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Research Article

Qualitative Behaviors of Functional Differential Equations of Third Order with Multiple Deviating Arguments

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This paper considers nonautonomous functional differential equations of the third order with multiple constant deviating arguments. Using the Lyapunov-Krasovskii functional approach, we find certain sufficient conditions for the solutions to be stable and bounded. We give an example to illustrate the theoretical analysis made in this work and to show the effectiveness of the method utilized here.

1. Introduction

In this paper, we consider nonautonomous differential equation of the third order with constant multiple deviating arguments, $\tau, \tau_i, (i = 1, 2, \ldots, n)$ as follows:

$$\begin{aligned} &x'''(t) + a(t)x''(t) + \sum_{i=1}^{n} b_i(t)g_i\left(x'(t-\tau_i)\right) + g\left(x'(t)\right) + h(x(t-\tau)) \\ &\quad = p\left(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_n), x', \ldots, x'(t-\tau_n) \ldots, x''(t), \ldots, x''(t-\tau_n)\right). \end{aligned} \tag{1.1}$$

Writing (1.1) as a system of first order equations, we have

$$x' = y, \qquad y' = z,$$

$$z' = -a(t)z - \sum_{i=0}^{n} b_i(t)g_i(y) - h(x) - g(y) + \sum_{i=1}^{n} b_i(t)\int_{t-\tau_i}^{t} g_i'(y(s))z(s)ds + \int_{t-\tau}^{t} h'(x(s))y(s)ds + p\left(t, x, \ldots, x(t-\tau_n), y, \ldots, y(t-\tau_n), z, \ldots, z(t-\tau_n)\right), \tag{1.2}$$

where τ and τ_i, $(i = 1, 2, \ldots, n)$, are positive constants, that is, fixed delays; the functions a, b_i, g, g_i, h, and p are continuous for their all respective arguments and the primes in (1.1) denote differentiation with respect to t, $t \in \Re^+ = [0, \infty)$. It is also assumed that the derivatives $a'(t) \equiv (d/dt)a(t)$, $b_i'(t) \equiv (d/dt)b_i(t)$, $g_i'(y) \equiv (d/dy)g_i(y)$, and $h'(x) \equiv (d/dx)h(x)$ exist and are continuous; throughout the paper $x(t)$, $y(t)$, and $z(t)$ are abbreviated as x, y, and z, respectively. Finally, the existence and uniqueness of solutions of (1.1) are assumed and all solutions considered are supposed to be real valued.

To the best of our knowledge from the literature, in the last five decades, there has been much attention paid to the discussion of stability and boundedness of solutions of nonlinear differential equations of the third order without a deviating argument. For a comprehensive treatment of the subject, we refer the readers to the book of Reissig et al. [1] as a survey and the papers of Ademola et al. [2], Afuwape et al. [3], Ezeilo [4–13], Ezeilo and Tejumola [14, 15], Mehri and Shadman [16], Ogundare [17], Ogundare and Okecha [18], Omeike [19, 20], Ponzo [21, 22], Swick [23–25], Tejumola [26, 27], Tunç [28–34], Tunç and Ateş [35], Tunç and Ayhan [36], and the references cited in these papers for some works on the topic.

Besides, first, in 1973, Sinha [37] studied the stability of solutions of a third order nonlinear differential equation with a deviating argument. Later, some authors dealt with the stability and boundedness of solutions for various third order nonlinear differential equations with a deviating argument. For some related works, one can refer to the papers of Afuwape and Omeike [38], Omeike [39], Sadek [40, 41], Tejumola and Tchegnani [42], Tunç [43–59], Yao and Meng [60], Zhu [61], and the references thereof.

It should be noted that throughout all the above mentioned papers, Lyapunov's functions or the Lyapunov-Krasovskii functionals have been used as a basic tool to prove the results established there. It is also worth mentioning that the most effective method to study the stability and boundedness of solutions of nonlinear differential equations of higher orders without or with a deviating argument in the literature is still the Lyapunov's direct method, despite its use for a past long period by now.

The motivation for this paper comes from the above mentioned papers and the books. Our results improve and include the results existing in the literature. This work makes also a contribution to the existing studies made in the literature.

2. Main Results

Let $p(\cdot) = 0$.

Our first result is given by the following theorem.

