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Research Article

On Some Solvable Difference Equations and Systems of Difference Equations

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Here, we give explicit formulae for solutions of some systems of difference equations, which extend some very particular recent results in the literature and give natural explanations for them, which were omitted in the previous literature.

1. Introduction

Recently, there has been a great interest in difference equations and systems (see, e.g., [1–25]), and among them in those ones which can be solved explicitly (see, e.g., [1–5, 9–11, 15, 16, 18–24] and the related references therein). For some classical results in the topic see, for example, [7].

Beside the above-mentioned papers, there are some papers which give formulae of some very particular equations and systems which are proved by induction, but without any explanation how these formulae are obtained and how these authors came across the equations and systems. Our explanation of such a formula that we gave in [10] has re-attracted attention to solvable difference equations.

Our aim here is to give theoretical explanations for some of the formulae recently appearing in the literature, as well as to give some extensions of their equations.

Before we formulate our results, we would like to say that the system of difference equations

$$x_{n+1} = \frac{a x_{n-1}}{b + c y_n x_{n-1}}, \qquad y_{n+1} = \frac{\alpha y_{n-1}}{\beta + \gamma x_n y_{n-1}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $a, b, c, \alpha, \beta,$ and γ are real numbers, was completely solved in [15], that is, we found formulae for all well-defined solutions of system (1.1).

2. Scaling Indices

In the recent paper [25] were given some formulae for the solutions of the following systems of difference equations:

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-3} y_{n-1}}, \qquad y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3} x_{n-1}}, \quad n \in \mathbb{N}_0. \tag{2.1}$$

Now we show that the results regarding system (2.1) easily follow from known ones. Indeed, if we use the change of variables

$$x_{2n+i} = u_n^{(i)}, \qquad y_{2n+i} = v_n^{(i)}, \quad n \geq -2, \ i = 1, 2, \tag{2.2}$$

then the systems in (2.1) are reduced to the next systems

$$u_{n+1}^{(i)} = \frac{u_{n-1}^{(i)}}{\pm 1 \pm u_{n-1}^{(i)} v_n^{(i)}}, \qquad v_{n+1}^{(i)} = \frac{v_{n-1}^{(i)}}{\pm 1 \pm v_{n-1}^{(i)} u_n^{(i)}}, \quad n \geq -1, \ i = 1, 2. \tag{2.3}$$

This means that $(u_n^{(i)}, v_n^{(i)})_{n \geq -2}$, $i = 1, 2$, are two (independent) solutions of the systems of difference equations

$$x_{n+1} = \frac{x_{n-1}}{\pm 1 \pm x_{n-1} y_n}, \qquad y_{n+1} = \frac{y_{n-1}}{\pm 1 \pm y_{n-1} x_n}, \quad n \geq -1, \ i = 1, 2. \tag{2.4}$$

However, all the systems of difference equations in (2.4) are particular cases of system (1.1). Hence, formulae for the solutions of systems (2.1) given in [25] follow directly from those in [15].

2.1. An Extension of Systems (2.1)

Systems (2.1) can be extended as follows:

$$x_{n+1} = \frac{ax_{n-2k+1}}{b + cx_{n-2k+1}y_{n-k+1}}, \qquad y_{n+1} = \frac{\alpha y_{n-2k+1}}{\beta + \gamma y_{n-2k+1}x_{n-k+1}}, \quad n \in \mathbb{N}_0, \tag{2.5}$$

where k is a fixed natural number.

If we use the change of variables

$$x_{mk+i} = u_m^{(i)}, \qquad y_{mk+i} = v_m^{(i)}, \quad m \geq -2, \ i = 1, \ldots, k, \tag{2.6}$$

then system (2.5) is reduced to the following k systems of difference equations:

$$u_{n+1}^{(i)} = \frac{au_{n-1}^{(i)}}{b + cu_{n-1}^{(i)}v_n^{(i)}}, \qquad v_{n+1}^{(i)} = \frac{\alpha v_{n-1}^{(i)}}{\beta + \gamma v_{n-1}^{(i)}u_n^{(i)}}, \quad n \geq -1, \tag{2.7}$$

$i = 1, \ldots, k$. This means that $(u_n^{(i)}, v_n^{(i)})_{n\geq -2}$, $i = 1, \ldots, k$, are k (independent) solutions of system (1.1), and solutions of system (2.5) are obtained by interlacing solutions of systems (2.7), $i = 1, \ldots, k$.

