

## *Research Article*

# **Random Attractors for Stochastic Retarded Lattice Dynamical Systems**

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This paper is devoted to a stochastic retarded lattice dynamical system with additive white noise. We extend the method of tail estimates to stochastic retarded lattice dynamical systems and prove the existence of a compact global random attractor within the set of tempered random bounded sets.

## **1. Introduction**

Lattice dynamical systems (LDSs) arise naturally in a wide variety of applications in science and engineering where the spatial structure has a discrete character. Among such examples are brain science [1], chemical reaction [2], material science [3], electrical engineering [4], laser systems [5], pattern recognition [6], complex network [7], and many others. On the other hand, LDSs also appear as spatial discretizations of partial differential equations on unbounded domains.

There are many works concerning deterministic LDSs. For example, the traveling wave solutions were studied in [8, 9], the chaotic properties of solutions were examined by [6, 10], the long-time behavior of LDSs was investigated by [11–17]. In particular, Bates et al. [11] established the first result on the existence of a global attractor for LDSs. Wang [13], Zhou and Shi [14] used the idea of tail estimates on solutions and obtained, respectively, some sufficient and necessary conditions for the existence of a global attractor for autonomous LDSs. Later, the method of tail estimates is extended to nonautonomous LDSs [15–17].

It is noted that an evolutionary system in reality is usually affected by external perturbations which in many cases are of great uncertainty or random influence. These

random effects are not only introduced to compensate for the defects in some deterministic models, but also are often rather intrinsic phenomena. Therefore, it is of prime importance to take into account these random effects in some models, and this has led to stochastic differential equations. Random attractors for stochastic partial differential equations were first introduced by Crauel and Flandoli [18], Flandoli and Schmalfuss [19], with notable developments given in [20–25] and others. Bates et al. [26] initiated the study of random attractors for stochastic LDSs. Since then, many works have been done for the existence of random attractors for stochastic LDSs, see, for example, [27–34] and the references therein. Similarly to deterministic LDSs, the method of tail estimates also plays a key role in the study of the existence of random attractors for stochastic LDSs.

On the other hand, in the natural world, the current rate of change of the state in an evolutionary system always depends on the historical status of the system. Then, it is more reasonable to describe the evolutionary systems by functional differential equations. Many papers are devoted to the study of the asymptotic behavior of deterministic functional differential equations, see, for example, [35–41] and the references therein. Especially, Zhao and Zhou [40, 41] considered the asymptotic behavior of some deterministic retarded LDSs and extended the method of tail estimates to deterministic retarded LDSs. More recently, Yan et al. [42, 43] discussed the asymptotic behavior of some stochastic retarded LDSs with global Lipschitz nonlinearities.

Consider the Hilbert space

$$\ell^2 = \left\{ u = (u_i)_{i \in \mathbb{Z}} : u_i \in \mathbb{R}, \sum_{i \in \mathbb{Z}} |u_i|^2 < \infty \right\}, \quad (1.1)$$

whose inner product and norm are given by

$$(u, v) = \sum_{i \in \mathbb{Z}} u_i v_i, \quad \|u\|^2 = \sum_{i \in \mathbb{Z}} u_i^2, \quad (1.2)$$

for all  $u = (u_i)_{i \in \mathbb{Z}}, v = (v_i)_{i \in \mathbb{Z}} \in \ell^2$ . For  $\nu > 0$ , let  $\mathcal{C} := C([- \nu, 0]; \ell^2)$  denote the Banach space of all continuous functions  $\xi : [- \nu, 0] \rightarrow \ell^2$  endowed with the supremum norm  $\|\xi\|_{\mathcal{C}} = \sup_{s \in [- \nu, 0]} \|\xi(s)\|$ . For any real numbers  $a \leq b, t \in [a, b]$  and any continuous function  $u : [a - \nu, b] \rightarrow \ell^2$ ,  $u^t$  denotes the element of  $\mathcal{C}$  given by  $u^t(s) = u(t + s)$  for  $s \in [- \nu, 0]$ .

In this paper, we investigate the long time behavior of the following stochastic retarded LDS:

$$du_i(t) = ((u_{i-1} - 2u_i + u_{i+1}) - \lambda_i u_i + f_i(u_i^t) + g_i)dt + a_i dw_i(t), \quad t > 0, i \in \mathbb{Z}, \quad (1.3)$$

with initial data

$$u_i(t) = u_i^0(t), \quad t \in [- \nu, 0], i \in \mathbb{Z}, \quad (1.4)$$

where  $u = (u_i)_{i \in \mathbb{Z}} \in \ell^2$ ,  $(\lambda_i)_{i \in \mathbb{Z}}$  is a bounded positive constant sequence,  $f = (f_i)_{i \in \mathbb{Z}} : \mathcal{C} \rightarrow \ell^2$  is a nonlinear mapping satisfying local Lipschitz condition,  $g = (g_i)_{i \in \mathbb{Z}} \in \ell^2$ ,  $a = (a_i)_{i \in \mathbb{Z}} \in \ell^2$ , and  $\{w_i : i \in \mathbb{Z}\}$  are independent two-sided real-valued Wiener processes on a probability space which will be specified later.

It is worth mentioning that in the absence of the white noise, the existence of a global attractor for (1.3)-(1.4) was established in [40]. The main contribution of this paper is to extend the method of tail estimates to stochastic retarded LDSs and prove the existence of a random attractor for the infinite dimensional random dynamical system generated by stochastic retarded LDS (1.3)-(1.4). It is clear that our method can be used for a variety of other stochastic retarded LDSs, as it was for the nonretarded case.

The paper is organized as follows. In the next section, we recall some fundamental results on the existence of a pullback random attractor for random dynamical systems. In Section 3, we establish a necessary and sufficient condition for the relative compactness of sequences in  $C([-v, 0]; \ell^2)$ . In Section 4, we define a continuous random dynamical system for stochastic retarded LDS (1.3)-(1.4). The existence of the random attractor for (1.3)-(1.4) is given in Section 5.

## 2. Preliminaries

In this section, we recall some basic concepts related to random attractors for random dynamical systems. The reader is referred to [18–21, 26, 44, 45] for more details.

Let  $(X, \|\cdot\|_X)$  be a separable Banach space with Borel  $\sigma$ -algebra  $\mathcal{B}(X)$  and  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space.

*Definition 2.1.*  $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t \in \mathbb{R}})$  is called a metric dynamical system if  $\vartheta : \mathbb{R} \times \Omega \rightarrow \Omega$  is  $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$ -measurable,  $\vartheta_0$  is the identity on  $\Omega$ ,  $\vartheta_{s+t} = \vartheta_t \circ \vartheta_s$  for all  $s, t \in \mathbb{R}$ , and  $\vartheta_t \mathbb{P} = \mathbb{P}$  for all  $t \in \mathbb{R}$ .

*Definition 2.2.* A set  $A \subset \Omega$  is called invariant with respect to  $(\vartheta_t)_{t \in \mathbb{R}}$ , if for all  $t \in \mathbb{R}$ , it holds

$$\vartheta_t^{-1} A = A. \quad (2.1)$$

*Definition 2.3.* A continuous random dynamical system on  $X$  over a metric dynamical system  $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t \in \mathbb{R}})$  is a mapping

$$\varphi : \mathbb{R}^+ \times \Omega \times X \longrightarrow X, \quad (t, \omega, x) \longrightarrow \varphi(t, \omega, x), \quad (2.2)$$

which is  $(\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$ -measurable, and for all  $\omega \in \Omega$ ,

- (i)  $\varphi(t, \omega, \cdot) : X \rightarrow X$  is continuous for all  $t \in \mathbb{R}^+$ ;
- (ii)  $\varphi(0, \omega, \cdot)$  is the identity on  $X$ ;
- (iii)  $\varphi(t + s, \omega, \cdot) = \varphi(t, \vartheta_s \omega, \cdot) \circ \varphi(s, \omega, \cdot)$  for all  $s, t \in \mathbb{R}^+$ .

*Definition 2.4.* A random set  $D$  is a multivalued mapping  $D : \Omega \rightarrow 2^X \setminus \emptyset$  such that for every  $x \in X$ , the mapping  $\omega \rightarrow d(x, D(\omega))$  is measurable, where  $d(x, B)$  is the distance between the element  $x$  and the set  $B \subset X$ . It is said that the random set is bounded (resp., closed or compact) if  $D(\omega)$  is bounded (resp., closed or compact) for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ .

*Definition 2.5.* A random variable  $r : \Omega \rightarrow (0, \infty)$  is called tempered with respect to  $(\vartheta_t)_{t \in \mathbb{R}}$ , if for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$

$$\lim_{t \rightarrow \infty} e^{-\beta t} r(\vartheta_{-t}\omega) = 0 \quad \forall \beta > 0. \quad (2.3)$$

A random set  $D$  is called tempered if  $D(\omega)$  is contained in a ball with center zero and tempered radius  $r(\omega)$  for all  $\omega \in \Omega$ .

*Remark 2.6.* If  $r > 0$  is tempered, then for any  $\tau \in \mathbb{R}$ ,  $\beta > 0$  and  $\mathbb{P}$ -a.e.  $\omega \in \Omega$

$$\lim_{t \rightarrow \infty} e^{-\beta t} r(\vartheta_{-t+\tau}\omega) = e^{-\beta \tau} \cdot \lim_{t \rightarrow \infty} e^{-\beta(t-\tau)} r(\vartheta_{-t+\tau}\omega) = 0. \quad (2.4)$$

Therefore, for any  $\tau \in \mathbb{R}$ ,  $r(\vartheta_\tau \cdot)$  is also tempered. Moreover, if for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ ,  $r(\vartheta_t\omega)$  is continuous in  $t$ , then for any  $\nu > 0$ ,  $\sup_{\sigma \in [-\nu, 0]} r(\vartheta_\sigma \cdot)$  is measurable and for all  $\beta > 0$  and  $\mathbb{P}$ -a.e.  $\omega \in \Omega$

$$\lim_{t \rightarrow \infty} e^{-\beta t} \sup_{\sigma \in [-\nu, 0]} r(\vartheta_{-t+\sigma}\omega) \leq \lim_{t \rightarrow \infty} e^{(\beta/2)(\nu-t)} \cdot \sup_{s \in (-\infty, 0]} \left\{ e^{(\beta/2)s} r(\vartheta_s\omega) \right\} = 0. \quad (2.5)$$

Hence, for any  $\nu > 0$ ,  $\sup_{\sigma \in [-\nu, 0]} r(\vartheta_\sigma \cdot)$  is also tempered.