Theorem 2.1. *In addition to the basic assumptions imposed to the functions $a(t)$, $b(t)$, g, g_i, and h appearing in* (1.1)*, we assume that there exist positive constants a, α, β, δ, b, b_i, B_i, c, c_1, and L_i such that the following conditions hold:*

(i) $g_i(0) = g(0) = h(0) = 0, a(t) \geq 2\alpha + a, \beta_i \leq b_i(t) \leq B_i,$

$$0 < c_1 \leq h'(x) \leq c, \qquad \alpha b - c > \delta, \qquad \frac{g_i(y)}{y} \geq b_i, \quad \frac{g(y)}{y} \geq b, \quad (y \neq 0), \tag{2.1}$$
$$\left|g_i'(y)\right| \leq L_i,$$

(ii) $[\sum_{i=1}^{n}\{\alpha b_i b_i(t)\} - c]y^2 \geq 2^{-1}\alpha a'(t)y^2 + \sum_{i=1}^{n} b_i'(t)\int_0^y g_i(\eta)d\eta.$

If

$$\tau_0 < \min\left\{\frac{\alpha b}{c + 2\alpha c + \alpha\sum_{i=1}^{n}(B_i L_i)}, \frac{2\alpha}{c + \sum_{i=1}^{n}[(2+\alpha)B_i L_i]}\right\}, \tag{2.2}$$

then the zero solution of (1.1) *is stable.*

Proof. Define the Lyapunov-Krasovskii functional $V(\cdot) = V(t, x_t, y_t, z_t)$ as

$$\begin{aligned} 2V(\cdot) = {} & z^2 + 2\alpha yz + 2\sum_{i=0}^{n} b_i(t)\int_0^y g_i(\eta)d\eta + 2\alpha\int_0^x h(\xi)d\xi + 2\int_0^y g(\eta)d\eta \\ & + \alpha a(t)y^2 + 2h(x)y + \lambda\int_{-\tau}^{0}\int_{t+s}^{t} y^2(\theta)d\theta\, ds + \sum_{i=1}^{n}\lambda_i\int_{-\tau_i}^{0}\int_{t+s}^{t} z^2(\theta)d\theta\, ds, \end{aligned} \tag{2.3}$$

where λ and λ_i are some positive constants to be chosen later.

Using the assumptions $g_i(y)/y \geq b_i$, $g(y)/y \geq b$, $(y \neq 0)$, $h(0) = 0$, and $0 < c_1 \leq h'(x) \leq c$, we have

$$\begin{aligned} 2b_i(t)\int_0^y g_i(\eta)d\eta &= 2b_i(t)\int_0^y \frac{g_i(\eta)}{\eta}\eta\, d\eta \geq (\beta_i b_i)y^2, \\ 2\int_0^y g(\eta)d\eta &= 2\int_0^y \frac{g(\eta)}{\eta}\eta\, d\eta \geq by^2, \\ h^2(x) &= 2\int_0^x h(\xi)h'(\xi)d\xi \leq 2c\int_0^x h(\xi)d\xi, \end{aligned} \tag{2.4}$$

so that

$$\begin{aligned}
2V(\cdot) &\geq (z+\alpha y)^2 + b\left[y + b^{-1}h(x)\right]^2 + 2\alpha\int_0^x h(\xi)d\xi - \frac{1}{b}h^2(x) + \sum_{i=1}^{n}(\beta_i b_i)y^2 \\
&\quad + \alpha(\alpha+a)y^2 + \lambda\int_{-\tau}^{0}\int_{t+s}^{t} y^2(\theta)d\theta\,ds + \sum_{i=1}^{n}\lambda_i\int_{-\tau_i}^{0}\int_{t+s}^{t} z^2(\theta)d\theta\,ds \\
&\geq (z+\alpha y)^2 + b\left[y + b^{-1}h(x)\right]^2 + 2\alpha\int_0^x h(\xi)d\xi - \frac{2c}{b}\int_0^x h(\xi)d\xi \\
&\quad + \sum_{i=1}^{n}(\beta_i b_i)y^2 + \alpha(\alpha+a)y^2 + \lambda\int_{-\tau}^{0}\int_{t+s}^{t} y^2(\theta)d\theta\,ds + \sum_{i=1}^{n}\lambda_i\int_{-\tau_i}^{0}\int_{t+s}^{t} z^2(\theta)d\theta\,ds.
\end{aligned} \tag{2.5}$$