For example, a natural extension of the systems in (2.1) is obtained for taking $k = 3$, $b/a = \pm 1$, $c/a = \pm 1$, $\beta/\alpha = \pm 1$, and $\gamma/\alpha = \pm 1$ in (2.5), that is, the system becomes

$$x_{n+1} = \frac{x_{n-5}}{\pm 1 \pm x_{n-5}y_{n-2}}, \qquad y_{n+1} = \frac{y_{n-5}}{\pm 1 \pm y_{n-5}x_{n-2}}, \quad n \in \mathbb{N}_0. \tag{2.8}$$

In this way it can be obtained countable many, at first sight different, systems of difference equations. Systems of difference equations in (2.1) are artificially obtained in this way. This method can be applied to any equation or system of difference equations, and one can get papers with putative "new" results.

3. Some Third-Order Systems of Difference Equations Related to (1.1)

The following third-order systems of difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \qquad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \qquad z_{n+1} = \frac{1}{z_n y_n}, \tag{3.1}$$

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \qquad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \qquad z_{n+1} = \frac{z_{n-1}}{y_n z_{n-1} - 1}, \tag{3.2}$$

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \qquad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \qquad z_{n+1} = \frac{x_n}{z_{n-1} y_n}, \tag{3.3}$$

$n \in \mathbb{N}_0$, have been studied recently (see [6] and the references therein).

As is directly seen, the first two equations in systems (3.1)–(3.3) are the same, and they form a particular case of system (1.1) which is solved in [15].

Since we know solutions for x_n and y_n, it is only needed to find explicit solutions for z_n in the third equations in systems (3.1)–(3.3), that is, in all three equations, the only unknown sequence is z_n. The joint feature for all three cases is that z_n can be solved in closed form.

Now we discuss systems of difference equations given in (3.1)–(3.3).

Case of System (3.1)

From the third equation in (3.1), we get

$$z_{n+1} = \frac{1}{z_n y_n} = \frac{y_{n-1}}{y_n} z_{n-1}, \quad n \in \mathbb{N}, \tag{3.4}$$

from which it follows that

$$z_{2n+i} = z_i \prod_{j=1}^{n} \frac{y_{2j+i-2}}{y_{2j+i-1}}, \quad n \in \mathbb{N}_0,\ i = 0, 1. \tag{3.5}$$

Case of System (3.3)

From the third equation in (3.3), we get

$$z_{n+1} = \frac{x_n}{z_{n-1} y_n} = \frac{x_n y_{n-2}}{x_{n-2} y_n} z_{n-3}, \quad n \geq 2, \tag{3.6}$$

from which it follows that

$$z_{4n+i} = z_i \prod_{j=1}^{n} \frac{x_{4j+i-1} y_{4j+i-3}}{x_{4j+i-3} y_{4j+i-1}}, \quad n \in \mathbb{N}_0,\ i = -1, 0, 1, 2. \tag{3.7}$$

Remark 3.1. The third equations in systems (3.1) and (3.3) are particular cases of the following difference equation (up to the shifting indices):

$$z_n = \frac{a_n}{z_{n-k}}, \quad n \in \mathbb{N}_0, \tag{3.8}$$

where $k \in \mathbb{N}$. From (3.8), it follows that

$$z_n = \frac{a_n}{a_{n-k}} z_{n-2k}, \quad n \geq k, \tag{3.9}$$

and consequently,

$$z_{2mk+i} = z_{i-2k} \prod_{j=0}^{m} \frac{a_{2kj+i}}{a_{2kj-k+i}}, \quad m \in \mathbb{N}_0,\ i = k, k+1, \ldots, 3k-1. \tag{3.10}$$

Case of System (3.2)