*Remark 2.7.* If  $r > 0$  is tempered, then for any  $\alpha > 0$  and  $\mathbb{P}$ -a.e.  $\omega \in \Omega$

$$R(\omega) = \int_{-\infty}^0 e^{\alpha s} r(\vartheta_s\omega) ds < \infty. \quad (2.6)$$

Moreover,  $R$  is tempered, and if for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ ,  $r(\vartheta_t\omega)$  is continuous in  $t$ , then  $R(\vartheta_t\omega)$  is also continuous in  $t$  for such  $\omega$ .

Hereafter, we always assume that  $\varphi$  is a continuous random dynamical system over  $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t \in \mathbb{R}})$ , and  $\mathfrak{D}$  is a collection of random subsets of  $X$ .

*Definition 2.8.* A random set  $K$  is called a random absorbing set in  $\mathfrak{D}$  if for every  $B \in \mathfrak{D}$  and  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , there exists  $t_B(\omega) > 0$  such that

$$\varphi(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega)) \subseteq K(\omega) \quad \forall t \geq t_B(\omega). \quad (2.7)$$

*Definition 2.9.* A random set  $\mathcal{A}$  is called a  $\mathfrak{D}$ -random attractor ( $\mathfrak{D}$ -pullback attractor) for  $\varphi$  if the following hold:

- (i)  $\mathcal{A}$  is a random compact set;
- (ii)  $\mathcal{A}$  is strictly invariant, that is, for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$  and all  $t \geq 0$ ,

$$\varphi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\vartheta_t\omega); \quad (2.8)$$

(iii)  $\mathcal{A}$  attracts all sets in  $\mathfrak{D}$ , that is, for all  $B \in \mathfrak{D}$  and  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ ,

$$\lim_{t \rightarrow \infty} d(\varphi(t, \vartheta_{-t}\omega, B(\vartheta_{-t}\omega)), \mathcal{A}(\omega)) = 0, \quad (2.9)$$

where  $d$  is the Hausdorff semimetric given by  $d(E, F) = \sup_{x \in E} \inf_{y \in F} \|x - y\|_X$  for any  $E \subseteq X$  and  $F \subseteq X$ .

**Definition 2.10.**  $\varphi$  is said to be  $\mathfrak{D}$ -pullback asymptotically compact in  $X$  if for all  $B \in \mathfrak{D}$  and  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ ,  $\{\varphi(t_n, \vartheta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$  has a convergent subsequence in  $X$  whenever  $t_n \rightarrow \infty$ , and  $x_n \in B(\vartheta_{-t_n}\omega)$ .

The following existence result on a random attractor for a continuous random dynamical system can be found in [19, 26]. First, recall that a collection  $\mathfrak{D}$  of random subsets of  $X$  is called inclusion closed if whenever  $E$  is an arbitrary random set, and  $F$  is in  $\mathfrak{D}$  with  $E(\omega) \subset F(\omega)$  for all  $\omega \in \Omega$ , then  $E$  must belong to  $\mathfrak{D}$ .

**Proposition 2.11.** Let  $\mathfrak{D}$  be an inclusion-closed collection of random subsets of  $X$  and  $\varphi$  a continuous random dynamical system on  $X$  over  $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t \in \mathbb{R}})$ . Suppose that  $K \in \mathfrak{D}$  is a closed random absorbing set for  $\varphi$  in  $\mathfrak{D}$  and  $\varphi$  is  $\mathfrak{D}$ -pullback asymptotically compact in  $X$ . Then  $\varphi$  has a unique  $\mathfrak{D}$ -random attractor  $\mathcal{A}$  which is given by

$$\mathcal{A}(\omega) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \varphi(t, \vartheta_{-t}\omega, K(\vartheta_{-t}))}. \quad (2.10)$$

In this paper, we will take  $\mathfrak{D}$  as the collection of all tempered random subsets of  $\mathcal{C}$  and prove the stochastic retarded LDS has a  $\mathfrak{D}$ -random attractor.

### 3. Compactness Criterion in $C([-v, 0]; \ell^2)$

In this section, we provide a necessary and sufficient condition for the relative compactness of sequences in  $C([-v, 0]; \ell^2)$ , which will be used to establish the asymptotic compactness of the retarded LDS.

**Lemma 3.1.** Let  $u \in C([-v, 0]; \ell^2)$ . Then for every  $\varepsilon > 0$ , there exists  $N(\varepsilon) > 0$  such that for all  $k \geq N(\varepsilon)$ ,

$$\sup_{s \in [-v, 0]} \sum_{|i| \geq k} |u_i(s)|^2 < \varepsilon. \quad (3.1)$$

*Proof.* For every  $\varepsilon > 0$ , by virtue of the uniform continuity of  $u$ , there exist  $-v = s_0 < s_1 < s_2 < \dots < s_p = 0$  such that

$$\|u(s) - u(s_j)\| < \frac{\sqrt{\varepsilon}}{2}, \quad \text{for } s \in [s_{j-1}, s_j], \quad j = 1, 2, \dots, p. \quad (3.2)$$

Since for each  $s_j$ ,  $u(s_j) \in \ell^2$ , there exists  $N_j(\varepsilon) > 0$  such that for all  $k \geq N_j(\varepsilon)$ ,

$$\sum_{|i| \geq k} |u_i(s_j)|^2 < \frac{\varepsilon}{4}. \quad (3.3)$$

Take  $N(\varepsilon) = \max_{1 \leq j \leq p} N_j(\varepsilon)$ . Then for each  $s \in [-v, 0]$ , there exists  $j \in \{1, 2, \dots, p\}$  such that  $s \in [s_{j-1}, s_j]$ . Therefore, we get from (3.2) and (3.3) that for all  $k \geq N(\varepsilon)$ ,

$$\begin{aligned} \sum_{|i| \geq k} |u_i(s)|^2 &\leq 2 \sum_{|i| \geq k} |u_i(s_j)|^2 + 2 \sum_{|i| \geq k} |u_i(s) - u_i(s_j)|^2 \\ &\leq 2 \sum_{|i| \geq k} |u_i(s_j)|^2 + 2 \|u(s) - u(s_j)\|^2 < \varepsilon, \end{aligned} \quad (3.4)$$

which completes the proof.  $\square$

**Theorem 3.2.** *Let  $\mathcal{S} \subset C([-v, 0]; \ell^2)$ . Then  $\mathcal{S}$  is relative compact in  $C([-v, 0]; \ell^2)$  if and only if the following conditions are satisfied:*

- (i)  $\mathcal{S}$  is bounded in  $C([-v, 0]; \ell^2)$ ;
- (ii)  $\mathcal{S}$  is equicontinuous;
- (iii)  $\lim_{k \rightarrow \infty} \sup_{u=(u_i)_{i \in \mathbb{Z}} \in \mathcal{S}} \sup_{s \in [-v, 0]} \sum_{|i| \geq k} |u_i(s)|^2 = 0$ .

*Proof.* The proof is divided into two steps. We first show the necessity of the conditions and then prove the sufficiency.

(1) Assume that  $\mathcal{S}$  is relative compact in  $C([-v, 0]; \ell^2)$ . Then we want to show conditions (i), (ii), and (iii) hold. Clearly, in this case, by the Ascoli-Arzelà theorem,  $\mathcal{S}$  must be bounded and equicontinuous. So we only need to prove condition (iii).

Given  $\varepsilon > 0$ , since  $\mathcal{S}$  is relative compact, there exists a finite subset  $\mathcal{E}$  of  $\mathcal{S}$  such that the balls of radii  $\varepsilon/2$  centered at  $\mathcal{E}$  form a finite covering of  $\mathcal{S}$ , that is, for each  $u \in \mathcal{S}$ , there exists  $v \in \mathcal{E}$  such that

$$\sup_{s \in [-v, 0]} \|u(s) - v(s)\| < \frac{\varepsilon}{2}. \quad (3.5)$$

By Lemma 3.1, there exists  $K^*(\varepsilon) > 0$  such that for all  $v \in \mathcal{E}$ ,

$$\sup_{s \in [-v, 0]} \sum_{|i| \geq K^*(\varepsilon)} |v_i(s)|^2 < \frac{\varepsilon^2}{4}. \quad (3.6)$$

By (3.5) and (3.6), we find that for each  $u \in \mathcal{S}$ , there exists  $v \in \mathcal{E}$  such that

$$\begin{aligned} \sup_{s \in [-v, 0]} \sum_{|i| \geq K^*(\varepsilon)} |u_i(s)|^2 &\leq 2 \sup_{s \in [-v, 0]} \sum_{|i| \geq K^*(\varepsilon)} |u_i(s) - v_i(s)|^2 \\ &\quad + 2 \sup_{s \in [-v, 0]} \sum_{|i| \geq K^*(\varepsilon)} |v_i(s)|^2 < \varepsilon^2. \end{aligned} \quad (3.7)$$

Therefore, for all  $k \geq K^*(\varepsilon)$ , we have

$$\sup_{u=(u_i)_{i \in \mathbb{Z}} \in \mathcal{S}} \sup_{s \in [-\nu, 0]} \sum_{|i| \geq k} |u_i(s)|^2 \leq \varepsilon^2, \quad (3.8)$$

which implies condition (iii).

