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Research Article

Asymptotic Properties of G-Expansive Homeomorphisms on a Metric G-Space

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We define and study the notions of positively and negatively G-asymptotic points for a homeomorphism on a metric G-space. We obtain necessary and sufficient conditions for two points to be positively/negatively G-asymptotic. Also, we show that the problem of studying G-expansive homeomorphisms on a bounded subset of a normed linear G-space is equivalent to the problem of studying linear G-expansive homeomorphisms on a bounded subset of another normed linear G-space.

1. Introduction

Expansiveness, introduced by Utz [1] in 1950 for homeomorphisms on metric spaces, is one of the important dynamical properties studied for dynamical systems. Expansive homeomorphisms have lots of applications in topological dynamics, ergodic theory, continuum theory, symbolic dynamics, and so forth. The notion of asymptotic points for a homeomorphism on a metric space was defined by Utz in [1]. On metric spaces, the existence of asymptotic points under expansive homeomorphisms is studied by Utz [1], Bryant [2, 3], Wine [4], Williams [5, 6], and others. In [7], authors have used this notion to classify all homeomorphisms of the circle without periodic points. Using the concept of generators, Bryant and Walters in [8] have obtained necessary and sufficient conditions for two points to be positively/negatively asymptotic under a homeomorphism on a compact metric space.

In [6], Williams has shown that the problem of studying expansive homeomorphisms on a bounded subset of a normed linear space is equivalent to the problem of studying linear expansive homeomorphisms on a bounded subset of another normed linear space. Using the above equivalence, Williams has obtained a necessary and sufficient condition for two points to be positively/negatively asymptotic under a homeomorphism on a bounded subset of a normed linear space. For study of expansive automorphisms on Banach spaces, one can refer to [9, 10].

With the intention of studying various dynamical properties of maps under the continuous action of a topological group, in [11], the notion of expansiveness termed as G-expansive homeomorphism is defined for a self-homeomorphism on a metric G-space. It is observed that the notion of expansiveness and the notion of G-expansiveness under a nontrivial action of G are independent of each other. Conditions under which an expansive homeomorphism on a metric G-space is G-expansive and viceversa are also obtained. Recently Choi and Kim in [12] have used this concept to generalize topological decomposition theorem proved in [13] due to Aoki and Hiraide for compact metric G-spaces. Further, in [14], the notion of generator in G-spaces termed as G-generator is defined and a characterization of G-expansive homeomorphisms is obtained using G-generator. Some interesting consequences have been obtained regarding existence of G-expansive homeomorphisms. In [15, 16] we have studied some more properties of G-expansive homeomorphisms. For some other dynamical properties on G-spaces, one can refer to [17, 18]. In Section 2, we give the preliminaries required for remaining sections. In Section 3, we define the notion of positively/negatively G-asymptotic points for a homeomorphism on a metric G-space. It is observed that this notion under the trivial action of G on X coincides with positively/negatively asymptotic points. However under a nontrivial action of G on X, while positively/negatively asymptotic points are positively/negatively G-asymptotic, examples are provided to justify that the converse is not true. Studying G-asymptotic points in relation to G-generators for a homeomorphism on a compact metric G-space, we obtain necessary and sufficient condition for two points to be positively/negatively G-asymptotic. In Section 4, we show that the problem of studying G-expansive homeomorphisms on a bounded subset of a normed linear G-space is equivalent to the problem of studying linear G-expansive homeomorphisms on a bounded subset of another normed linear G-space. Using the above equivalence, we obtain a necessary and sufficient condition for two points to be positively/negatively G-asymptotic under a homeomorphism on a bounded subset of a normed linear G-space extending William's result [6].