On the other hand, it is obvious that

$$\begin{aligned}
2\alpha\int_0^x h(\xi)d\xi - \frac{2c}{b}\int_0^x h(\xi)d\xi &= 2b^{-1}(\alpha b - c)\int_0^x h(\xi)d\xi \\
&\geq c_1 b^{-1}(\alpha b - c)x^2,
\end{aligned} \tag{2.6}$$

so that

$$\begin{aligned}
2V(\cdot) &\geq (z+\alpha y)^2 + b\left[y + b^{-1}h(x)\right]^2 + c_1 b^{-1}(\alpha b - c)x^2 + \alpha(\alpha+a)y^2 \\
&\quad + \sum_{i=1}^{n}(\beta_i b_i)y^2 + \lambda\int_{-\tau}^{0}\int_{t+s}^{t} y^2(\theta)d\theta\,ds + \sum_{i=1}^{n}\lambda_i\int_{-\tau_i}^{0}\int_{t+s}^{t} z^2(\theta)d\theta\,ds.
\end{aligned} \tag{2.7}$$

Hence, we can obtain some positive constants D_j, $(j = 1, 2, 3)$, such that

$$V(\cdot) \geq D_1 x^2 + D_2 y^2 + D_3 z^2 \geq D_4\left(x^2 + y^2 + z^2\right), \tag{2.8}$$

where $D_4 = \min\{D_1, D_2, D_3\}$, since the integrals $\lambda\int_{-\tau}^{0}\int_{t+s}^{t} y^2(\theta)d\theta\,ds$ and $\sum_{i=1}^{n}\lambda_i\int_{-\tau_i}^{0}\int_{t+s}^{t} z^2(\theta)d\theta\,ds$ are nonnegative.

Let (x, y, z) be a solution of (1.2). Differentiating the Lyapunov-Krasovskii functional $V(\cdot)$ along this solution, we find

$$\begin{aligned}\frac{d}{dt}V(\cdot) = & -\left[\alpha\sum_{i=1}^{n}\{b_i(t)g_i(y)\}y^{-1} + \alpha g(y)y^{-1} - h'(x) - 2^{-1}\alpha a'(t)\right]y^2 \\ & + \sum_{i=1}^{n} b_i'(t)\int_0^y g_i(\eta)d\eta - (a(t)-\alpha)z^2 + z\sum_{i=1}^{n} b_i(t)\int_{t-\tau_i}^{t} g_i'(y(s))z(s)ds \\ & + z\int_{t-\tau}^{t} h'(x(s))y(s)ds \\ & + \alpha y\sum_{i=1}^{n} b_i(t)\int_{t-\tau_i}^{t} g_i'(y(s))z(s)ds + \alpha y\int_{t-\tau}^{t} h'(x(s))y(s)ds \\ & + \lambda\tau y^2 - \lambda\int_{t-\tau}^{t} y^2(s)ds + \sum_{i=1}^{n}(\lambda_i\tau_i)z^2 - \sum_{i=1}^{n}\lambda_i\int_{t-\tau_i}^{t} z^2(s)ds.\end{aligned} \tag{2.9}$$

Using the assumptions of Theorem 2.1 and the estimate $2|mn| \le m^2 + n^2$, we get