(2) Assume that conditions (i), (ii), and (iii) are valid. We want to prove that  $\mathcal{S}$  is relative compact in  $C([- \nu, 0]; \ell^2)$ . That is, given  $\varepsilon > 0$ , we want to show that  $\mathcal{S}$  has a finite covering of balls of radii  $\varepsilon$ . By condition (iii), we find that there exists  $K(\varepsilon) > 0$  such that for all  $u = (u_i)_{i \in \mathbb{Z}} \in \mathcal{S}$ ,

$$\sup_{s \in [-\nu, 0]} \sum_{|i| \geq K(\varepsilon)} |u_i(s)|^2 < \frac{\varepsilon^2}{4}. \quad (3.9)$$

Consider the set  $\mathcal{S}|_K = \{u|_K = (u_i)_{|i| \leq K(\varepsilon)} : u = (u_i)_{i \in \mathbb{Z}} \in \mathcal{S}\}$  in  $C([- \nu, 0]; \mathbb{R}^{2K(\varepsilon)+1})$ . By conditions (i) and (ii), we know that  $\mathcal{S}|_K$  is bounded and equicontinuous in  $C([- \nu, 0]; \mathbb{R}^{2K(\varepsilon)+1})$ . Then, by the Ascoli-Arzelà theorem, we obtain that  $\mathcal{S}|_K$  is relative compact in  $C([- \nu, 0]; \mathbb{R}^{2K(\varepsilon)+1})$  and hence there exists a finite subset  $\overline{\mathcal{L}}$  of  $\mathcal{S}|_K$  such that the balls of radii  $\varepsilon/2$  centered at  $\overline{\mathcal{L}}$  form a finite covering of  $\mathcal{S}|_K$ , that is, for each  $u|_K \in \mathcal{S}|_K$ , there exists  $v|_K \in \overline{\mathcal{L}}$  such that

$$\sup_{s \in [-\nu, 0]} \sum_{|i| \leq K(\varepsilon)} |u_i(s) - v_i(s)|^2 < \frac{\varepsilon^2}{4}. \quad (3.10)$$

Now for each  $v|_K = (v_i)_{|i| \leq K(\varepsilon)} \in \overline{\mathcal{L}}$ , we choose  $\tilde{v} = (\tilde{v}_i)_{i \in \mathbb{Z}}$  such that  $\tilde{v}_i = v_i$  for  $|i| \leq K(\varepsilon)$  and  $\tilde{v}_i = 0$  for  $|i| > K(\varepsilon)$ . Then by (3.9) and (3.10), we find that for each  $u \in \mathcal{S}$ , there exists  $\tilde{v} \in \overline{\mathcal{L}} = \{\tilde{v} : v|_K \in \overline{\mathcal{L}}\}$  such that

$$\sup_{s \in [-\nu, 0]} \|u(s) - \tilde{v}(s)\|^2 \leq \sup_{s \in [-\nu, 0]} \sum_{|i| \leq K(\varepsilon)} |u_i(s) - v_i(s)|^2 + \sup_{s \in [-\nu, 0]} \sum_{|i| > K(\varepsilon)} |u_i(s)|^2 < \varepsilon^2, \quad (3.11)$$

which implies that the set  $\mathcal{S}$  has a finite covering of balls with radii  $\varepsilon$ . The proof is complete.  $\square$

The next result is a variant of Theorem 3.2 which shows that condition (iii) in Theorem 3.2 has an equivalent form which is easier to verify for asymptotic compactness of dynamical systems associated with retarded LDSs.

**Theorem 3.3.** *Let  $\{u^n\}_{n=1}^\infty = \{(u_i^n)_{i \in \mathbb{Z}}\}_{n=1}^\infty \subset C([- \nu, 0]; \ell^2)$ . Then  $\{u^n\}_{n=1}^\infty$  is relative compact in  $C([- \nu, 0]; \ell^2)$  if and only if the following conditions are satisfied:*

- (i)  $\{u^n\}_{n=1}^\infty$  is bounded in  $C([- \nu, 0]; \ell^2)$ ;
- (ii)  $\{u^n\}_{n=1}^\infty$  is equicontinuous;
- (iii)  $\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{s \in [-\nu, 0]} \sum_{|i| \geq k} |u_i^n(s)|^2 = 0$ .

*Proof.* If  $\{u^n\}_{n=1}^\infty$  is relative compact in  $C([-v, 0]; \ell^2)$ , then it follows from Theorem 3.2 that the above conditions (i), (ii), and (iii) are satisfied. So, to complete the proof, we only need to show that the above conditions (i), (ii), and (iii) imply the conditions in Theorem 3.2. Given  $\varepsilon > 0$ , it follows from condition (iii) that there exists  $K_1(\varepsilon) > 0$  such that

$$\limsup_{n \rightarrow \infty} \sup_{s \in [-v, 0]} \sum_{|i| \geq K_1(\varepsilon)} |u_i^n(s)|^2 < \frac{\varepsilon^2}{2}, \quad (3.12)$$

which implies that there exists  $N(\varepsilon) > 0$  such that

$$\sup_{s \in [-v, 0]} \sum_{|i| \geq K_1(\varepsilon)} |u_i^n(s)|^2 < \varepsilon^2, \quad \forall n > N(\varepsilon). \quad (3.13)$$

By Lemma 3.1, we find that there exists  $K_2(\varepsilon) > 0$  such that

$$\sup_{s \in [-v, 0]} \sum_{|i| \geq K_2(\varepsilon)} |u_i^n(s)|^2 < \varepsilon^2, \quad \forall 1 \leq n \leq N(\varepsilon). \quad (3.14)$$

Take  $K(\varepsilon) = \max\{K_1(\varepsilon), K_2(\varepsilon)\}$ . It follows from (3.13) and (3.14) that

$$\sup_{s \in [-v, 0]} \sum_{|i| \geq K(\varepsilon)} |u_i^n(s)|^2 < \varepsilon^2, \quad \forall n \geq 1, \quad (3.15)$$

which implies that

$$\sup_{\{u^n\}_{n=1}^\infty} \sup_{s \in [-v, 0]} \sum_{|i| \geq k} |u_i^n(s)|^2 \leq \varepsilon^2, \quad \forall k \geq K(\varepsilon). \quad (3.16)$$

Therefore,

$$\limsup_{k \rightarrow \infty} \sup_{n \in \mathbb{N}} \sup_{s \in [-v, 0]} \sum_{|i| \geq k} |u_i^n(s)|^2 = 0, \quad (3.17)$$

which together with conditions (i) and (ii) shows that the conditions in Theorem 3.2 are satisfied with  $\mathcal{S} = \{u^n\}_{n=1}^\infty$ . The proof is complete.  $\square$

#### 4. Stochastic Retarded Lattice Differential Equations

In this section, we show that there is a continuous random dynamical system generated by stochastic retarded LDS (1.3)-(1.4).



For convenience, we now formulate (1.3)-(1.4) as a stochastic functional differential equation in  $\ell^2$ . Define the linear operators  $A, B, B^*, \lambda$  from  $\ell^2$  to  $\ell^2$  as follows. For  $u = (u_i)_{i \in \mathbb{Z}} \in \ell^2$ ,

$$\begin{aligned} (Au)_i &= -u_{i-1} + 2u_i - u_{i+1}, & (\lambda u)_i &= \lambda_i u_i, \\ (Bu)_i &= u_{i+1} - u_i, & (B^*u)_i &= u_{i-1} - u_i, \end{aligned} \quad (4.1)$$

for each  $i \in \mathbb{Z}$ . Then  $A = BB^* = B^*B$  and  $(B^*u, v) = (u, Bv)$  for all  $u, v \in \ell^2$ . Therefore,  $(Au, u) \geq 0$  for all  $u \in \ell^2$ . Let  $e_i \in \ell^2$  denote the element having 1 at position  $i$  and all the other components 0. Then

$$w(t) = \sum_{i \in \mathbb{Z}} a_i w_i(t) e_i \quad \text{with } (a_i)_{i \in \mathbb{Z}} \in \ell^2, \quad (4.2)$$

is an  $\ell^2$ -valued two-sided Wiener process with a symmetric nonnegative finite trace covariance operator  $Q$  such that  $Qe_i = a_i e_i$ . For  $\xi \in \mathcal{C}$ , let  $f(\xi) = (f_i(\xi_i))_{i \in \mathbb{Z}}$ . Then stochastic retarded LDS (1.3)-(1.4) can be rewritten as a stochastic functional equation in  $\ell^2$

$$du = [-(A + \lambda)u + f(u^t) + g]dt + dw, \quad t > 0, \quad (4.3)$$

with the initial data

$$u(t) = u^0(t), \quad t \in [-\nu, 0]. \quad (4.4)$$

In the sequel, we consider the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  where

$$\Omega = \left\{ \omega \in C(\mathbb{R}, \ell^2) : \omega(0) = 0 \right\}, \quad (4.5)$$

$\mathcal{F}$  is the Borel  $\sigma$ -algebra induced by the compact-open topology of  $\Omega$ , and  $\mathbb{P}$  the corresponding Wiener measure on  $(\Omega, \mathcal{F})$  with respect to the covariance operator  $Q$ . Let

$$\vartheta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}. \quad (4.6)$$

Then  $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t \in \mathbb{R}})$  is an ergodic metric dynamical system. Since the above probability space is canonical, we have

$$w(t, \omega) = \omega(t), \quad w(t, \vartheta_s \omega) = w(t + s, \omega) - w(s, \omega). \quad (4.7)$$

By Proposition A.1 in [26], there exists a  $\{\vartheta_t\}_{t \in \mathbb{R}}$ -invariant set  $\tilde{\Omega} \in \mathcal{F}$  of full  $\mathbb{P}$ -measure such that

$$\lim_{t \rightarrow \pm\infty} \frac{\|\omega(t)\|}{t} = 0 \quad \forall \omega \in \tilde{\Omega}. \quad (4.8)$$

Let  $\overline{\mathcal{F}}$  be the  $\mathbb{P}$ -completion of  $\mathcal{F}$  and let

$$\mathcal{F}_t = \bigvee_{s \leq t} \mathcal{F}_s^t, \quad t \in \mathbb{R}, \quad (4.9)$$

with

$$\mathcal{F}_s^t = \sigma\{w(\tau_2) - w(\tau_1) : s \leq \tau_1 \leq \tau_2 \leq t\} \vee \mathcal{N}, \quad (4.10)$$

where  $\sigma\{w(\tau_2) - w(\tau_1) : s \leq \tau_1 \leq \tau_2 \leq t\}$  is the smallest  $\sigma$ -algebra generated by the random variable  $w(\tau_2) - w(\tau_1)$  for all  $\tau_1, \tau_2$  such that  $s \leq \tau_1 \leq \tau_2 \leq t$  and  $\mathcal{N}$  is the collection of  $\mathbb{P}$ -null sets of  $\overline{\mathcal{F}}$ . Note that

$$\vartheta_\tau^{-1} \mathcal{F}_s^t = \mathcal{F}_{s+\tau}^{t+\tau}, \quad (4.11)$$

so  $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t \in \mathbb{R}}, (\mathcal{F}_s^t)_{s \leq t})$  is a filtered metric dynamical system.