2. Preliminaries

Throughout $H(X)$ denotes the collection of all self-homeomorphisms of a topological space X, $\mathbb{Z}$ denotes the set of integers, and $\mathbb{N}$ denotes the set of positive integers. By a G-space [19, 20] we mean a triple (X, G, θ), where X is a Hausdorff space, G is a topological group, and $\theta : G \times X \to X$ is a continuous action of G on X. Henceforth, $\theta(g, x)$ will be denoted by gx. For $x \in X$, the set $G(x) = \{gx \mid g \in G\}$ is called the *G-orbit* of x in X. Note that G-orbits $G(x)$ and $G(y)$ of points x, y in X are either disjoint or equal. A subset S of X is called G-invariant if $\theta(G \times S) \subseteq S$. Let $X/G = \{G(x) \mid x \in X\}$ and $p_X : X \to X/G$ be the natural quotient map taking x to $G(x)$, $x \in X$, then X/G endowed with the quotient topology is called the orbit space of X (with respect to G). The map p_X which is called the orbit map, is continuous and open and if G is compact then p_X is also a closed map. An action of G on X is called trivial if $gx = x$, for every $g \in G$ and $x \in X$. If X, Y are G-spaces, then a continuous map $h : X \to Y$ is called equivariant if $h(gx) = gh(x)$ for each g in G and each x in X. We call h pseudoequivariant if $h(G(x)) = G(h(x))$ for each x in X. An equivariant map is clearly pseudoequivariant but converse need not be true [11]. We have studied properties of such maps in detail in [21]. By a normed linear G-space, we mean a normed linear space on which a topological group G acts.

Recall that if X is a metric space with metric d and h is a self homeomorphism of X then h is called expansive, if there exists a $\delta > 0$ such that whenever $x, y \in X$, $x \neq y$ then there exists an integer n satisfying $d(h^n(x), h^n(y)) > \delta$; δ is then called an expansive constant for h. Distinct points $x, y \in X$ are called positively (resp., negatively) asymptotic under h if for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ (resp., $n \leq N$) implies $d(f^n(x), f^n(y)) < \epsilon$. Given a compact Hausdorff space X and a self-homeomorphism h of X, a finite open cover $\mathbb{U}$ of X is called a generator for (X, h) [22] if for each bisequence $(U_i)_{i\in\mathbb{Z}}$ of members of $\mathbb{U}$, $\cap_{i=-\infty}^{\infty} h^{-i}(clU_i)$ contains at most one point. If X is a metric G-space with metric d then a self-homeomorphism h of X is called G-expansive with G-expansive constant $\delta > 0$ if whenever $x, y \in X$ with $G(x) \neq G(y)$ then there exists an integer n satisfying $d(h^n(u), h^n(v)) > \delta$, for all $u \in G(x)$ and $v \in G(y)$. Given a compact Hausdorff G-space X and a self-homeomorphism h of X, a finite cover $\mathbb{U}$ of X consisting of G-invariant open sets is called a G-generator for (X, h) if for each bisequence $(U_i)_{i\in\mathbb{Z}}$ of members of $\mathbb{U}$, $\cap_{i=-\infty}^{\infty} h^{-i}(clU_i)$ contains at most one G-*orbit*. Under the trivial action of G on X, a G-generator is equivalent to a generator but in [14] examples are provided to justify that under a nontrivial action both are independent.

3. G-Generators and G-Asymptotic Points

Definition 3.1. Let (X, d) be a metric G-space and $h : X \to X$ be a homeomorphism. Then $x, y \in X$ are called positively G-asymptotic (resp., negatively G-asymptotic) points with respect to h if for given $\varepsilon > 0$ there exists an integer N such that whenever $n \geq N$ (resp., $n \leq N$), $d(h^n(gx), h^n(ky)) < \varepsilon$, for some $g, k \in G$.

Remark 3.2. Under the trivial action of a G on X the notion of positively (resp., negatively) G-asymptotic points coincides with the notion of positively (resp., negatively) asymptotic points. On the other hand, under a nontrivial action of G on X, clearly positively (resp., negatively) asymptotic points with respect to a homeomorphism on X are positively (resp., negatively) G-asymptotic points: in fact take $g = k =$ the identity element of G. However, the fact that the converse need not be true can be seen from the following example.

Example 3.3. Let $X = \{\pm(1/m), \pm(1-1/m) \mid m \in \mathbb{N}\}$ under usual metric and define $h : X \to X$ defined by

$$h(x) = \begin{cases} x, & \text{if } x \in \{-1, 0, 1\}, \\ \text{the point of } X \text{ which is immediate next to right (left)of } x, \\ \text{if } x > 0 \ (x < 0), \end{cases} \tag{3.1}$$

then $h \in H(X)$. Let discrete group $G = \{-1, 1\}$ act on X by $-1 \cdot x = -x$ and $1 \cdot x = x$, $x \in X$. Then the points $x = -1/8$ and $y = 1/4$ are seen to be positively G-asymptotic but are not positively asymptotic with respect to h.