$$\begin{aligned}&\left[\alpha\sum_{i=1}^{n}(b_i(t)g_i(y))y^{-1} + \alpha g(y)y^{-1} - h'(x) - 2^{-1}\alpha a'(t)\right]y^2 - \sum_{i=1}^{n} b_i'(t)\int_0^y g_i(\eta)d\eta \\ &\quad \ge \left[\alpha\sum_{i=1}^{n}(b_ib_i(t)) + \alpha b - c - 2^{-1}\alpha a'(t)\right]y^2 - \sum_{i=1}^{n} b_i'(t)\int_0^y g_i(\eta)d\eta \\ &\quad = \left[\sum_{i=1}^{n}\{\alpha b_ib_i(t)\} - c\right]y^2 - 2^{-1}\alpha a'(t)y^2 - \sum_{i=1}^{n} b_i'(t)\int_0^y g_i(\eta)d\eta + (\alpha b)y^2 \\ &\quad \ge (\alpha b)y^2, \\ &\qquad [a(t)-\alpha]z^2 \ge (\alpha + a)z^2, \\ &z\sum_{i=1}^{n} b_i(t)\int_{t-\tau_i}^{t} g_i'(y(s))z(s)ds \le \frac{1}{2}\sum_{i=1}^{n}(B_iL_i\tau_i)z^2 + \frac{1}{2}\sum_{i=1}^{n}(B_iL_i)\int_{t-\tau_i}^{t} z^2(s)ds, \\ &\alpha y\sum_{i=1}^{n} b_i(t)\int_{t-\tau_i}^{t} g_i'(y(s))z(s)ds \le \frac{1}{2}\sum_{i=1}^{n}(\alpha B_iL_i\tau_i)y^2 + \frac{1}{2}\sum_{i=1}^{n}(\alpha B_iL_i)\int_{t-\tau_i}^{t} z^2(s)ds, \\ &z\int_{t-\tau}^{t} h'(x(s))y(s)ds \le \frac{c}{2}\tau z^2 + \frac{c}{2}\int_{t-\tau}^{t} y^2(s)ds, \\ &\alpha y\int_{t-\tau}^{t} h'(x(s))y(s)ds \le \frac{\alpha c}{2}\tau y^2 + \frac{\alpha c}{2}\int_{t-\tau}^{t} y^2(s)ds,\end{aligned} \tag{2.10}$$

so that

$$\begin{aligned}\frac{d}{dt}V(\cdot) \leq & -\frac{1}{2}\alpha b y^2 - az^2 - \frac{1}{2}\left[\alpha b - \left\{\alpha\sum_{i=1}^{n}(B_iL_i\tau_i) + (2\lambda+\alpha c)\tau\right\}\right]y^2 \\ & - \frac{1}{2}\left[2\alpha - \left\{\sum_{i=1}^{n}(B_iL_i\tau_i) + 2\sum_{i=1}^{n}(\lambda_i\tau_i) \ + c\tau\right\}\right]z^2 \\ & + \left[2^{-1}(1+\alpha)c - \lambda\right]\int_{t-\tau}^{t} y^2(s)ds \\ & + \left[2^{-1}(1+\alpha)\sum_{i=1}^{n}(B_iL_i) - \sum_{i=1}^{n}\lambda_i\right]\int_{t-\tau_i}^{t} z^2(s)ds.\end{aligned} \tag{2.11}$$

Let $\lambda = (1/2)(1+\alpha)c$ and $\sum_{i=1}^{n}\lambda_i = (1/2)(1+\alpha)\sum_{i=1}^{n}(B_iL_i)$. Hence,

$$\begin{aligned}\frac{d}{dt}V(\cdot) \leq & -\frac{1}{2}\alpha b y^2 - az^2 - \frac{1}{2}\left[\alpha b - \left\{\alpha\sum_{i=1}^{n}(B_iL_i\tau_i) + (2\lambda+\alpha c)\tau\right\}\right]y^2 \\ & - \frac{1}{2}\left[2\alpha - \left\{\sum_{i=1}^{n}(B_iL_i\tau_i) + 2\sum_{i=1}^{n}(\lambda_i\tau_i) + c\tau\right\}\right]z^2.\end{aligned} \tag{2.12}$$

Let $\tau_0 = \max\{\tau, \tau_1, \tau_2, \ldots, \tau_n\}$. Then

$$\begin{aligned}\frac{d}{dt}V(\cdot) \leq & -\frac{1}{2}\alpha b y^2 - az^2 - \frac{1}{2}\left[\alpha b - \left\{\alpha\sum_{i=1}^{n}(B_iL_i) \ + (2\lambda+\alpha c)\right\}\tau_0\right]y^2 \\ & - \frac{1}{2}\left[2\alpha - \left\{\sum_{i=1}^{n}(B_iL_i) + 2\sum_{i=1}^{n}\lambda_i + c\right\}\tau_0\right]z^2.\end{aligned} \tag{2.13}$$

Thus, if

$$\tau_0 < \min\left\{\frac{\alpha b}{c + 2\alpha c + \alpha\sum_{i=1}^{n}(B_iL_i)}, \frac{2\alpha}{c + \sum_{i=1}^{n}[(2+\alpha)B_iL_i]}\right\}, \tag{2.14}$$

then

$$\frac{d}{dt}V(\cdot) \leq -\frac{1}{2}\alpha b y^2 - az^2. \tag{2.15}$$

This completes the proof of Theorem 2.1 (Burton [62], Hale [63], and Krasovskiĭ [64]). □

Let $p(\cdot) = 0$.