Note that problem (4.3)-(4.4) is interpreted as an integral equation as follows:

$$\begin{aligned} u(t) &= u^0(0) + \int_0^t (-(A + \lambda)u + f(u^s) + g) ds + w(t), \quad t > 0, \\ u(t) &= u^0(t), \quad t \in [-v, 0]. \end{aligned} \quad (4.12)$$

$\mathbb{P}$ -a.s. for any  $u^0 \in \mathcal{C}$ . By the theory in [46], we deal with (4.12) on the complete probability space  $(\Omega, \overline{\mathcal{F}}, \mathbb{P})$ . For  $\lambda$  and  $f$ , we make the following assumptions.

(A<sub>1</sub>) There exist positive constants  $\lambda^l$  and  $\lambda^u$  such that

$$0 < \lambda^l \leq \lambda_i \leq \lambda^u < \infty, \quad i \in \mathbb{Z}. \quad (4.13)$$

(A<sub>2</sub>)  $f(0) = 0$ .

(A<sub>3</sub>) For any  $r > 0$ , there exists a constant  $l(r) > 0$  such that

$$\|f(\xi) - f(\eta)\| \leq l(r) \|\xi - \eta\|_{\mathcal{C}}, \quad (4.14)$$

for all  $\xi, \eta \in C([-v, 0]; \ell^2)$  with  $\|\xi\|_{\mathcal{C}}, \|\eta\|_{\mathcal{C}} \leq r$ .

(A<sub>4</sub>) There exist positive constants  $\alpha_0$  and  $c_f$  such that

$$\int_0^t e^{\alpha s} |f_i(u_i^s)|^2 ds \leq c_f^2 \int_{-v}^t e^{\alpha s} |u_i(s)|^2 ds, \quad (4.15)$$

for all  $\alpha \in (0, \alpha_0)$ ,  $t > 0$ ,  $u \in C([-v, t]; \ell^2)$ ,  $i \in \mathbb{Z}$ .

(A<sub>5</sub>)  $\lambda^l > c_f$ .

We now associate a continuous random dynamical system with the stochastic retarded lattice differential equations over  $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t \in \mathbb{R}})$ . To this end, we introduce an auxiliary Ornstein-Uhlenbeck process on  $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t \in \mathbb{R}})$  and transform the stochastic retarded lattice differential equations into a random one. Let

$$z(t, \omega) = \begin{cases} \int_{-\infty}^t (A + \lambda) e^{-(A+\lambda)(t-s)} (w(t, \omega) - w(s, \omega)) ds, & \omega \in \tilde{\Omega}, \\ 0, & \omega \notin \tilde{\Omega}, \end{cases} \quad (4.16)$$

where  $e^{-(A+\lambda)t}$  is the uniformly continuous semigroup on  $\ell^2$  generated by bounded linear operator  $-A - \lambda$ . Then by (4.8), (4.16) is well defined. The process  $z(t)$ ,  $t \in \mathbb{R}$  is a stationary, Gaussian process. Moreover, the random variable  $\|z(0, \omega)\|$  is tempered and for each  $\omega \in \Omega$ , the mapping  $t \rightarrow z(t, \omega)$  is continuous. Furthermore, by Lemma 5.13 in [46], we find that for all  $t \in \mathbb{R}$  and  $\mathbb{P}$ -a.s.,

$$z(t) = \int_{-\infty}^t e^{-(A+\lambda)(t-s)} dw(s). \quad (4.17)$$

Noticing that

$$\int_{-\infty}^t e^{-(A+\lambda)(t-s)} dw(s) = e^{-(A+\lambda)t} z(0) + \int_0^t e^{-(A+\lambda)(t-s)} dw(s), \quad (4.18)$$

and using the Itô formula, we get from (4.17) that for all  $t > 0$  and  $\mathbb{P}$ -a.s.,

$$z(t) = z(0) - \int_0^t (A + \lambda) z(s) ds + w(t). \quad (4.19)$$

Setting  $v(t) = u(t) - z(t)$  for  $t \geq -\nu$  in (4.12), then by (4.19), we obtain a deterministic equation,  $\mathbb{P}$ -a.s.

$$\begin{aligned} v(t) &= v^0(0) + \int_0^t (-(A + \lambda)v + f(v^s + z^s) + g) ds, \quad t > 0, \\ v(t) &= v^0(t), \quad t \in [-\nu, 0], \end{aligned} \quad (4.20)$$

which is equivalent to the functional differential equation

$$\frac{dv}{dt} = -(A + \lambda)v + f(v^t + z^t) + g, \quad t > 0, \quad (4.21)$$

with initial condition

$$v(t) = v^0(t), \quad t \in [-\nu, 0]. \quad (4.22)$$

Here  $v^0(t) = u^0(t) - z^0(t, \omega)$ ,  $t \in [-\nu, 0]$ .

Problem (4.21)-(4.22) is a deterministic functional differential equation with random coefficients, which can be solved pathwise. We now establish the following result for problem (4.21)-(4.22).

**Theorem 4.1.** *Let  $T > 0$  and  $\omega \in \Omega$  be fixed. Then the following properties hold.*

- (1) *For each  $v^0 \in \mathcal{C}$ , problem (4.21)-(4.22) has a unique solution  $v(\cdot, \omega, v^0) \in C([- \nu, T]; \ell^2)$ .*
- (2) *Let  $v_1(\cdot, \omega, v_1^0)$  and  $v_2(\cdot, \omega, v_2^0)$  be the solutions of problem (4.21)-(4.22) for the initial data  $v_1^0$  and  $v_2^0$ , respectively. Then there exists a constant  $c(T) > 0$  such that for all  $t \in [0, T]$*

$$\|v_1^t(\cdot, \omega, v_1^0) - v_2^t(\cdot, \omega, v_2^0)\|_{\mathcal{C}} \leq \|v_1^0 - v_2^0\|_{\mathcal{C}} e^{c(T)t}. \quad (4.23)$$

*Proof.* (1) Denote

$$F(t, \xi, \omega) = -A\xi(0) - \lambda\xi(0) + f(\xi + z^t(\cdot, \omega)) + g, \quad (4.24)$$

for all  $t \geq 0$ ,  $\xi \in \mathcal{C}$  and  $\omega \in \Omega$ . Then by (A<sub>1</sub>)–(A<sub>3</sub>), we have that

$$\|F(t, \xi, \omega) - F(t, \eta, \omega)\| \leq [4 + \lambda^u + l_f(r)] \|\xi - \eta\|_{\mathcal{C}}, \quad (4.25)$$

for any  $\xi, \eta \in \mathcal{C}$  with  $\|\xi\|_{\mathcal{C}} \leq r$ ,  $\|\eta\|_{\mathcal{C}} \leq r$ . Therefore,  $F$  satisfies local Lipschitz condition and maps the bounded sets of  $\mathcal{C}$  into the bounded sets of  $\ell^2$ . Then by using a standard argument, one can show that for each  $v^0 \in \mathcal{C}$ , there exists a  $T_{\max} \leq \infty$  such that problem (4.21)-(4.22) has a unique solution  $v$  on  $[0, T_{\max})$ . Moreover, if  $T_{\max} < \infty$  then

$$\limsup_{t \uparrow T_{\max}} \|v^t\|_{\mathcal{C}} = \infty. \quad (4.26)$$

We prove now that this local solution is a global one. Let  $T \in (0, T_{\max})$ . By (A<sub>5</sub>), we can choose  $\beta > 0$  small enough such that  $2\lambda^l > 2c_f + \beta$ . Taking the inner product of (4.21) with  $v$  in  $\ell^2$ , we get

$$\frac{1}{2} \frac{d}{dt} \|v\|^2 + (Av, v) + (\lambda v, v) = (f(v^t + z^t), v) + (g, v). \quad (4.27)$$

Clearly,

$$(Av, v) = (Bv, Bv) \geq 0, \quad (\lambda v, v) = \sum_{i \in \mathbb{Z}} \lambda_i v_i^2 \geq \lambda^l \|v\|^2. \quad (4.28)$$

Using the Young inequality, we find that

$$\begin{aligned} (f(v^t + z^t), v) &\leq \|f(v^t + z^t)\| \|v\| \leq \frac{c_f}{2} \|v\|^2 + \frac{1}{2c_f} \|f(v^t + z^t)\|^2, \\ (g, v) &\leq \|g\| \|v\| \leq \frac{\beta}{2} \|v\|^2 + \frac{1}{2\beta} \|g\|^2. \end{aligned} \quad (4.29)$$

Then it follows from (4.27), (4.28), (4.29) that

$$\frac{d}{dt} \|v\|^2 \leq -(2\lambda^l - c_f - \beta) \|v\|^2 + \frac{1}{c_f} \|f(v^t + z^t(\vartheta, \omega))\|^2 + \frac{1}{\beta} \|g\|^2. \quad (4.30)$$

Choose  $\alpha \in (0, \alpha_0)$  small enough such that  $2\lambda^l > 2c_f + \alpha + \beta$ . Then by (4.30), we obtain

$$\frac{d}{dt} (e^{\alpha t} \|v\|^2) \leq -(2\lambda^l - c_f - \alpha - \beta) e^{\alpha t} \|v\|^2 + \frac{e^{\alpha t}}{c_f} \|f(v^t + z^t)\|^2 + \frac{e^{\alpha t}}{\beta} \|g\|^2. \quad (4.31)$$

