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Research Article

On Elliptic Equations in Orlicz Spaces Involving Natural Growth Term and Measure Data

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The paper deals with the existence of solutions of elliptic equations in the framework of Orlicz spaces with right-hand side measure and natural growth term.

1. Introduction

We deal with boundary value problems

$$\begin{aligned} A(u) &= g(u)M(|\nabla u|) + \mu \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{P_μ}$$

where

$$A(u) = -\operatorname{div}(a(\cdot, u, \nabla u)), \tag{1.1}$$

Ω is a bounded domain of $\mathbf{R}^N$, with the segment property. $a : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory function (i.e., measurable with respect to x in Ω for every (s, ξ) in $\mathbf{R} \times \mathbf{R}^N$, and continuous with respect to (s, ξ) in $\mathbf{R} \times \mathbf{R}^N$ for almost every x in Ω) such that there exist two N-functions $P \ll M$ and for all $\xi, \xi^* \in \mathbf{R}^N$, $\xi \neq \xi^*$, the following hypotheses are true

$$a(x, s, \xi)\xi \geq \alpha M\left(\frac{|\xi|}{\lambda}\right), \tag{1.2}$$

$$[a(x, s, \xi) - a(x, s, \xi^*)][\xi - \xi^*] > 0, \tag{1.3}$$

$$|a(x, s, \xi)| \leq c(x) + k_1 \overline{P}^{-1} M(k_2|s|) + k_3 \overline{M}^{-1} M(k_4|\xi|), \tag{1.4}$$

where $c(x)$ belongs to $E_{\overline{M}}(\Omega)$, $c \geq 0$, k_i $(i = 1,2,3,4)$ to $\mathbf{R}^+$, and α, λ to $\mathbf{R}^+_*$.

$$M(t) \ll t^N, \qquad \int_{.}^{+\infty} \frac{\overline{M}(t)}{t^{1+N'}} dt = +\infty, \tag{1.5}$$

$$g : \mathbf{R}^+ \longrightarrow \mathbf{R}^+ \quad \text{an integrable function on } \mathbf{R}^+, \tag{1.6}$$

$$\mu \in M_b^+(\Omega). \tag{1.7}$$

For the sake of simplicity, we suppose in (1.2) that $\alpha = \lambda = 1$.

This paper is devoted to study the Dirichlet problem for some nonlinear elliptic equations whose simplest model is

$$-\Delta_p u = g(u)|\nabla u|^p + \mu. \tag{1.8}$$

This kind of problems has been widely studied. Many authors have proved results for second-order elliptic problems with lower-order terms depending on the gradient; these works include, for instance, [1–6]. After the classical example by Kazdan and Kramer (see [7]), which shows that (1.8) cannot always have solutions, two different kind of questions have been considered. On the one hand, in some papers, the existence of solutions when the source f is small in a suitable norm are proved. On the other hand, conditions on which the function g have been considered in order to get a solution for all f in a given Lebesgue space. This is the way chosen in [1, 5, 6] under the hypothesis $g \in L^1$.

In [8], the authors present some results concerning existence, nonexistence, multiplicity, and regularity of positive solutions for two elliptic quasilinear problems with Dirichlet data in a bounded domain. The first problem is similar to (1.8) and the other is

$$-\Delta_p v = \lambda f(x)(1 + b(v))^{p-1}, \tag{1.9}$$

where λ, f, b, g required some specified conditions. The first one, of unknown u, involves a gradient term with natural growth. The second one, of unknown v, presents a source term of order 0. They gave and established a precise connection between problems in u and v. Also, they proved a result of existence for the problem in v with general bounded Radon measures data and obtained some results for the problem in u by using the connection between these two problems. Other authors have established this connection between the two problems (1.8) and (1.9); one can see, for example, [9].

Many researchers have investigated the possibility to find solutions of (1.8) under the sign condition $g(s)s \geq 0$, in which case the term $g(u)|\nabla u|^p$ is said to be an absorption term. In [5], the author treated the problem (1.8) without using the above sign condition but by supposing the summability of the function g. The principal tools used is the Lebesgue decomposition theorem (see [10]) and the cut functions with respect to the measure μ. The result given is optimal.

In the spirit of the work [5], our purpose in this paper is to prove existence results in the setting of the Orlicz Sobolev space $W^1L_M(\Omega)$ when the operator does not satisfy the classical polynomial growth.

2. Preliminaries

Let $M : \mathbf{R}^+ \to \mathbf{R}^+$ be an N-function, that is, M is continuous, convex, with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$ and $M(t)/t \to \infty$ as $t \to \infty$. The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \sup\{st - M(s) : s > 0\}$.

