, that is, $W_{a,k} \subset Y_{a,k}$. It follows that

$$D(B_{a,k}) := \{x \in D(A) : x, Ax \in W_{a,k}\} \subset \{x \in D(A) : x, Ax \in Y_{a,k}\} := D\left(A_{Y_{a,k}}\right). \tag{3.21}$$

Hence

$$W_{a,k} = \overline{D(B_{a,k})}^{\|\cdot\|_{a,k}} \subset \overline{D\left(A_{Y_{a,k}}\right)}^{\|\cdot\|_{a,k}} \subset \overline{D(B_{a,k})}^{|\cdot|_{a,k}} = Z_{a,k}. \tag{3.22}$$

Finally, this implies that $D(B_{a,k}) \subset D(A_{a,k})$ and $B_{a,k} \subset A_{a,k}$. □

The next result treats the "maximal property". In order to obtain the analogous result to the resolvent families case, we need more information about the function $k(t)$.

Theorem 3.8. *Let A be a linear operator defined in a Banach space X and $k \in C^1(\mathbb{R}_+)$, $k \ne 0$ with absolutely convergent Laplace transform for $\lambda > 0$. Assume that $1/\widehat{a}(\lambda) \in \rho(A)$ for all $\lambda > 0$. Suppose that $1 \le k(0)$. Then $A_{a,k}$ generates a strongly continuous semigroup of contractions on $Z_{a,k}$.*

Proof. By Hille-Yosida theorem it is sufficient to have $(\alpha, \infty) \subset \rho(A_{a,k})$ and $\|\lambda(\lambda - A_{a,k})\|_{B(Z_{a,k})} \le 1$ for all $\lambda > \alpha$, for some real α.

In order to show this, we take $n = 0$ in the second part of Lemma 3.4 and obtain

$$\left\|\frac{\widehat{k}(\lambda)}{\widehat{a}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - A_{a,k}\right)^{-1}\right\|_{B(Z_{a,k})} \le \frac{1}{\lambda}. \tag{3.23}$$

Let $\mu := 1/\widehat{a}(\lambda)$. Then (3.23) gives

$$\left\|\mu\left(\mu - A_{a,k}\right)^{-1}\right\| \le \frac{1}{\lambda\widehat{k}(\lambda)}, \quad (\lambda > 0). \tag{3.24}$$

Since $\widehat{a}(\lambda) \to 0$ as $\lambda \to \infty$, and $\lim_{\lambda\to\infty}\lambda\widehat{k}(\lambda) = k(0)$ by the initial value theorem, we obtain

$$\left\|\mu(\mu - A_{a,k})^{-1}\right\|_{B(Z_{a,k})} \leq \frac{1}{k(0)}, \tag{3.25}$$

for all μ sufficiently large. Since $k(0) \geq 1$, we get $\|\mu(\mu - A_{a,k})\|_{B(Z_{a,k})} \leq 1$ for μ sufficiently large, which concludes the proof. □

Remark 3.9. From the maximal uniqueness of $Z_{1,1}$ and under the same hypothesis of preceding theorem we obtain that $Z_{a,k} \subset Z_{1,1}$, $|\cdot|_{1,1} \leq |\cdot|_{a,k}$ and $A_{a,k} \subset A_{1,1}$. Note that, in particular, $Z_{a,1} \subset Z_{1,1}$ which include [3, Remark 2.10].

4. Applications

Taking $k(t) \equiv 1$ we obtain the main result in [3, Theorem 2.5].

Corollary 4.1 (see [6], Theorem 2.5). *Let A be a linear operator defined in a Banach space X and $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, $a \neq 0$ with absolutely convergent Laplace transform for $\lambda > 0$. Assume that $1/\widehat{a}(\lambda) \in \rho(A)$ for all $\lambda > 0$. Then there exist a linear subspace $Z_{a,1}$ and a norm $|\cdot|_{a,1}$ such that $(Z_{a,1}, |\cdot|_{a,1})$ is a Banach space and the equation*

$$u(t) = f(t) + \int_0^t a(s-t)A_{k,1}u(s)ds \tag{4.1}$$

admits resolvent family of contractions on $Z_{a,1}$, where $A_{k,1}$ is defined in Lemma 3.4.

Taking $k(t) \equiv 1$ or $k(t) \equiv t$, we obtain from the preceding corollary the following.