Now, we can also choose  $\gamma > 0$  small enough such that  $2\lambda^l > (2 + \gamma)c_f + \alpha + \beta$ . Integrating (4.31) over  $[0, t]$  ( $t \in [0, T]$ ) leads to

$$\begin{aligned} e^{\alpha t} \|v(t)\|^2 &\leq \|v(0)\|^2 - (2\lambda^l - c_f - \alpha - \beta) \int_0^t e^{\alpha s} \|v(s)\|^2 ds \\ &\quad + \frac{1}{c_f} \int_0^t e^{\alpha s} \|f(v^s + z^s)\|^2 ds + \frac{\|g\|^2}{\beta} \int_0^t e^{\alpha s} ds. \end{aligned} \quad (4.32)$$

Using the Young inequality and (A<sub>4</sub>), we find that

$$\begin{aligned} \frac{1}{c_f} \int_0^t e^{\alpha s} \|f(v^s + z^s)\|^2 ds &\leq c_f \int_{-\nu}^t e^{\alpha s} \|v(s) + z(s)\|^2 ds \\ &\leq c_f \int_0^t e^{\alpha s} \left[ (1 + \gamma) \|v(s)\|^2 + (1 + \gamma^{-1}) \|z(s)\|^2 \right] ds \\ &\quad + c_f \int_{-\nu}^0 e^{\alpha s} \|v(s) + z(s)\|^2 ds. \end{aligned} \quad (4.33)$$

Then by (4.32) and (4.33), we obtain

$$\begin{aligned}
e^{\alpha t} \|v(t)\|^2 &\leq -\left(2\lambda^l - (2 + \gamma)c_f - \alpha - \beta\right) \int_0^t e^{\alpha s} \|v(s)\|^2 ds + \|v(0)\|^2 \\
&\quad + \frac{\|g\|^2}{\alpha\beta} e^{\alpha t} + c_f \int_{-\nu}^0 e^{\alpha s} \|v(s) + z(s)\|^2 ds + c_1 \int_0^t e^{\alpha s} \|z(\mathfrak{D}_s \omega)\|^2 ds \\
&\leq (1 + 2\nu c_f) \|v^0\|_C^2 + 2\nu c_f \|z^0\|_C^2 + c_1 \int_0^t e^{\alpha s} \|z(s)\|^2 ds + \frac{\|g\|^2}{\alpha\beta} e^{\alpha t},
\end{aligned} \tag{4.34}$$

where  $c_1 = c_f(1 + \gamma^{-1})$ . Consequently,

$$\|v(t)\|^2 \leq \left[(1 + 2\nu c_f) \|v^0\|_C^2 + 2\nu c_f \|z^0\|_C^2\right] e^{-\alpha t} + c_1 \int_0^t e^{\alpha(s-t)} \|z(s)\|^2 ds + \frac{\|g\|^2}{\alpha\beta}. \tag{4.35}$$

Hence, for fixed  $\sigma \in [-\nu, 0]$ , we get that for  $t \in (-\sigma, T]$ ,

$$\begin{aligned}
\|v(t + \sigma)\|^2 &\leq \left[(1 + 2\nu c_f) \|v^0\|_C^2 + 2\nu c_f \|z^0\|_C^2\right] e^{-\alpha(t+\sigma)} + c_1 \int_0^{t+\sigma} e^{\alpha(s-t-\sigma)} \|z(s)\|^2 ds + \frac{\|g\|^2}{\alpha\beta} \\
&\leq \left[(1 + 2\nu c_f) \|v^0\|_C^2 + 2\nu c_f \|z^0\|_C^2\right] e^{\alpha(\nu-t)} + c_1 e^{\alpha\nu} \int_0^t e^{\alpha(s-t)} \|z(s)\|^2 ds + \frac{\|g\|^2}{\alpha\beta},
\end{aligned} \tag{4.36}$$

and for  $t \in [0, -\sigma]$ ,

$$\|v(t + \sigma)\|^2 \leq \|v^0\|_C^2 \leq \left[(1 + 2\nu c_f) \|v^0\|_C^2 + 2\nu c_f \|z^0\|_C^2\right] e^{\alpha(\nu-t)}. \tag{4.37}$$

In view of (4.36) and (4.37), we find that for all  $t \in [0, T]$ ,

$$\|v^t\|_C^2 \leq \left[(1 + 2\nu c_f) \|v^0\|_C^2 + 2\nu c_f \|z^0\|_C^2\right] e^{\alpha(\nu-t)} + c_1 e^{\alpha\nu} \int_0^t e^{\alpha(s-t)} \|z(s)\|^2 ds + \frac{\|g\|^2}{\alpha\beta}. \tag{4.38}$$

Therefore, for all  $t \in [0, T]$ ,

$$\|v^t\|_C^2 \leq \left[(1 + 2\nu c_f) \|v^0\|_C^2 + 2\nu c_f \|z^0\|_C^2\right] e^{\alpha\nu} + c_1 e^{\alpha\nu} \int_0^T \|z(s)\|^2 ds + \frac{\|g\|^2}{\alpha\beta}, \tag{4.39}$$

which, together with (4.26), implies that  $T_{\max} = \infty$ . This proves the property (1).

(2) Let  $\tilde{v}(t, \omega) = v_1(t, \omega, v_1^0) - v_1(t, \omega, v_2^0)$ . By (4.38), there exists a constant  $r(T) > 0$  such that

$$\|v_1^t\|_C \leq r(T), \quad \|v_2^t\|_C \leq r(T). \tag{4.40}$$

Then from (4.20) and (4.25), we have that for  $t \in [0, T]$

$$\|\tilde{v}(t)\| \leq \|\tilde{v}(0)\| + [4 + \lambda^u + l_f(r(T))] \int_0^t \|\tilde{v}^s\|_{\mathcal{C}} ds. \quad (4.41)$$

Hence, for fixed  $\sigma \in [-\nu, 0]$ , we get that for  $t \in (-\sigma, T]$ ,

$$\begin{aligned} \|\tilde{v}(t + \sigma)\| &\leq \|\tilde{v}(0)\| + [4 + \lambda^u + l_f(r(T))] \int_0^{t+\sigma} \|\tilde{v}^s\|_{\mathcal{C}} ds \\ &\leq \|\tilde{v}^0\|_{\mathcal{C}} + [4 + \lambda^u + l_f(r(T))] \int_0^t \|\tilde{v}^s\|_{\mathcal{C}} ds, \end{aligned} \quad (4.42)$$

and for  $t \in [0, -\sigma]$ ,

$$\|\tilde{v}(t + \sigma)\| \leq \|\tilde{v}^0\|_{\mathcal{C}}. \quad (4.43)$$

In view of (4.42) and (4.43), we find that for all  $t \in [0, T]$ ,

$$\|\tilde{v}^t\|_{\mathcal{C}} \leq \|\tilde{v}^0\|_{\mathcal{C}} + [4 + \lambda^u + l_f(r(T))] \int_0^t \|\tilde{v}^s\|_{\mathcal{C}} ds. \quad (4.44)$$

The Gronwall inequality implies that for all  $t \in [0, T]$ ,

$$\|\tilde{v}^t\|_{\mathcal{C}} \leq \|\tilde{v}^0\|_{\mathcal{C}} e^{[4 + \lambda^u + l_f(r(T))]t}. \quad (4.45)$$

This proves the property (2). The proof is complete.  $\square$

Conversely, if for each  $\omega \in \Omega$ ,  $v(t, \omega, v^0)$  is a solution of problem (4.21)-(4.22) with  $v^0(\cdot) = u^0(\cdot) - z^0(\cdot, \omega)$ , then the process

$$u(t, \omega, u^0) = v(t, \omega, v^0) + z(t, \omega) \quad (4.46)$$

is a solution of problem (4.3)-(4.4). And if  $u^0$  is a  $\mathcal{C}$ -valued  $\mathcal{F}_0$ -measurable random variable, then  $u(t, \omega, u^0)$  is an  $\mathcal{F}_t$ -adapted process.

**Theorem 4.2.** *Problem (4.21)-(4.22) generates a continuous random dynamical system  $\phi$  over  $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in \mathbb{R}})$ , where*

$$\phi(t, \omega, v^0) = v^t(\cdot, \omega, v^0), \quad \text{for } t \geq 0, \omega \in \Omega, v^0 \in \mathcal{C}. \quad (4.47)$$

Moreover, if one defines  $\psi$  by

$$\psi(t, \omega, u^0) = u^t(\cdot, \omega, u^0), \quad \text{for } t \geq 0, \omega \in \Omega, u^0 \in \mathcal{C}, \quad (4.48)$$

then  $\psi$  is another continuous random dynamical system associated to problem (4.3)-(4.4).