If we use the change of variables $v_n = 1/z_n$, the third equation in (3.2) becomes

$$v_{n+1} = -v_{n-1} + y_n, \quad n \in \mathbb{N}_0. \tag{3.11}$$

Hence,

$$v_{2n+i} = -v_{2(n-1)+i} + y_{2n+i-1}, \quad n \in \mathbb{N}_0,\ i = 1,2, \tag{3.12}$$

from which it follows that

$$v_{2n+i} = (-1)^{n+1} v_{i-2} + \sum_{j=0}^{n} (-1)^{n-j} y_{2j+i-1}, \tag{3.13}$$

so that

$$z_{2n+i} = \frac{z_{i-2}}{(-1)^{n+1} + z_{i-2} \sum_{j=0}^{n} (-1)^{n-j} y_{2j+i-1}}. \tag{3.14}$$

Remark 3.2. The third equation in system (3.2) is a particular case of the following difference equation:

$$z_n = \frac{a_n z_{n-k}}{b_n z_{n-k} + c_n}, \quad n \in \mathbb{N}_0, \tag{3.15}$$

which, by the change of variables $z_n = 1/v_n$, is transformed into

$$v_n = \frac{c_n}{a_n} v_{n-k} + \frac{b_n}{a_n}, \quad n \in \mathbb{N}_0, \tag{3.16}$$

from which it follows that

$$v_{km+i} = \frac{c_{km+i}}{a_{km+i}} v_{k(m-1)+i} + \frac{b_{km+i}}{a_{km+i}}, \quad m \in \mathbb{N}_0,\ i = 0,1,\dots,k-1, \tag{3.17}$$

so by a well-known formula, we have that

$$v_{km+i} = v_{i-k} \prod_{j=0}^{m} \frac{c_{kj+i}}{a_{kj+i}} + \sum_{j=0}^{m} \frac{b_{kj+i}}{a_{kj+i}} \prod_{l=j+1}^{m} \frac{c_{kl+i}}{a_{kl+i}}, \tag{3.18}$$

for $m \in \mathbb{N}_0$ and $i = 0,1,\dots,k-1$, and consequently,

$$z_{mk+i} = \frac{z_{i-k}}{\prod_{j=0}^{m} (c_{kj+i}/a_{kj+i}) + z_{i-k} \sum_{j=0}^{m} (b_{kj+i}/a_{kj+i}) \prod_{l=j+1}^{m} (c_{kl+i}/a_{kl+i})}, \tag{3.19}$$

for $m \in \mathbb{N}_0$ and $i = 0,1,\dots,k-1$.

Remark 3.3. Note that (3.16) suggests that the third equation in (3.2) can be also of the form

$$z_n = a_n z_{n-k} + b_n, \quad n \in \mathbb{N}_0, \tag{3.20}$$

where $k \in \mathbb{N}$ is fixed, that is, to be an equation which consists of k (independent) linear first-order difference equation, which is solvable. In fact, the third equation in systems (3.1)–(3.3) can be any difference equation which can be solved in z_n, and in this way we can obtain numerous putative "new" results.

4. A Generalization of System (1.1)

Consider the following system of difference equations:

$$x_{n+1} = g^{-1}\left(\frac{g(x_{n-1})}{ah(y_n)g(x_{n-1}) + b}\right), \qquad y_{n+1} = h^{-1}\left(\frac{h(y_{n-1})}{cg(x_n)h(y_{n-1}) + d}\right), \quad n \in \mathbb{N}_0, \tag{4.1}$$

where $g, h : \mathbb{R} \to \mathbb{R}$ are increasing functions such that

$$g(0) = h(0) = 0, \tag{4.2}$$

$$g(x)x > 0, \qquad h(x)x > 0, \quad \text{for } x \neq 0. \tag{4.3}$$

Now we will find formulae for all well-defined solutions of system (4.1), that is, for the solutions (x_n, y_n), $n \geq -1$, such that

$$ah(y_n)g(x_{n-1}) + b \neq 0, \qquad cg(x_n)h(y_{n-1}) + d \neq 0, \tag{4.4}$$

for every $n \in \mathbb{N}_0$.