We obtain a necessary and sufficient condition for two points to be positively/negatively G-asymptotic with respect to a homeomorphism on a compact metric G-space having a G-generator. We first prove the following lemma for G- generators.

Lemma 3.4. *Let X be a compact metric G-space, $h \in H(X)$, and $\Im$ be a G-generator for (X,h). Then for each nonnegative integer n, there exists $\varepsilon > 0$ such that for $x, y \in X$ with $G(x) \neq G(y)$, $d(gx, ky) < \varepsilon$ for some $g, k \in G$ implies the existence of $A_{-n}, \dots, A_0, \dots, A_n$ in $\Im$ such that $gx, ky \in \cap_{i=-n}^{n} h^{-i}(A_i)$. Conversely, for each $\varepsilon > 0$, there exists a positive integer n such that $x, y \in \cap_{i=-n}^{n} h^{-i}(A_i)$ with $G(x) \neq G(y)$ and $A_{-n}, \dots, A_0, \dots, A_n$ in $\Im$ implies $d(gx, ky) < \varepsilon$ for some $g, k \in G$.*

Proof. Since X is compact and $\Im$ being a G-generator is an open cover of X, $\Im$ has a Lebesgue number, say η. Fix a nonnegative integer, say, n. Since X is a compact metric space therefore h^i, $|i| \leq n$, are uniformly continuous. Thus for above η, there exists an $\varepsilon > 0$ such that $d(x, y) < \varepsilon$ implies $d(h^i(x), h^i(y)) < \eta$ for all i, $|i| \leq n$. Now if for some $g, k \in G$, $d(gx, ky) < \varepsilon$ then using the fact that η is a Lebesgue number for $\Im$, for each i, $|i| \leq n$, we find an $A_i \in \Im$ such that $h^i(gx)$, $h^i(ky) \in A_i$ and hence

$$gx, ky \in \bigcap_{i=-n}^{n} h^{-i}(A_i). \tag{3.2}$$

Conversely, suppose $\varepsilon > 0$ is given. If the required result is not true, then for each positive integer j, there exist $x_j, y_j \in X$ with distinct G-orbits and $\{A_{j,i}\}_{-j \leq i \leq j} \subset \Im$ such that

$$x_j, y_j \in \bigcap_{i=-j}^{j} h^{-i}(A_{j,i}), \qquad d(gx_j, ky_j) \geq \varepsilon \tag{$*$}$$

for all $g, k \in G$. Since X is compact, sequences $\{x_j\}$ and $\{y_j\}$ will converge. Suppose they converge to x and y, respectively, then $(*)$ implies $G(x) \neq G(y)$. Since $\Im$ is a finite open cover, infinitely many of $A_{j,0}$ are same, say A_0 and therefore for infinitely many $j's$, $x_j, y_j \in A_0$. But this gives $x, y \in ClA_0$. Similarly, for each integer n, infinitely many of $A_{j,n}$ coincide and hence one gets A_n in $\Im$ such that $x, y \in h^{-n}(ClA_n)$. Thus

$$x, y \in \bigcap_{n=-\infty}^{\infty} h^{-n}(ClA_n). \tag{3.3}$$

This contradicts the fact that $\Im$ be a G-generator for (X, h). □

Theorem 3.5. *Let X be a compact metric G-space, $h \in H(X)$ be equivariant and $\Im$ be a G-generator for (X,h). Then $x, y \in X$ with distinct G-orbits are positively G-asymptotic with respect to h if and only if there exists an $N \in \mathbb{N}$ such that for each $i \geq N$, there exists an $A_i \in \Im$ with $x, y \in \cap_{i=N}^{\infty} h^{-i}(A_i)$.*

Proof. Suppose $x, y \in X$ with distinct G-orbits are positively G-asymptotic points. Then for a given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d\Big(h^i(gx), h^i(ky)\Big) < \varepsilon \quad \text{for some } g, k \in G, \tag{3.4}$$

wherein $i \geq N$. Take ε to be a Lebesgue number of $\Im$. Then for each $i \geq N$, there exists A_i in $\Im$ such that $h^i(gx), h^i(ky) \in A_i$ for some $g, k \in G$ and hence using equivariancy of h, we obtain $x, y \in \cap_{i=N}^{\infty} h^{-i}(A_i)$.