*Proof.* From property (2) of Theorem 4.1, it follows that  $\phi(\cdot, \omega, \cdot) : [0, \infty) \times \mathcal{C} \rightarrow \mathcal{C}$  is continuous for all  $\omega \in \Omega$ . By (4.20), we have that for  $s, t \geq 0$  and  $\sigma \in [-\nu, 0]$ ,

$$\phi(t, \vartheta_s \omega, \phi(s, \omega, v^0))(\sigma) = \phi(s, \omega, v^0)(0) + \int_0^{t+\sigma} F(\tau, \phi(\tau, \vartheta_s \omega, \phi(s, \omega, v^0)), \vartheta_s \omega) d\tau. \quad (4.49)$$

Then again by (4.20) and noticing that

$$F(t, \xi, \vartheta_s \omega) = F(t + s, \xi, \omega), \quad \forall s, t \geq 0, \xi \in \mathcal{C}, \quad (4.50)$$

we get that

$$\begin{aligned} \phi(t, \vartheta_s \omega, \phi(s, \omega, v^0))(\sigma) &= v^0(0) + \int_0^s F(\tau, \phi(\tau, \omega, v^0), \omega) d\tau \\ &\quad + \int_s^{t+s+\sigma} F(\tau, \phi(\tau - s, \vartheta_s \omega, \phi(s, \omega, v^0)), \omega) d\tau. \end{aligned} \quad (4.51)$$

For each  $\omega \in \Omega$  consider

$$\Phi(\tau, \omega, v^0) = \begin{cases} \phi(\tau, \omega, v^0), & \text{if } 0 \leq \tau \leq s, \\ \phi(\tau - s, \vartheta_s \omega, \phi(s, \omega, v^0)), & \text{if } s < \tau \leq t + s. \end{cases} \quad (4.52)$$

Then for  $\tau = t + s$ , we have

$$\Phi(t + s, \omega, v^0) = \phi(t, \vartheta_s \omega, \phi(s, \omega, v^0)) \quad \text{for } s, t \geq 0. \quad (4.53)$$

It follows from (4.51) that

$$\Phi(t + s, \omega, v^0)(\sigma) = v^0(0) + \int_0^{t+s+\sigma} F(\tau, \Phi(\tau, \omega, v^0), \omega) d\tau, \quad (4.54)$$

for all  $\sigma \in [-\nu, 0]$ . By the uniqueness of the solution of (4.20), we find that

$$\Phi(t + s, \omega, v^0) = \phi(t + s, \omega, v^0), \quad (4.55)$$



while (4.53) implies

$$\phi(t+s, \omega, v^0) = \phi(t, \vartheta_s \omega, \phi(s, \omega, v^0)) \quad \text{for } s, t \geq 0. \quad (4.56)$$

Hence,  $\phi$  is a continuous random dynamical system.

As for  $\psi$ , noticing that

$$\psi(t, \omega, u^0) = \phi(t, \omega, u^0 - z^0(\omega)) + z^t(\omega), \quad \text{for } t \geq 0, \omega \in \Omega \text{ and } u^0 \in \mathcal{C}, \quad (4.57)$$

we get from (4.56) that for  $s, t \geq 0$ ,

$$\begin{aligned} \psi(t, \vartheta_s \omega, \psi(s, \omega, u^0)) &= \phi(t, \vartheta_s \omega, \phi(s, \omega, u^0 - z^0(\omega))) + z^t(\vartheta_s \omega) \\ &= \phi(t+s, \omega, u^0 - z^0(\omega)) + z^{t+s}(\omega) \\ &= \psi(t+s, \omega, u^0). \end{aligned} \quad (4.58)$$

Therefore,  $\psi$  is also a continuous random dynamical system. Furthermore,  $\phi$  and  $\psi$  are conjugated random dynamical systems, that is

$$\psi(t, \omega, T(\omega, \xi)) = T(\vartheta_t \omega, \phi(t, \omega, \xi)), \quad \text{for any } \xi \in \mathcal{C}, \quad (4.59)$$

where for every  $\omega \in \Omega$ ,  $T(\omega, \xi) = \xi + z^0(\omega)$  is a homeomorphism of  $\mathcal{C}$ . The proof is complete.  $\square$

## 5. Existence of Random Attractors

In this section, we prove the existence of a  $\mathfrak{D}$ -random attractor for the random dynamical system  $\psi$  associated with (4.3)-(4.4). We first establish the existence of a  $\mathfrak{D}$ -random attractor for its conjugated random dynamical system  $\phi$ , then the existence of a  $\mathfrak{D}$ -random attractor for  $\psi$  follows from the conjugation relation between  $\phi$  and  $\psi$ . To this end, we will derive uniform estimates on the solutions of problem (4.21)-(4.22) when  $t \rightarrow \infty$  with the purpose of proving the existence of a bounded random absorbing set and the asymptotic compactness for  $\phi$ .

From now on, we always assume that  $\mathfrak{D}$  is the collection of all tempered subsets of  $\mathcal{C}$  with respect to  $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t \in \mathbb{R}})$ . The next lemma shows that  $\phi$  has a random absorbing set in  $\mathfrak{D}$ .

**Lemma 5.1.** *There exists  $K \in \mathfrak{D}$  such that  $K$  is a random absorbing set for  $\phi$  in  $\mathfrak{D}$ , that is, for any  $B \in \mathfrak{D}$  and  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , there exists  $T_B(\omega) > 0$  such that*

$$\phi(t, \vartheta_{-t} \omega, B(\vartheta_{-t} \omega)) \subseteq K(\omega) \quad \forall t \geq T_B(\omega). \quad (5.1)$$

*Proof.* Replacing  $\omega$  by  $\vartheta_{-t}\omega$  in (4.38), we get that for all  $t \geq 0$ ,

$$\begin{aligned} \left\| v^t(\vartheta_{-t}\omega, v^0(\vartheta_{-t}\omega)) \right\|_C^2 &\leq \left[ (1 + 2\nu c_f) \left\| v^0(\vartheta_{-t}\omega) \right\|_C^2 + 2\nu c_f \left\| z^0(\vartheta_{-t}\omega) \right\|_C^2 \right] e^{\alpha(v-t)} \\ &\quad + c_1 e^{\alpha v} \int_0^t e^{\alpha(s-t)} \|z(s, \vartheta_{-t}\omega)\|^2 ds + \frac{\|g\|^2}{\alpha\beta} \\ &\leq \left[ (1 + 2\nu c_f) \left\| v^0(\vartheta_{-t}\omega) \right\|_C^2 + 2\nu c_f \left\| z^0(\vartheta_{-t}\omega) \right\|_C^2 \right] e^{\alpha(v-t)} \\ &\quad + c_1 e^{\alpha v} \int_{-\infty}^0 e^{\alpha s} \|z(0, \vartheta_s\omega)\|^2 ds + \frac{\|g\|^2}{\alpha\beta}. \end{aligned} \quad (5.2)$$

By assumption,  $B \in \mathfrak{D}$  is tempered. On the other hand, by Remark 2.6,  $\|z^0(\omega)\|_C^2$  is also tempered. Therefore, if  $v^0(\vartheta_{-t}\omega) \in B(\vartheta_{-t}\omega)$ , then there exists  $T_B(\omega) > 0$  such that for all  $t \geq T_B(\omega)$ ,

$$\left[ (1 + 2\nu c_f) \left\| v^0(\vartheta_{-t}\omega) \right\|_C^2 + 2\nu c_f \left\| z^0(\vartheta_{-t}\omega) \right\|_C^2 \right] e^{\alpha(v-t)} \leq 1 + r(\omega), \quad (5.3)$$

where

$$r(\omega) = \int_{-\infty}^0 e^{\alpha s} \|z(0, \vartheta_s\omega)\|^2 ds \quad (5.4)$$

is tempered by Remark 2.7. Then it follows from (5.2) and (5.3) that for all  $t \geq T_B(\omega)$ ,

$$\left\| v^t(\vartheta_{-t}\omega, v^0(\vartheta_{-t}\omega)) \right\|_C^2 \leq (c_1 e^{\alpha v} + 1)r(\omega) + \frac{\|g\|^2}{\alpha\beta} + 1. \quad (5.5)$$

Given  $\omega \in \Omega$ , denote by

$$K(\omega) = \left\{ \xi \in \mathcal{C} : \|\xi\|_C^2 \leq r_1(\omega) \right\}, \quad (5.6)$$

where

$$r_1(\omega) = (c_1 e^{\alpha v} + 1)r(\omega) + \frac{\|g\|^2}{\alpha\beta} + 1 \quad (5.7)$$

is tempered. Then  $K \in \mathfrak{D}$ . Further, (5.5) indicates that  $K$  is a random absorbing set for  $\phi$  in  $\mathfrak{D}$ , which completes the proof.  $\square$

**Lemma 5.2.** *Let  $B \in \mathfrak{D}$  and  $v^0(\omega) \in B(\omega)$ . Then for every  $\varepsilon > 0$  and  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , there exist  $T^* = T^*(B, \omega, \varepsilon) > 0$  and  $N^* = N^*(\omega, \varepsilon) > 0$  such that the solution  $v(t, \omega, v^0(\omega))$  of problem (4.21)-(4.22) satisfies, for all  $t \geq T^*$ ,*

$$\sup_{s \in [-v, 0]} \sum_{|i| \geq N^*} \left| v_i^t \left( s, \mathfrak{D}_{-t}\omega, u^0(\mathfrak{D}_{-t}\omega) \right) \right|^2 \leq \varepsilon. \quad (5.8)$$

*Proof.* Let  $\rho$  be a smooth function defined on  $\mathbb{R}^+$  such that  $0 \leq \rho(s) \leq 1$  for all  $s \geq 0$ , and

$$\rho(s) = \begin{cases} 0, & 0 \leq s \leq 1, \\ 1, & s \geq 2. \end{cases} \quad (5.9)$$

Then there exists a positive deterministic constant  $c_2$  such that  $|\rho'(s)| \leq c_2$  for all  $s \geq 0$ . Taking the inner product of (4.21) with  $x = (\rho(|i|/k)v_i)$  in  $\ell^2$ , we obtain that

$$\frac{1}{2} \frac{d}{dt} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{k} \right) |v_i|^2 + (Av, x) + (\lambda v, x) = (f(v^t + z^t), x) + (g, x). \quad (5.10)$$

We now estimate terms in (5.10) as follows. First, we get from (A<sub>1</sub>) that

$$(\lambda v, x) = \sum_{i \in \mathbb{Z}} \lambda_i \rho \left( \frac{|i|}{k} \right) v_i^2 \geq \lambda^l \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{k} \right) |v_i|^2. \quad (5.11)$$