If $x_{-1} = 0$, then from (4.1), (4.2), and (4.3), and by the method of induction, we get $x_{2n+1} = 0$, $n \in \mathbb{N}_0$. Also, if $x_0 = 0$, then from (4.1), (4.2), and (4.3), and by the method of induction, we get $x_{2n} = 0$, $n \in \mathbb{N}_0$. Similarly, if $y_{-1} = 0$, then we get $y_{2n+1} = 0$, $n \in \mathbb{N}_0$, while if $y_0 = 0$, then we get $y_{2n} = 0$, $n \in \mathbb{N}_0$.

If $x_{n_0} = 0$ for some $n_0 \in \mathbb{N}$, then from (4.1)–(4.3) it follows that $x_{n_0-2k} = 0$, for each $k \in \mathbb{N}_0$ such that $n_0 - 2k \geq -1$. Hence, in this case we have that $x_{-1} = 0$ or $x_0 = 0$. Similarly, if $y_{n_1} = 0$ for some $n_1 \in \mathbb{N}$, then from (4.1)–(4.3) it follows that $y_{n_1-2k} = 0$, for each $k \in \mathbb{N}_0$ such that $n_1 - 2k \geq -1$. Hence, in this case we have that $y_{-1} = 0$ or $y_0 = 0$. Thus, in both cases we arrive at a situation explained in the previous paragraph.

Hence, from now on, we assume that none of the initial values x_{-1}, x_0, y_{-1}, and y_0 is equal to zero. Then, for every well-defined solution of system (4.1), we have that $x_n \neq 0$ and $y_n \neq 0$, for every $n \geq -1$, and consequently $g(x_n) \neq 0$ and $h(y_n) \neq 0$, for every $n \geq -1$.

Let

$$v_n = \frac{1}{g(x_n)h(y_{n-1})}, \qquad u_n = \frac{1}{g(x_{n-1})h(y_n)}, \tag{4.5}$$

then by taking function g to the first equation in system (4.1) and function h to the second one, then multiplying the first equation in such obtained system by $h(y_n)$ and the second by $g(x_n)$, system (4.1) is transformed into:

$$v_{n+1} = bu_n + a, \qquad u_{n+1} = dv_n + c, \quad n \in \mathbb{N}_0, \tag{4.6}$$

from which it follows that

$$\begin{aligned} v_{n+1} &= bdv_{n-1} + bc + a, \quad n \in \mathbb{N}, \\ u_{n+1} &= bdu_{n-1} + ad + c, \quad n \in \mathbb{N}. \end{aligned} \tag{4.7}$$

Hence, if $bd \neq 1$, we have that

$$\begin{aligned} v_{2n} &= \frac{(bd)^n}{g(x_0)h(y_{-1})} + (bc + a)\frac{1-(bd)^n}{1-bd}, \quad n \in \mathbb{N}_0, \\ v_{2n+1} &= \frac{(bd)^n}{g(x_1)h(y_0)} + (bc + a)\frac{1-(bd)^n}{1-bd}, \quad n \in \mathbb{N}_0, \end{aligned} \tag{4.8}$$

while if $bd = 1$, we have that

$$\begin{aligned} v_{2n} &= \frac{1}{g(x_0)h(y_{-1})} + (bc + a)n, \quad n \in \mathbb{N}_0, \\ v_{2n+1} &= \frac{1}{g(x_1)h(y_0)} + (bc + a)n, \quad n \in \mathbb{N}_0. \end{aligned} \tag{4.9}$$

We have also that

$$\begin{aligned} u_{2n} &= \frac{(bd)^n}{h(y_0)g(x_{-1})} + (ad + c)\frac{1-(bd)^n}{1-bd}, \quad n \in \mathbb{N}_0, \\ u_{2n+1} &= \frac{(bd)^n}{h(y_1)g(x_0)} + (ad + c)\frac{1-(bd)^n}{1-bd}, \quad n \in \mathbb{N}_0, \end{aligned} \tag{4.10}$$

while if $bd = 1$, we have that

$$\begin{aligned} u_{2n} &= \frac{1}{h(y_0)g(x_{-1})} + (ad + c)n, \quad n \in \mathbb{N}_0, \\ u_{2n+1} &= \frac{1}{h(y_1)g(x_0)} + (ad + c)n, \quad n \in \mathbb{N}_0. \end{aligned} \tag{4.11}$$