Conversely, suppose that there exists an integer N such that for each $i \geq N$, there exists an $A_i \in \Im$ such that $x, y \in \cap_{i=N}^{\infty} h^{-i}(A_i)$. Let $\varepsilon > 0$. Then by Lemma 3.4, obtain a positive integer n such that if $x, y \in \cap_{i=N}^{\infty} h^{-i}(A_i)$ with $G(x) \neq G(y)$ and $A_{-n}, \ldots, A_0, \ldots, A_n$ in $\Im$ then $d(gx, ky) < \varepsilon$ for some $g, k \in G$. Let $p \geq N + n$. Then $x, y \in \cap_{i=N}^{\infty} h^{-i}(A_i)$ implies

$$x, y \in \bigcap_{i=p-n}^{p+n} h^{-i}(A_i). \tag{3.5}$$

Therefore,

$$h^p(x), h^p(y) \in \bigcap_{i=p-n}^{p+n} h^{-(i-p)}(A_i) = \bigcap_{j=-n}^{n} h^{-j}(A_{j+p}). \tag{3.6}$$

Also $G(x) \neq G(y)$ implies $h^p(G(x)) \cap h^p((G(y)) = \emptyset$ and from equivariancy of h, we obtain that $G(h^p(x)) \neq G(h^p(y))$ and hence for some g, $k \in G$, $d(gh^p(x), kh^p(y)) < \epsilon$. Now equivariancy of h gives $d(h^p(gx), h^p(ky)) < \epsilon$. Thus given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$, for some $g, k \in G$, we have $d(h^n(gx), h^n(ky)) < \epsilon$ which proves that x, y are positively G-asymptotic points with respect to h. □

The following result concerning negatively G-asymptotic points can be proved similarly.

Theorem 3.6. *Let X be a compact metric G-space, $h \in H(X)$ be equivariant and $\Im$ be a G-generator for (X, h). Then $x, y \in X$ with distinct G-orbits are negatively G-asymptotic with respect to h if and only if there exists an integer N such that for each $i \leq N$, there exists an $A_i \in \Im$ with $x, y \in \cap_{i=-\infty}^{N} h^{-i}(A_i)$.*

4. Linearization of G-Expansive Homeomorphisms

We show that the problem of studying G-expansive homeomorphisms on a bounded subset of a normed linear G-space is equivalent to the problem of studying linear G-expansive homeomorphisms on a bounded subset of another normed linear G-space.

Let H be a normed linear G-Space with norm $||$ and G act on H in such a way that $T_k : H \longrightarrow H$ defined by $T_k(x) = kx,\ x \in H$ is linear for every $k \in G$.

Let

$$S(H) = \{z : \mathbb{Z} \longrightarrow H\} \tag{4.1}$$

and for $z \in S(H)$, let

$$z(i) = z_i, \quad i \in \mathbb{Z},$$
$$N_G(H) = \left\{ z \in S(H) \mid \sum_{i=-\infty}^{\infty} 2^{-|i|} |kz_i|^2 < \infty,\ k \in G \right\}. \tag{4.2}$$

Let $h : N_G(H) \to N_G(H)$ be defined by $(h(z))_i = (z_{i+1})$, for every $z \in N_G(H)$ and for every $i \in \mathbb{Z}$. For $z, w \in N_G(H)$, define $(z+w)_i = z_i + w_i$ and for a scalar c, define cz by $(cz)_i = cz_i$. Define $\|z\|$ by $(\sum_{i=-\infty}^{\infty} 2^{-|i|}|z_i|^2)^{-1/2}$. With this norm $N_G(H)$ is a normed linear space.

Using the above notations we have the following results.