Secondly, by the property of the cutoff function  $\rho$ , we estimate

$$\begin{aligned} (Av, x) &= (Bv, Bx) \\ &= \sum_{i \in \mathbb{Z}} (v_{i+1} - v_i) \left[ \rho \left( \frac{|i+1|}{k} \right) v_{i+1} - \rho \left( \frac{|i|}{k} \right) v_i \right] \\ &= \sum_{i \in \mathbb{Z}} (v_{i+1} - v_i) \left[ \left( \rho \left( \frac{|i+1|}{k} \right) - \rho \left( \frac{|i|}{k} \right) \right) v_{i+1} + \rho \left( \frac{|i|}{k} \right) (v_{i+1} - v_i) \right] \\ &= \sum_{i \in \mathbb{Z}} \left[ \rho \left( \frac{|i+1|}{k} \right) - \rho \left( \frac{|i|}{k} \right) \right] (v_{i+1} - v_i) v_{i+1} + \rho \left( \frac{|i|}{k} \right) (v_{i+1} - v_i)^2 \\ &\geq \sum_{i \in \mathbb{Z}} \left[ \rho \left( \frac{|i+1|}{k} \right) - \rho \left( \frac{|i|}{k} \right) \right] (v_{i+1} - v_i) v_{i+1} \\ &\geq - \sum_{i \in \mathbb{Z}} \frac{|\rho'(\xi_i)|}{k} |v_{i+1} - v_i| |v_{i+1}| \\ &\geq - \frac{c_2}{k} \sum_{i \in \mathbb{Z}} (|v_{i+1}|^2 + |v_i| |v_{i+1}|) \geq - \frac{2c_2}{k} \|v\|^2. \end{aligned} \quad (5.12)$$

Thirdly, using the Young inequality and (A<sub>3</sub>), we find that

$$\begin{aligned} (f(v^t + z^t), x) &= \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) f(v_i^t + z_i^t(\vartheta, \omega)) \\ &\leq \frac{c_f}{2} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i|^2 + \frac{1}{2c_f} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |f(v_i^t + z_i^t(\vartheta, \omega))|^2. \end{aligned} \quad (5.13)$$

Finally, using the Young inequality again, we obtain that

$$(g, x) = \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) g_i v_i \leq \frac{\beta}{2} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) v_i^2 + \frac{1}{2\beta} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) g_i^2. \quad (5.14)$$

Taking into account (5.10), (5.11), (5.12), (5.13), and (5.14), we obtain that

$$\begin{aligned} \frac{d}{dt} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i|^2 &\leq - (2\lambda^l - c_f - \beta) \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i|^2 \\ &\quad + \frac{1}{c_f} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |f(v_i^t + z_i^t)|^2 \\ &\quad + \frac{1}{\beta} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) g_i^2 + \frac{4c_2}{k} \|v\|^2, \end{aligned} \quad (5.15)$$

which implies

$$\begin{aligned} \frac{d}{dt} \left( e^{\alpha t} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i|^2 \right) &\leq - (2\lambda^l - c_f - \alpha - \beta) e^{\alpha t} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i|^2 \\ &\quad + \frac{1}{c_f} e^{\alpha t} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |f(v_i^t + z_i^t)|^2 \\ &\quad + \frac{1}{\beta} e^{\alpha t} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) g_i^2 + \frac{4c_2}{k} e^{\alpha t} \|v\|^2. \end{aligned} \quad (5.16)$$

Using the Young inequality and (A<sub>4</sub>), we get that

$$\begin{aligned}
& \frac{1}{c_f} \int_0^t e^{\alpha s} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |f(v_i^s + z_i^s)|^2 ds \\
& \leq c_f \int_{-\nu}^t e^{\alpha s} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i(s) + z_i(s)|^2 ds \\
& \leq c_f \int_0^t e^{\alpha s} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) \left[ (1 + \gamma) |v_i(s)|^2 + (1 + \gamma^{-1}) |z_i(s)|^2 \right] ds \\
& \quad + c_f \int_{-\nu}^0 e^{\alpha s} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i(s) + z_i(s)|^2 ds.
\end{aligned} \tag{5.17}$$

Integrating (5.16) over  $[0, t]$  ( $t \geq 0$ ) leads to

$$\begin{aligned}
& e^{\alpha t} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i(t)|^2 - \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i(0)|^2 \\
& \leq -\left(2\lambda^l - c_f - \alpha - \beta\right) \int_0^t e^{\alpha s} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i(s)|^2 ds \\
& \quad + \frac{1}{c_f} \int_0^t e^{\alpha s} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |f(v_i^s + z_i^s)|^2 ds \\
& \quad + \frac{1}{\beta} \int_0^t e^{\alpha s} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) g_i^2 ds + \frac{4c_2}{k} \int_0^t e^{\alpha s} \|v(s)\|^2 ds.
\end{aligned} \tag{5.18}$$

It follows from (5.17) and (5.18) that

$$\begin{aligned}
& e^{\alpha t} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i(t)|^2 - \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i(0)|^2 \\
& \leq -\left(2\lambda^l - (2 + \gamma)c_f - \alpha - \beta\right) \int_0^t e^{\alpha s} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i(s)|^2 ds \\
& \quad + c_f \int_{-\nu}^0 e^{\alpha s} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i(s) + z_i(s)|^2 ds + \frac{e^{\alpha t}}{\alpha\beta} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) g_i^2 \\
& \quad + c_1 \int_0^t e^{\alpha s} \sum_{|i| > k-1} |z_i(s)|^2 ds + \frac{4c_2}{k} \int_0^t e^{\alpha s} \|v(s)\|^2 ds \\
& \leq c_f \int_{-\nu}^0 e^{\alpha s} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i(s) + z_i(s)|^2 ds + \frac{e^{\alpha t}}{\alpha\beta} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) g_i^2 \\
& \quad + c_1 \int_0^t e^{\alpha s} \sum_{|i| > k-1} |z_i(s)|^2 ds + \frac{4c_2}{k} \int_0^t e^{\alpha s} \|v(s)\|^2 ds,
\end{aligned} \tag{5.19}$$

which implies

$$\begin{aligned} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i(t)|^2 &\leq \left[ (1 + 2\nu c_f) \|v^0\|_C^2 + 2\nu c_f \|z^0\|_C^2 \right] e^{-\alpha t} + \frac{1}{\alpha\beta} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) g_i^2 \\ &\quad + c_1 \int_0^t e^{\alpha(s-t)} \sum_{|i| > k-1} |z_i(s)|^2 ds + \frac{4c_2}{k} \int_0^t e^{\alpha(s-t)} \|v(s)\|^2 ds. \end{aligned} \quad (5.20)$$

If we take  $t \geq \nu$ , we have that, for all  $\sigma \in [-\nu, 0]$ ,

$$\begin{aligned} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i(t + \sigma)|^2 &\leq \left[ (1 + 2\nu c_f) \|v^0\|_C^2 + 2\nu c_f \|z^0\|_C^2 \right] e^{-\alpha(t+\sigma)} + \frac{1}{\alpha\beta} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) g_i^2 \\ &\quad + c_1 \int_0^{t+\sigma} e^{\alpha(s-t-\sigma)} \sum_{|i| > k-1} |z_i(s)|^2 ds + \frac{4c_2}{k} \int_0^{t+\sigma} e^{\alpha(s-t-\sigma)} \|v(s)\|^2 ds \\ &\leq \left[ (1 + 2\nu c_f) \|v^0\|_C^2 + 2\nu c_f \|z^0\|_C^2 \right] e^{\alpha(\nu-t)} + \frac{1}{\alpha\beta} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) g_i^2 \\ &\quad + c_1 e^{\alpha\nu} \int_0^t e^{\alpha(s-t)} \sum_{|i| > k-1} |z_i(s)|^2 ds + \frac{4c_2 e^{\alpha\nu}}{k} \int_0^t e^{\alpha(s-t)} \|v(s)\|^2 ds, \end{aligned} \quad (5.21)$$

whence for all  $t \geq \nu$ ,

$$\begin{aligned} \sup_{\sigma \in [-\nu, 0]} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) |v_i^t(\sigma)|^2 &\leq \left[ (1 + 2\nu c_f) \|v^0\|_C^2 + 2\nu c_f \|z^0\|_C^2 \right] e^{\alpha(\nu-t)} \\ &\quad + \frac{1}{\alpha\beta} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) g_i^2 + c_1 e^{\alpha\nu} \int_0^t e^{\alpha(s-t)} \sum_{|i| > k-1} |z_i(s)|^2 ds \\ &\quad + \frac{4c_2 e^{\alpha\nu}}{k} \int_0^t e^{\alpha(s-t)} \|v(s, \omega, v^0(\omega))\|^2 ds. \end{aligned} \quad (5.22)$$

Replacing  $\omega$  by  $\vartheta_{-t}\omega$ , we find that

$$\begin{aligned}
& \sup_{\sigma \in [-v, 0]} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{k} \right) \left| v_i^t \left( \sigma, \vartheta_{-t}\omega, v^0(\vartheta_{-t}\omega) \right) \right|^2 \\
& \leq \left[ (1 + 2\nu c_f) \left\| v^0(\vartheta_{-t}\omega) \right\|_C^2 + 2\nu c_f \left\| z^0(\vartheta_{-t}\omega) \right\|_C^2 \right] e^{\alpha(v-t)} \\
& \quad + \frac{1}{\alpha\beta} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{k} \right) g_i^2 + c_1 e^{\alpha v} \int_0^t e^{\alpha(s-t)} \sum_{|i| > k-1} |z_i(s, \vartheta_{-t}\omega)|^2 ds \\
& \quad + \frac{4c_2 e^{\alpha v}}{k} \int_0^t e^{\alpha(s-t)} \left\| v(s, \vartheta_{-t}\omega, v^0(\vartheta_{-t}\omega)) \right\|^2 ds.
\end{aligned} \tag{5.23}$$

We now estimate terms in (5.23) as follows. Since  $B \in \mathfrak{D}$  is tempered set, and  $\|z^0(\omega)\|_C^2$  is tempered function, if  $v^0(\vartheta_{-t}\omega) \in B(\vartheta_{-t}\omega)$ , then for every  $\varepsilon > 0$ , there exists  $T_1 = T_1(B, \omega, \varepsilon) > 0$  such that for all  $t \geq T_1$ ,

$$\left[ (1 + 2\nu c_f) \left\| v^0(\vartheta_{-t}\omega) \right\|_C^2 + 2\nu c_f \left\| z^0(\vartheta_{-t}\omega) \right\|_C^2 \right] e^{\alpha(v-t)} \leq \frac{\varepsilon}{4}. \tag{5.24}$$