From (4.5), we have that

$$\begin{aligned} g(x_n) &= \frac{1}{v_n h(y_{n-1})} = \frac{u_{n-1}}{v_n} g(x_{n-2}), \quad n \in \mathbb{N}, \\ h(y_n) &= \frac{1}{u_n g(x_{n-1})} = \frac{v_{n-1}}{u_n} h(y_{n-2}), \quad n \in \mathbb{N}. \end{aligned} \tag{4.12}$$

Using the relations (4.12), we get

$$x_{2n} = g^{-1}\left(g(x_0)\prod_{j=1}^{n}\frac{u_{2j-1}}{v_{2j}}\right), \quad n \in \mathbb{N}_0,$$
$$x_{2n+1} = g^{-1}\left(g(x_{-1})\prod_{j=0}^{n}\frac{u_{2j}}{v_{2j+1}}\right), \quad n \in \mathbb{N}_0,$$
$$y_{2n} = h^{-1}\left(h(y_0)\prod_{j=1}^{n}\frac{v_{2j-1}}{u_{2j}}\right), \quad n \in \mathbb{N}_0,$$
$$y_{2n+1} = h^{-1}\left(h(y_{-1})\prod_{j=0}^{n}\frac{v_{2j}}{u_{2j+1}}\right), \quad n \in \mathbb{N}_0. \tag{4.13}$$

Example 4.1. If we choose $g(t) = t^{2k+1}$ and $h(t) = t^{2l+1}$ for some $k, l \in \mathbb{N}_0$, then conditions (4.2) and (4.3) are obviously satisfied, and system (4.1) can be written in the form

$$x_{n+1} = \left(\frac{x_{n-1}^{2k+1}}{ay_n^{2l+1}x_{n-1}^{2k+1} + b}\right)^{1/(2k+1)}, \qquad y_{n+1} = \left(\frac{y_{n-1}^{2l+1}}{cx_n^{2k+1}y_{n-1}^{2l+1} + d}\right)^{1/(2l+1)}, \quad n \in \mathbb{N}_0, \tag{4.14}$$

and from (4.13), we have that its solutions are given by

$$x_{2n} = x_0\prod_{j=1}^{n}\left(\frac{u_{2j-1}}{v_{2j}}\right)^{1/(2k+1)}, \quad n \in \mathbb{N}_0,$$
$$x_{2n+1} = x_{-1}\prod_{j=0}^{n}\left(\frac{u_{2j}}{v_{2j+1}}\right)^{1/(2k+1)}, \quad n \in \mathbb{N}_0,$$
$$y_{2n} = y_0\prod_{j=1}^{n}\left(\frac{v_{2j-1}}{u_{2j}}\right)^{1/(2l+1)}, \quad n \in \mathbb{N}_0,$$
$$y_{2n+1} = y_{-1}\prod_{j=0}^{n}\left(\frac{v_{2j}}{u_{2j+1}}\right)^{1/(2l+1)}, \quad n \in \mathbb{N}_0. \tag{4.15}$$

Remark 4.2. System (4.1) can be generalized by using the method of scaling indices from Section 2.1, that is, the following system is also solvable:

$$x_{n+1} = g^{-1}\left(\frac{g(x_{n-2k+1})}{ah(y_{n-k+1})g(x_{n-2k+1}) + b}\right), \qquad y_{n+1} = h^{-1}\left(\frac{h(y_{n-2k+1})}{cg(x_{n-k+1})h(y_{n-2k+1}) + d}\right), \tag{4.16}$$

$n \in \mathbb{N}_0$, where $k \in \mathbb{N}$ and $g, h : \mathbb{R} \to \mathbb{R}$ are increasing functions satisfying conditions (4.2) and (4.3).