Theorem 4.1. *Let H be a normed linear G-Space, X be a bounded subset of H and $f : X \to X$ be an equivariant homeomorphism. Then $g : X \to S(H)$ defined by $(g(x))_i = f^i(x)$, for each $x \in X$ and each integer i, satisfies $g(X) \subseteq N_G(H)$.*

Proof. Let $x \in X$ and $k \in G$ then X being bounded and f being equivariant, we have

$$\sum_{i=-\infty}^{\infty} 2^{-|i|}\left|k(g(x))_i\right|^2 = \sum_{i=-\infty}^{\infty} 2^{-|i|}\left|kf^i(x)\right|^2 = \sum_{i=-\infty}^{\infty} 2^{-|i|}\left|f^i(kx)\right|^2 < \infty. \tag{4.3}$$

Hence $g(X) \subseteq N_G(H)$. □

Theorem 4.2. *Let H be a normed linear G-Space, X be a bounded subset of H and $f : X \to X$ be an equivariant homeomorphism. The map h is a linear homeomorphism of $N_G(H)$ onto itself under which $g(X)$ is invariant. Moreover, $g(X)$ is bounded and g is a homeomorphism of X onto $g(X)$. Also, h is G-expansive on $g(X)$ if and only if f is G-expansive on X.*

Proof. Let $z, w \in N_G(H)$. Then

$$(h(z+w))_i = (z+w)_{i+1} = z_{i+1} + w_{i+1} = (h(z))_i + (h(w))_i = (h(z)+h(w))_i, \tag{4.4}$$

for every $i \in Z$. Therefore $h(z+w) = h(z) + h(w)$. Also, $(h(cz))_i = (cz)_{i+1} = cz_{i+1} = c(h(z))_i$ implies $h(cz) = c(h(z))$. Hence h is linear. If $z \neq w$ in $N_G(H)$ then for some $i \in \mathbb{Z}, z_i \neq w_i$ which implies $(h(z))_{i-1} \neq (h(w))_{i-1}$ and hence $h(z) \neq h(w)$. Thus h is one-one. If $w \in N_G(H)$ then $w' \in N_G(H)$, where $(w')_i = w_{i-1}$ and $h(w') = w$, which proves that h is onto. If $z_n \to 0$ then $h(z_n) \to 0$ therefore h is continuous. Similarly h^{-1} is continuous. Next, we show that $h(g(X)) \subseteq g(X)$. Let $x \in X$ then

$$\left(h(g(x))\right)_i = \left(g(x)\right)_{i+1} = f^{i+1}(x) = f^i\left(f(x)\right) = \left(g\left(f(x)\right)\right)_i \tag{4.5}$$

which implies $h(g(X)) \subseteq g(X)$. Clearly $g(X)$ is bounded. It is easy to observe that g is a homeomorphism of X onto $g(X)$. Suppose f is G-expansive on X with G-expansive constant δ. Let $z, w \in g(X)$ with $G(z) \neq G(w)$. Let $z = g(z_0)$, $w = g(w_0)$, $z_0, w_0 \in X$. Since f is equivariant, g is also equivariant and hence $G(z_0) \neq G(w_0)$. Further G-expansivity of f on X gives existence of an integer n such that

$$\left|f^n(kz_0) - f^n(pw_0)\right| > \delta, \tag{4.6}$$

for all $k, p \in G$. Now h being linear and g being equivariant, we get

$$\begin{aligned}
\left\|h^n(kz) - h^n(pw)\right\| &= \left\|h^n(kz - pw)\right\| \\
&= \sqrt{\sum_{i=-\infty}^{\infty} 2^{-|i|}\left|\left(h^n(kz - pw)\right)_i\right|^2} \\
&= \sqrt{\sum_{i=-\infty}^{\infty} 2^{-|i|}\left|\left(h^n(kz)\right)_i - \left(h^n(pw)\right)_i\right|^2} \\
&= \sqrt{\sum_{i=-\infty}^{\infty} 2^{-|i|}\left|(kz)_{n+i} - (pw)_{n+i}\right|^2} \\
&\geq \left|(kz)_n - (pw)_n\right| \\
&= \left|\left(kg(z_0)\right)_n - \left(pg(w_0)\right)_n\right| \\
&= \left|\left(g(kz_0)\right)_n - \left(g(pw_0)\right)_n\right| \\
&= \left|f^n(kz_0) - f^n(pw_0)\right| \\
&> \delta.
\end{aligned} \tag{4.7}$$

Therefore h is G-expansive on $g(X)$ with G-expansive constant δ.