Secondly, since  $g \in \ell^2$ , there exists  $N_1 = N_1(\omega, \varepsilon) > 0$  such that, for all  $k \geq N_1$ ,

$$\frac{1}{\alpha\beta} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{k} \right) g_i^2 \leq \frac{1}{\alpha\beta} \sum_{|i| > k} \rho \left( \frac{|i|}{k} \right) g_i^2 \leq \frac{\varepsilon}{4}. \tag{5.25}$$

Thirdly, note that

$$\int_{-\infty}^0 e^{\alpha(s)} \|z(0, \vartheta_s \omega)\|^2 ds < \infty. \tag{5.26}$$

Then by the Lebesgue theorem of dominated convergence, there exists  $N_2 = N_2(\omega, \varepsilon) > 0$  such that for all  $k \geq N_2$ ,

$$\int_{-\infty}^0 e^{\alpha s} \sum_{|i| > k-1} |z_i(0, \vartheta_s \omega)|^2 ds \leq \frac{\varepsilon}{4c_1 e^{\alpha v}}. \tag{5.27}$$

Then it follows from (5.27) that for all  $t \geq 0$  and  $k \geq N_2$ ,

$$c_1 e^{\alpha v} \int_0^t e^{\alpha(s-t)} \sum_{|i| > k-1} |z_i(s, \vartheta_{-t}\omega)|^2 ds \leq c_1 e^{\alpha v} \int_{-\infty}^0 e^{\alpha s} \sum_{|i| > k-1} |z_i(\vartheta_s \omega)|^2 ds \leq \frac{\varepsilon}{4}. \tag{5.28}$$

Next, we get from (4.35) that

$$\begin{aligned}
& \frac{4c_2 e^{\alpha v}}{k} \int_0^t e^{\alpha(s-t)} \left\| v\left(s, \vartheta_{-t}\omega, v^0(\vartheta_{-t}\omega)\right) \right\|^2 ds \\
& \leq \frac{4c_2 \|g\|^2 e^{\alpha v}}{k\alpha^2 \beta} + \frac{4c_1 c_2 e^{\alpha v}}{k} \int_0^t \int_0^s e^{\alpha(\tau-t)} \|z(\tau, \vartheta_{-t}\omega)\|^2 d\tau ds \\
& \quad + \frac{4c_2 e^{\alpha v}}{k} \left[ (1 + 2\nu c_f) \left\| v^0(\vartheta_{-t}\omega) \right\|_C^2 + 2\nu c_f \left\| z^0(\vartheta_{-t}\omega) \right\|_C^2 \right] t e^{-\alpha t}.
\end{aligned} \tag{5.29}$$

For the integral on the right side of (5.29), we have that for all  $t \geq 0$ ,

$$\begin{aligned}
\int_0^t \int_0^s e^{\alpha(\tau-t)} \|z(\tau, \vartheta_{-t}\omega)\|^2 d\tau ds &= \int_0^t s e^{-\alpha s} \|z(0, \vartheta_{-s}\omega)\|^2 ds \\
&\leq \int_0^\infty s e^{-\alpha s} \|z(0, \vartheta_{-s}\omega)\|^2 ds < \infty.
\end{aligned} \tag{5.30}$$

Since  $B \in \mathfrak{D}$  is tempered set, and  $\|z^0(\omega)\|_C^2$  is tempered function, there exists  $T_3 = T_3(B, \omega, \varepsilon) > 0$  such that, for all  $t \geq T_3$  and  $k \in \mathbb{N}$ ,

$$\frac{4c_2 e^{\alpha v}}{k} \left[ (1 + 2\nu c_f) \left\| v^0(\vartheta_{-t}\omega) \right\|_C^2 + 2\nu c_f \left\| z^0(\vartheta_{-t}\omega) \right\|_C^2 \right] t e^{-\alpha t} \leq \frac{\varepsilon}{8}. \tag{5.31}$$

At the same time, there exists  $N_3 = N_3(\omega, \varepsilon) > 0$  such that, for all  $k \geq N_3$ ,

$$\frac{4c_2 \|g\|^2 e^{\alpha v}}{k\alpha^2 \beta} + \frac{4c_1 c_2 e^{\alpha v}}{k} \int_0^\infty s e^{-\alpha s} \|z(0, \vartheta_{-s}\omega)\|^2 ds \leq \frac{\varepsilon}{8}. \tag{5.32}$$

It follows from (5.29), (5.30), (5.31), and (5.32) that, for all  $t \geq T_3(\omega, \varepsilon)$  and  $k \geq N_3(\omega, \varepsilon)$ ,

$$\frac{4c_2 e^{\alpha v}}{k} \int_0^t e^{\alpha(s-t)} \left\| v\left(s, \vartheta_{-t}\omega, v^0(\vartheta_{-t}\omega)\right) \right\|^2 ds \leq \frac{\varepsilon}{4}. \tag{5.33}$$

Let  $T_4 = T_4(B, \omega, \varepsilon) = \max\{T_1, T_2, T_3\}$ ,  $N_4 = N_4(\omega, \varepsilon) = \max\{N_1, N_2, N_3\}$ . Then it follows from (5.23), (5.24), (5.25), (5.28), and (5.33) that, for all  $t \geq T_4$  and  $k \geq N_4$ ,

$$\begin{aligned}
& \sup_{\sigma \in [-\nu, 0]} \sum_{|i| \geq 2k} \left| v_i^t(\sigma, \vartheta_{-t}\omega, v^0(\vartheta_{-t}\omega)) \right|^2 \\
& < \sup_{\sigma \in [-\nu, 0]} \sum_{i \in \mathbb{Z}} \rho\left(\frac{|i|}{k}\right) \left| v_i^t(\sigma, \vartheta_{-t}\omega, v^0(\vartheta_{-t}\omega)) \right|^2 \leq \varepsilon.
\end{aligned} \tag{5.34}$$

The proof is complete.  $\square$



**Lemma 5.3.** *The random dynamical system  $\phi$  is  $\mathfrak{D}$ -pullback asymptotically compact in  $\mathcal{C}$ , that is, for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , the sequence  $\{\phi(t_n, \vartheta_{-t_n}\omega, v_n^0(\vartheta_{-t_n}\omega))\}_{n=1}^\infty$  has a convergent subsequence in  $\mathcal{C}$  provided  $t_n \rightarrow \infty$ ,  $B \in \mathfrak{D}$  and  $v_n^0(\vartheta_{-t_n}\omega) \in B(\vartheta_{-t_n}\omega)$ .*

*Proof.* By (4.21) and Lemma 5.1, we find that, for every  $t \geq T_B(\omega) + \nu$ , and  $\sigma_1, \sigma_2 \in [-\nu, 0]$ ,

$$\begin{aligned} & \left\| \phi\left(t, \vartheta_{-t}\omega, v^0(\vartheta_{-t}\omega)\right)(\sigma_1) - \phi\left(t, \vartheta_{-t}\omega, v^0(\vartheta_{-t}\omega)\right)(\sigma_2) \right\| \\ &= \left\| v\left(t + \sigma_1, \vartheta_{-t}\omega, v^0(\vartheta_{-t}\omega)\right) - v\left(t + \sigma_2, \vartheta_{-t}\omega, v^0(\vartheta_{-t}\omega)\right) \right\| \\ &\leq \left\| v'\left(t + \xi, \vartheta_{-t}\omega, v^0(\vartheta_{-t}\omega)\right) \right\| |\sigma_1 - \sigma_2| \leq r_2(\omega) |\sigma_1 - \sigma_2|, \end{aligned} \quad (5.35)$$

where

$$r_2(\omega) = 4 + \lambda^u + l_f \left( \sup_{\sigma \in [-\nu, 0]} \left\{ \sqrt{r_1(\vartheta_\sigma \omega)} + \|z(0, \vartheta_\sigma \omega)\| \right\} \right) + \|g\|, \quad (5.36)$$

and  $\xi$  is between  $\sigma_1$  and  $\sigma_2$ .

By Lemma 5.1, (5.35), and Lemma 5.2,  $\{\phi(t_n, \vartheta_{-t_n}\omega, v_n^0(\vartheta_{-t_n}\omega))\}_{n=1}^\infty$  satisfies conditions (i)–(iii) in Theorem 3.3. Therefore,  $\{\phi(t_n, \vartheta_{-t_n}\omega, v_n^0(\vartheta_{-t_n}\omega))\}_{n=1}^\infty$  is relative compact in  $\mathcal{C}$  and hence has a convergent subsequence in  $\mathcal{C}$ .  $\square$

We are now in a position to present our main result about the existence of a  $\mathfrak{D}$ -random attractor for  $\psi$  in  $\mathcal{C}$ .

**Theorem 5.4.** *The random dynamical system  $\psi$  has a unique  $\mathfrak{D}$ -random attractor in  $\mathcal{C}$ .*

*Proof.* Notice that  $\phi$  has a closed absorbing set  $K$  in  $\mathfrak{D}$  by Lemma 5.1 and is  $\mathfrak{D}$ -pullback asymptotically compact in  $\mathcal{C}$  by Lemma 5.3. Hence, the existence of a unique  $\mathfrak{D}$ -random attractor  $\{\mathcal{A}_1(\omega)\}_{\omega \in \Omega}$  for  $\phi$  follows from Proposition 2.11 immediately.

Since  $\psi$  and  $\phi$  are conjugated by the random homeomorphism  $T(\omega, \xi) = \xi + z^0(\omega)$ , and  $z^0(\omega) \in \mathcal{C}$  is tempered, then by Proposition 1.8.3 in [45],  $\psi$  has a unique  $\mathfrak{D}$ -random attractor  $\{\mathcal{A}_2(\omega)\}_{\omega \in \Omega}$  in  $\mathcal{C}$  which is given by

$$\mathcal{A}_2(\omega) = \left\{ \xi(\omega) + z^0(\omega) : \xi(\omega) \in \mathcal{A}_1(\omega) \right\}. \quad (5.37)$$

The proof is complete.  $\square$

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