Corollary 4.2 (see [1]). *Let A be a linear operator on X such that $(0,\infty) \subset \rho(A)$. Then there exist a linear subspace $Z_1 \subset X$ and a norm $|\cdot|_1$ such that $(Z_1, |\cdot|_1)$ is a Banach space and the restriction A_1 of A to Z_1 is the infinitesimal generator of a C_0-semigroup of contractions on Z_1.*

Corollary 4.3 (see [2]). *Let A be a linear operator on X such that $(0,\infty) \subset \rho(A)$. Let A_t be the operator in Z_t defined as above. Then A_t is the infinitesimal generator of a strongly continuous cosine family of contractions on Z_t.*

Remark 4.4. Applying Theorem 3.5 with $k(t) \equiv t^{\beta}/(\Gamma(\beta+1))$ we obtain corresponding results for α-times integrated semigroups and β-times integrated cosine families taking $a(t) \equiv 1$ and $a(t) \equiv t$, respectively.

Suppose that $\sigma_p(A) \neq \emptyset$ and $\mu \in \sigma_p(A)$. In the following examples we search conditions under which the function $t \to r(t,\mu)$ is bounded. This ensure the non-triviality of the subspace $Z_{a,k}$.

Example 4.5. Let us consider $k(t) := t^{\beta}/(\Gamma(\beta+1))$ and $a(t) := t^{\alpha-1}/\Gamma(\alpha)$. Let A be a closed linear and densely defined operator on a Banach space X such that

(a) $(0,\infty) \subset \rho(A)$,

(b) $\sigma_p(A) \cap \mathbb{R}_- \neq \emptyset$,

(c) $0 < \alpha < 2$ and $\alpha \geq \beta$.

Let $\mu \in \sigma_p(A) \cap \mathbb{R}_-$. Then applying Laplace transform, we have that

$$\widehat{r}(\lambda,\mu) = \frac{\lambda^{\alpha-(\beta+1)}}{\lambda^{\alpha}-\mu}. \tag{4.2}$$

It follows from (2.4) that

$$r(t,\mu) = t^{\beta}E_{\alpha,\beta+1}(\mu t^{\alpha}), \quad (\mu < 0). \tag{4.3}$$

Take $z = \mu t^{\alpha}$ with $t > 0$, then $\arg(z) = \pi$. By Remark 2.4, for each $\pi(\alpha/2) < \omega < \min\{\pi, \pi\alpha\}$, there is a constant $C := C(\omega) > 0$ such that

$$\left|E_{\alpha,\beta+1}(\mu t^{\alpha})\right| \leq \frac{C}{1+|\mu|t^{\alpha}}, \quad t > 0. \tag{4.4}$$

Therefore

$$|r(t,\mu)| = \left|t^{\beta}E_{\alpha,\beta+1}(\mu t^{\alpha})\right| \leq \frac{Ct^{\beta}}{1+|\mu|t^{\alpha}}, \quad t > 0. \tag{4.5}$$

From here and part (c), it follows that $r(t,\mu)$ is a bounded function for $t > 0$. In particular, $Z_{a,k} \equiv Z_{\alpha,\beta} \neq \{0\}$ by Proposition 3.6. Since $(0,\infty) \subset \rho(A)$, we have that $1/\widehat{a}(\lambda) = \lambda^2 \in \rho(A)$ for $\lambda > 0$. It follows from Theorem 3.5 that A is the infinitesimal generator of a strongly continuous Laplace transformable (α,β)-resolvent family of contractions on $Z_{\alpha,\beta}$. Note in particular that if $\alpha = 1$ and $\beta > 0$, then A is the generator of β-times integrated semigroup $(S_{\beta}(t))_{t\geq 0}$ on each subspace $Z_{1,\beta}$ for $0 < \beta \leq 1$. It means that the initial value problem

$$\begin{aligned} u'(t) &= Au(t), \quad t \geq 0, \\ u(0) &= u_0 \end{aligned} \tag{4.6}$$

is well posed in the sense that there exists a strongly continuous family of linear operators $(S_{\beta}(t))_{t\geq 0}$ on a subspace $Z_{1,\beta}$ of X such that for all initial values $u_0 \in Z_{1,\beta}$ there exists a unique classical solution of (4.6).

Example 4.6. Let us consider $k(t) := t^{\beta}/(\Gamma(\beta+1))$ and $a(t) := t^{\alpha-1}/\Gamma(\alpha)$ when $\alpha = 2$, that is, $a(t) := t$. Let A be a closed linear and densely defined operator on a Banach space X such that

(a) $(0,\infty) \subset \rho(A)$,

(b) $\sigma_p(A) \cap \mathbb{R}_- \neq \emptyset$,

(c) $0 < \beta \leq 1$.