Our second main result is given by the following theorem.

Theorem 2.2. *In addition to all the assumptions of Theorem 2.1, we assume that the condition*

$$\left|p(\cdot)\right| \leq \left|q(t)\right| \tag{2.16}$$

holds, where $|q| \in L^1(0,\infty)$. *If*

$$\tau_0 < \min\left\{ \frac{\alpha b}{c + 2\alpha c + \alpha \sum_{i=1}^{n}(B_i L_i)}, \frac{2\alpha}{c + \sum_{i=1}^{n}[(2+\alpha)B_i L_i]} \right\}, \tag{2.17}$$

then, there exists a finite positive constant K *such that the solution* $x(t)$ *of* (1.1) *defined by the initial function*

$$x(t) = \phi(t), \qquad x'(t) = \phi'(t), \qquad x''(t) = \phi''(t) \tag{2.18}$$

satisfies

$$|x(t)| \leq K, \qquad \left|x'(t)\right| \leq K, \qquad \left|x''(t)\right| \leq K \tag{2.19}$$

for all $t \geq t_0$, *where* $\phi \in C^2([t_0 - r, t_0], \Re)$.

Proof. Under the assumptions of Theorem 2.2, the time derivative of the Lyapunov-Krasovskii functional $V(\cdot)$ satisfies

$$\frac{d}{dt}V(\cdot) \leq -\frac{1}{2}\alpha b y^2 - az^2 + (\alpha y + z)p(\cdot). \tag{2.20}$$

Using the estimates $|m| < 1 + m^2$ and $D_4(x^2 + y^2 + z^2) \leq V(\cdot)$, it follows that

$$\begin{aligned} \frac{d}{dt}V(\cdot) &\leq (\alpha|y| + |z|)\left|p(\cdot)\right| \\ &\leq D_5\left(2 + y^2 + z^2\right)\left|q(t)\right| \\ &\leq 2D_5\left|q(t)\right| + D_5 D_4^{-1} V(\cdot)\left|q(t)\right|, \end{aligned} \tag{2.21}$$

where $D_5 = \max\{1, \alpha\}$.

Integrating the above estimate from 0 to t, using the assumption $|q| \in L^1(0,\infty)$ and the Gronwall-Bellman inequality (see Gronwall [65] and Mitrinović [66]), we can conclude that all solutions of (1.1) are bounded. □

Example 2.3. Consider the nonlinear differential equation of the third order with two deviating arguments as follows:

$$\begin{aligned} x'''(t) + \left(11 + \frac{1}{1+t^2}\right)x''(t) + 2\left(1+e^{-t}\right)x'(t-\tau_1) + 2\left(2+e^{-t}\right)x'(t-\tau_2) \\ + 4x'(t) + x(t-\tau) + \operatorname{arc}tgx(t-\tau) = \frac{4}{1+t^2+x^2(t)+x^2(t-\tau_1)+x'^2(t-\tau_2)}. \end{aligned} \tag{2.22}$$

Writing (2.22) as a system of first order equations, we obtain

$$\begin{aligned} x' &= y, \qquad y' = z, \\ z' &= -\left(11 + \frac{1}{1+t^2}\right)z - 2\left(5+2e^{-t}\right)y - x - \operatorname{arc}tgx \\ &\quad + 2\left(1+e^{-t}\right)\int_{t-\tau_1}^{t} z(s)ds + 2\left(2+e^{-t}\right)\int_{t-\tau_2}^{t} z(s)ds + \int_{t-\tau}^{t} y(s)ds \\ &\quad + \int_{t-\tau}^{t} \frac{y(s)}{1+x^2(s)}ds + \frac{4}{1+t^2+\cdots+y^2(t-\tau_2)}. \end{aligned} \tag{2.23}$$