It is easy to see that the change of variables in (2.6) leads to the following k systems of difference equations:

$$u_{n+1}^{(i)} = g^{-1}\left(\frac{g\left(u_{n-1}^{(i)}\right)}{ah\left(v_n^{(i)}\right)g\left(u_{n-1}^{(i)}\right)+b}\right), \qquad v_{n+1}^{(i)} = h^{-1}\left(\frac{h\left(v_{n-1}^{(i)}\right)}{cg\left(u_n^{(i)}\right)h\left(v_{n-1}^{(i)}\right)+d}\right), \quad n \geq -1, \tag{4.17}$$

for $i = 1, \dots, k$.

Remark 4.3. Well-defined solutions of the following system of difference equations:

$$x_{n+1} = g^{-1}\left(\frac{g(x_{n-1})}{a_n h(y_n) g(x_{n-1}) + b_n}\right), \qquad y_{n+1} = h^{-1}\left(\frac{h(y_{n-1})}{c_n g(x_n) h(y_{n-1}) + d_n}\right), \quad n \in \mathbb{N}_0, \tag{4.18}$$

where $g, h : \mathbb{R} \to \mathbb{R}$ are increasing functions satisfying conditions (4.2) and (4.3) and a_n, b_n, c_n, and d_n, $n \in \mathbb{N}_0$, are real sequences, can be found similarly. We omit the details.

5. Solutions of a Generalization of a Recent Equation

Explaining some recent formulae appearing in the literature, in our recent paper [24], we have found formulae for well-defined solutions of the following difference equation:

$$x_{n+1} = \frac{x_n x_{n-k}}{x_{n-k+1}(a + b x_n x_{n-k})}, \quad n \in \mathbb{N}_0, \tag{5.1}$$

where $k \in \mathbb{N}$ and the parameters a, b as well as initial values x_{-i}, $i = \overline{0, k}$ are real numbers.

Equation (5.1) can be extended naturally in the following way:

$$x_{n+1} = \frac{x_n x_{n-k}}{x_{n-k+1}(a_n + b_n x_n x_{n-k})}, \quad n \in \mathbb{N}_0, \tag{5.2}$$

where $k \in \mathbb{N}$ and the sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$, as well as initial values x_{-i}, $i = \overline{0, k}$, are real numbers.

Employing the change of variables

$$y_n = \frac{1}{x_n x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{5.3}$$

Equation (5.2) is transformed into the linear first-order difference equation

$$y_{n+1} = a_n y_n + b_n, \quad n \in \mathbb{N}_0, \tag{5.4}$$

whose general solution is

$$y_n = y_0 \prod_{l=0}^{n-1} a_l + \sum_{s=0}^{n-1} b_s \prod_{l=s+1}^{n-1} a_l, \quad n \in \mathbb{N}_0. \tag{5.5}$$

From (5.3), we have that

$$x_n = \frac{1}{y_n x_{n-k}} = \frac{y_{n-k}}{y_n} x_{n-2k}, \tag{5.6}$$

for $n \ge k$, which yields

$$x_{2km+i} = x_{i-2k} \prod_{j=0}^{m} \frac{y_{(2j-1)k+i}}{y_{2jk+i}}, \tag{5.7}$$

for every $m \in \mathbb{N}_0$ and $i \in \{k, k+1, \ldots, 3k-1\}$.

Using (5.5) in (5.7), we get

$$x_{2km+i} = x_{i-2k} \prod_{j=0}^{m} \frac{y_0 \prod_{l=0}^{(2j-1)k+i-1} a_l + \sum_{i=s}^{(2j-1)k+i-1} b_s \prod_{l=s+1}^{(2j-1)k+i-1} a_l}{y_0 \prod_{l=0}^{2jk+i-1} a_l + \sum_{s=0}^{2jk+i-1} b_s \prod_{l=s+1}^{2jk+i-1} a_l}, \tag{5.8}$$

for every $m \in \mathbb{N}_0$ and $i \in \{k, k+1, \ldots, 3k-1\}$.

Formula (5.8) generalizes the main formulae obtained in our paper [24].

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