Let $\mu \in \sigma_p(A) \cap \mathbb{R}_-$. Then

$$\widehat{r}(\lambda,\mu) = \frac{\lambda^{1-\beta}}{\lambda^2-\mu} = \frac{\lambda}{\lambda^2-\mu}\frac{1}{\lambda^\beta}. \tag{4.7}$$

Let $\omega^2 := -\mu$. First, we consider $\beta = 1$. In this case, if we take inverse Laplace transform, we obtain

$$r(t,\mu) = \frac{1}{\omega}\sin(\omega t). \tag{4.8}$$

Obviously, this function is bounded for $t \geq 0$ and $\mu = -\omega^2 < 0$.

Now, we consider $0 < \beta < 1$. In this case

$$\begin{aligned} r(t,\mu) &= \int_0^t \frac{s^{\beta-1}}{\Gamma(\beta)}\cos(\omega(t-s))ds \\ &= \frac{1}{\Gamma(\beta)}\left\{\cos(\omega t)\int_0^t s^{\beta-1}\cos(\omega s)ds + \sin(\omega t)\int_0^t s^{\beta-1}\sin(\omega s)ds\right\}. \end{aligned} \tag{4.9}$$

Remember that the incomplete Gamma function is defined by

$$\Gamma(a,z) := \int_z^\infty e^{-t}t^{a-1}dt. \tag{4.10}$$

About the asymptotic behavior, we know that (see [23, formula (8.357)])

$$\Gamma(a,z) := z^{a-1}e^{-z}\left[1 + \mathcal{O}\left(|z|^{-1}\right)\right], \tag{4.11}$$

as $|z| \to \infty$ and $-3\pi/2 < \arg(z) < 3\pi/2$.

It can be verified that the following formulas holds (see [23, formula (2.632)]):

$$\begin{aligned} \int_0^t s^{\beta-1}\sin(\omega s)ds = -\frac{1}{2\omega^\beta}&\left[\exp\left[\frac{i\pi}{2}(\beta-1)\right]\Gamma(\beta,-i\omega t)\right. \\ &\left.+\exp\left[\frac{i\pi}{2}(1-\beta)\right]\right]\Gamma(\beta,i\omega t), \quad (\beta<1), \\ \int_0^t s^{\beta-1}\cos(\omega s)ds = -\frac{1}{2\omega^\beta}&\left[\exp\left(\frac{i\beta\pi}{2}\right)\Gamma(\beta,-i\omega t) + \exp\left(\frac{i\beta\pi}{2}\right)\Gamma(\beta,i\omega t)\right]. \end{aligned} \tag{4.12}$$

Note that $\arg(i\omega t) = \pi/2$ and $\arg(i\omega t) = -\pi/2$, so we can apply (4.11). Then, for t sufficiently large, we obtain that

$$\left|\Gamma(\beta,i\omega t)\right| = \left|\Gamma(\beta,-i\omega t)\right| \leq \omega^{\beta-1}t^{\beta-1}\left[1+\mathcal{O}\left(|\omega t|^{-1}\right)\right]. \tag{4.13}$$

The boundedness follows from the fact that $\beta < 1$. Therefore, for t sufficiently large

$$\left|r(t,\mu)\right| \leq M\left[\left|\Gamma(\beta, i\omega t)\right| + \left|\Gamma(\beta, -i\omega t)\right|\right] \leq K, \tag{4.14}$$

where K depends on ω and β. Since $(0,\infty) \subset \rho(A)$, we have that $\lambda^2 \in \rho(A)$ for $\lambda > 0$. It follows from Theorem 3.5 that A is the infinitesimal generator of β-times integrated cosine function $(C_\beta(t))_{t\geq 0}$ of contractions on $Z_{2,\beta}$ for $0 < \beta \leq 1$. As in the above example, it now means that the initial value problem

$$\begin{aligned} &u''(t) = Au(t), \quad t \geq 0, \\ &u(0) = u_0, \\ &u'(0) = 0 \end{aligned} \tag{4.15}$$

is well posed in the sense that there exists a strongly continuous family of linear operators $(C_\beta(t))_{t\geq 0}$ on a nontrivial subspace $Z_{2,\beta}$ of X such that for all initial values $u_0 \in Z_{2,\beta}$ there exists a unique classical solution of (4.15).

Acknowledgments

The first author is partially supported by a research grant of Banco Santander and Universidad Tecnológica de Bolívar. The second author is partially supported by Proyecto FONDECYT 1100485.