It follows that (2.22) is special case of (1.1), and when we compare (2.22) with (1.1) we obtain the following estimates:

$$\begin{aligned} &a(t) = 11 + \frac{1}{1+t^2} \geq 11 = 2 \times 5 + 1, \quad \alpha = 5,\ a = 1, \\ &b_1(t) = 1 + \frac{1}{e^t}, \quad 1 \leq 1 + \frac{1}{e^t} \leq 2, \quad \beta_1 = 1, \quad B_1 = 2, \\ &g_1(y) = 2y, \quad g_1(0) = 0, \quad \frac{g_1(y)}{y} = 2 = b_1, \quad (y \neq 0), \\ &g_1'(y) = 2 = L_1, \qquad \int_0^y g_1(\eta)d\eta = \int_0^y 2\eta\, d\eta = y^2, \\ &b_2(t) = 2 + \frac{1}{e^t}, \quad 2 \leq 2 + \frac{1}{e^t} \leq 3, \quad \beta_2 = 2, \quad B_2 = 3, \\ &g_2(y) = 2y, \quad g_2(0) = 0, \quad \frac{g_2(y)}{y} = 2 = b_2, \quad (y \neq 0), \end{aligned}$$

$$
\begin{gathered}
g_2'(y) = 2 = L_2, \qquad \int_0^y g_2(\eta)d\eta = \int_0^y 2\eta\, d\eta = y^2, \\
g(y) = 4y, \quad g(0) = 0, \quad \frac{g(y)}{y} = 4 = b, \quad (y \neq 0), \\
h(x) = x + \operatorname{arc}tgx, \qquad h(0) = 0, \qquad h'(x) = 1 + \frac{1}{1+x^2}, \\
0 < 2^{-1} < h'(x) \leq 2, \qquad c_1 = 2^{-1}, \qquad c = 2, \\
a'(t) = -\frac{2t}{(1+t^2)^2}, \quad b_1'(t) = -\frac{1}{e^t}, \quad b_2'(t) = -\frac{1}{e^t}, \quad (t \geq 0), \\
p(t,x,\ldots,z) = \frac{4}{1+t^2+\cdots+y^2(t-\tau_2)} \leq \frac{4}{1+t^2} = q(t).
\end{gathered}
\tag{2.24}
$$

In view of the above discussion, it follows that

$$
\begin{gathered}
\alpha b - c = 4 > 0, \\
\left[\sum_{i=1}^{2}\{\alpha b_i b_i(t)\} - c\right] y^2 \geq 2^{-1}\alpha a'(t)y^2 + \sum_{i=1}^{2} b_i'(t)\int_0^y g_i(\eta)d\eta \\
= [\alpha b_1 b_1(t) + \alpha b_2 b_2(t) - c]y^2 = \left[5\left(1+e^{-t}\right) + 10\left(2+e^{-t}\right) - 4\right]y^2 \\
= \left(21 + 15e^{-t}\right)y^2, \quad (t \geq 0), \\
\frac{1}{2}\alpha a'(t)y^2 + b_1'(t)\int_0^y g_1(\eta)d\eta + b_1'(t)\int_0^y g_1(\eta)d\eta \\
= -\left[\frac{5t}{(1+t^2)^2}\right]y^2 - 2e^{-t}y^2, \quad (t \geq 0), \\
\left(21 + 15e^{-t}\right)y^2 \geq -\left[\frac{5t}{(1+t^2)^2}\right] - 2e^{-t}y^2, \\
\int_0^\infty |q(s)|ds = \int_0^\infty \frac{4}{1+s^2}ds = 2\pi < \infty,
\end{gathered}
\tag{2.25}
$$

that is, $|q| \in L^1(0,\infty)$, and

$$
\tau_0 < \min\left\{\frac{\alpha b}{c + 2\alpha c + \alpha\sum_{i=1}^{2}(B_i L_i)}, \frac{2\alpha}{c + \sum_{i=1}^{2}[(2+\alpha)B_i L_i]}\right\} = \min\left\{\frac{5}{18}, \frac{5}{36}\right\} = \frac{5}{36}. \tag{2.26}
$$

Thus, all the assumptions of Theorems 2.1 and 2.2 hold. This shows that the zero solution of (2.22) is stable and all solutions of the same equation are bounded, when $p(\cdot) = 0$ and $\neq 0$, respectively.

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