Conversely, suppose h is G-expansive on $g(X)$ with G-expansive constant δ. We show that f is G-expansive on X with G-expansive constant $\delta/\sqrt{3}$. Suppose not. Then there exist $z_0, w_0 \in X$ with $G(z_0) \neq G(w_0)$ such that

$$\left|f^n(kz_0) - f^n(pw_0)\right| \leq \frac{\delta}{\sqrt{3}}, \tag{4.8}$$

for some $k, p \in G$ and for all $n \in \mathbb{Z}$. Let $z = g(z_0)$, $w = g(w_0)$ then g being equivariant homeomorphism, $G(z) \neq G(w)$. Now h being linear and g being equivariant, we have for all $n \in \mathbb{Z}$

$$\begin{aligned}
\left\|h^n(kz) - h^n(pw)\right\| &= \left\|h^n\left(g(kz_0) - g(pw_0)\right)\right\| \\
&= \sqrt{\sum_{i=-\infty}^{\infty} 2^{-|i|}\left|\left(g(kz_0)\right)_{n+i} - \left(g(pw_0)\right)_{n+i}\right|^2} \\
&= \sqrt{\sum_{i=-\infty}^{\infty} 2^{-|i|}\left|f^{n+i}(kz_0) - f^{n+i}(pw_0)\right|^2} \\
&\leq \sqrt{\sum_{i=-\infty}^{\infty} \frac{2^{-|i|}\delta^2}{3}} \\
&= \delta,
\end{aligned} \tag{4.9}$$

a contradiction to the fact that h is G-expansive with G-expansive constant δ. Thus $\delta/\sqrt{3}$ is a G-expansive constant for f. □

Theorem 4.3. *Let H be a normed linear G-Space, X be a bounded subset of H and $f : X \to X$ be an equivariant homeomorphism. Points $x, y \in X$ are positively (negatively) G-asymptotic under f if and only if $g(x)$ and $g(y)$ are positively (negatively) G-asymptotic under h.*

Proof. Suppose $g(x), g(y)$ are positively G-asymptotic under h. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and for some $k, p \in G$, we have

$$\left\| h^n(kg(x)) - h^n(pg(y)) \right\| = \left\| h^n(g(kx)) - h^n(g(py)) \right\| < \epsilon. \quad (4.10)$$

Since

$$\left| f^n(kx) - f^n(py) \right| \leq \left\| h^n(g(kx)) - h^n(g(py)) \right\|, \quad (4.11)$$

we get

$$\left| f^n(kx) - f^n(py) \right| < \epsilon. \quad (4.12)$$

Thus x, y are positively G-asymptotic under f.

Conversely, suppose x, y are positively G-asymptotic under f. Let $\epsilon > 0$ be given then there exist $N_1 \in \mathbb{N}$ and $k, p \in G$ such that for all $n \geq N_1$,

$$\left| f^n(kx) - f^n(py) \right| < \frac{\epsilon}{2}. \quad (4.13)$$

Choose $N_2 \in \mathbb{N}$, $N_2 < N_1$ such that

$$\sum_{i<N_2} 2^{-|i|} (\text{diam } X)^2 < \left(\frac{\epsilon^2}{4}\right). \quad (4.14)$$

Then for $n > (N_1 - N_2)$, we have

$$\begin{aligned}
&\left\| h^n(g(kx)) - h^n(g(py)) \right\|^2 \\
&\quad = \sum_{i \leq N_2} 2^{-|i|} \left| f^{n+i}(kx) - f^{n+i}(py) \right|^2 + \sum_{i \geq N_2} 2^{-|i|} \left| f^{n+i}(kx) - f^{n+i}(py) \right|^2 \\
&\quad < \left(\frac{\epsilon^2}{4}\right) + \left(\frac{\epsilon^2}{4}\right) \sum_{i \geq N_2} 2^{-|i|} \\
&\quad < \left(\frac{\epsilon^2}{4}\right) + \left(\frac{\epsilon^2}{4}\right) \sum_{i=-\infty}^{\infty} 2^{-|i|} \\
&\quad = \left(\frac{\epsilon^2}{4}\right) + \left(\frac{3\epsilon^2}{4}\right) \\
&\quad = \epsilon^2.
\end{aligned} \quad (4.15)$$

Hence for $n > (N_1 - N_2)$ and for above $k, p \in G$, g being equivariant we get,

$$\|h^n(kg(x)) - h^n(pg(y))\| < \epsilon, \tag{4.16}$$

implying $g(x), g(y)$ are positively G-asymptotic under h.

The proof for the case of negatively asymptotic points is similar. □

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