References

[1] S. Kantorovitz, "The Hille-Yosida space of an arbitrary operator," *Journal of Mathematical Analysis and Applications*, vol. 136, no. 1, pp. 107–111, 1988.
[2] I. Cioranescu, "On the second order Cauchy problem associated with a linear operator," *Journal of Mathematical Analysis and Applications*, vol. 154, no. 1, pp. 238–242, 1991.
[3] C. Lizama, "On volterra equations associated with a linear operator," *Proceedings of the American Mathematical Society*, vol. 118, no. 4, pp. 1159–1166, 1993.
[4] G. Da Prato and M. Iannelli, "Linear integro-differential equations in Banach spaces," *Rendiconti del Seminario Matematico dell'Università di Padova*, vol. 62, pp. 207–219, 1980.
[5] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems*, vol. 96 of *Monographs in Mathematics*, Birkhäuser, Basel, Switzerland, 2001.
[6] C. Lizama, "Regularized solutions for abstract Volterra equations," *Journal of Mathematical Analysis and Applications*, vol. 243, no. 2, pp. 278–292, 2000.
[7] M. Kostic, "(a,k)—regularized C-resolvent families: regularity and local properties," *Abstract and Applied Analysis*, vol. 2009, Article ID 858242, 27 pages, 2009.
[8] C. Lizama and P. J. Miana, "A Landau-Kolmogorov inequality for generators of families of bounded operators," *Journal of Mathematical Analysis and Applications*, vol. 371, no. 2, pp. 614–623, 2010.
[9] C. Lizama and J. Sánchez, "On perturbation of K—regularized resolvent families," *Taiwanese Journal of Mathematics*, vol. 7, no. 2, pp. 217–227, 2003.
[10] C. Lizama and H. Prado, "On duality and spectral properties of (a,k)—regularized resolvents," *Proceedings of the Royal Society of Edinburgh A*, vol. 139, no. 3, pp. 505–517, 2009.
[11] C. Lizama and H. Prado, "Rates of approximation and ergodic limits of regularized operator families," *Journal of Approximation Theory*, vol. 122, no. 1, pp. 42–61, 2003.
[12] S.-Y. Shaw and J.-C. Chen, "Asymptotic behavior of (a,k)-regularized resolvent families at zero," *Taiwanese Journal of Mathematics*, vol. 10, no. 2, pp. 531–542, 2006.

[13] S.-Y. Shaw and H. Liu, "Continuity of restrictions of (a, k)—regularized resolvent families to invariant subspaces," *Taiwanese Journal of Mathematics*, vol. 13, no. 2A, pp. 535–544, 2009.
[14] M. Kostić, *Generalized Semigroups and Cosine Functions*, vol. 23 of *Posebna Izdanja*, Matematički Institut SANU, Belgrade, 2011.
[15] C. Lizama and G. M. N'Guérékata, "Bounded mild solutions for semilinear integro differential equations in Banach spaces," *Integral Equations and Operator Theory*, vol. 68, no. 2, pp. 207–227, 2010.
[16] H. Kellerman and M. Hieber, "Integrated semigroups," *Journal of Functional Analysis*, vol. 84, no. 1, pp. 160–180, 1989.
[17] W. Arendt and H. Kellerman, *Integrated Solutions of Volterra Integrodifferential Equations and Applications*, vol. 190 of *Pitman Research Notes in Mathematical*, 1987.
[18] R. Gorenflo and F. Mainardi, "Fractional calculus: integral and differential equations of fractional order," in *Fractals and Fractional Calculus in Continuum Mechanics*, A. Carpinteri and F. Mainardi-Paraaaa, Eds., pp. 223–276, Springer, New York, NY, USA, 1997.
[19] F. Mainardi and R. Gorenflo, "On Mittag-Leffler-type functions in fractional evolution processes," *Journal of Computational and Applied Mathematics*, vol. 118, no. 1-2, pp. 283–299, 2000.
[20] R. Gorenflo and F. Mainardi, *Fractional Calculus: Integral and Differential Equations of Fractional Order*, Fractals and Fractional Calculus in Continuum Mechanics, Springer, New York, NY, USA, 1997.
[21] H. J. Haubold, A. M. Mathai, and R. K. Saxena, "Mittag-Leffler functions and their applications," *Journal of Applied Mathematics*, vol. 2011, Article ID 298628, 51 pages, 2011.
[22] C. Chen, M. Li, and F.-B. Li, "On boundary values of fractional resolvent families," *Journal of Mathematical Analysis and Applications*, vol. 384, no. 2, pp. 453–467, 2011.
[23] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Elsevier, 2007.