Let P and Q be two N-functions. $P \ll Q$ means that P grows essentially less rapidly than Q; that is, for each $\varepsilon > 0$,

$$\frac{P(t)}{Q(\varepsilon t)} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{2.1}$$

The N-function M is said to satisfy the Δ_2 condition if for some $k > 0$: $M(2t) \leq kM(t)$ for all $t \geq 0$; when this inequality holds only for $t \geq t_0 > 0$, M is said to satisfy the Δ_2 condition near infinity.

Let Ω be an open subset of $\mathbf{R}^N$. The Orlicz class $\mathcal{L}_M(\Omega)$ (resp., the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that $\int_\Omega M(u(x))dx < +\infty$ (resp. $\int_\Omega M(u(x)/\lambda)dx < +\infty$ for some $\lambda > 0$).

Note that $L_M(\Omega)$ is a Banach space under the norm $\|u\|_{M,\Omega} = \inf\{\lambda > 0 : \int_\Omega M(u(x)/\lambda)dx \leq 1\}$ and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. In general $E_M(\Omega) \neq L_M(\Omega)$ and the dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$.

We now turn to the Orlicz-Sobolev space. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). This is a Banach space under the norm $\|u\|_{1,M,\Omega} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_{M,\Omega}$. Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathfrak{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathfrak{D}(\Omega)$ in $W^1L_M(\Omega)$. We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$, $\int_\Omega M((D^\alpha u_n - D^\alpha u)/\lambda)dx \to 0$ for all $|\alpha| \leq 1$. This implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. If M satisfies the Δ_2 condition on $\mathbf{R}^+$ (near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence.

Fore more details about the Orlicz spaces and their properties one can see [11, 12].

For $k > 0$, we define the truncation at height k, $T_k : \mathbf{R} \to \mathbf{R}$ by: $T_k(s) = \max(-k, \min(k, s))$.

3. Main Result

3.1. Useful Results

First, we give the following definitions and results which will be used in our main result.

The p-capacity $C_p(B, \Omega)$ of any set $B \subset \Omega$ with respect to Ω is defined in the following classical way. The p-capacity of any compact set $K \subset \Omega$ is first defined as

$$C_p(K, \Omega) = \inf\left\{ \int_\Omega |\nabla\varphi|^p dx : \varphi \in D(\Omega), \varphi \geq \chi_K \right\}, \tag{3.1}$$

where χ_K is the characteristic function of K; we will use the convention that $\inf \phi = +\infty$. The p-capacity of any open subset $U \subset \Omega$ is then defined by

$$C_p(U,\Omega) = \sup\{C_p(K,\Omega), K \text{ compact } K \subset \Omega\}. \tag{3.2}$$

Finally, the p-capacity of any subset $B \subset \Omega$ is defined by

$$C_p(B,\Omega) = \inf\{C_p(U,\Omega), U \text{ open } B \subset U\}. \tag{3.3}$$

Definition 3.1. We say that u is a weak solution of the problem (P_μ) if

$$\begin{gathered} u \text{ is measurable}, \quad T_k(u) \in W_0^1 L_M(\Omega), \\ \int_\Omega a(\cdot,u,\nabla u)\nabla v\,dx = \int_\Omega g(u)M(|\nabla u|)v\,dx + \int_\Omega v\,d\mu \quad \forall v \in D(\Omega). \end{gathered} \tag{3.4}$$

We define $M_b(\Omega)$ as the space of all Radon measures on Ω with bounded total variation, and $C_b(\Omega)$ as the space of all bounded, continuous functions on Ω, so that $\int_\Omega \varphi\,d\mu$ is defined for $\varphi \in C_b(\Omega)$ and $\mu \in M_b(\Omega)$.

We say that a sequence (μ_n) of measures in $M_b(\Omega)$ converges to a measure μ in $M_b(\Omega)$ if

$$\lim_{n\to+\infty} \int_\Omega \varphi\,d\mu_n = \int_\Omega \varphi\,d\mu_n \tag{3.5}$$

for every $\varphi \in C_b(\Omega)$. If this convergence holds only for all the continuous functions φ with compact support in Ω, then we have the usual weak $*$ convergence in $M_b(\Omega)$.

Lemma 3.2. *Under the hypotheses* (1.2)–(1.6), $f \in L^\infty(\Omega)$ *and* $f \geq 0$, *there exists at least one positive weak solution of the problem*

$$\begin{gathered} A(u) = g(u)M(|\nabla u|) + f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{P_f}$$

For the proof see [12].

Let M be a fixed N-function, we define K as the set of N functions D satisfying the following conditions:

(i) $M(D^{-1}(s))$ is a convex function,

(ii) $\int_0^{\cdot} DoB^{-1}(1/r^{1-1/N})dr < +\infty, B(t) = M(t)/t$,

(iii) there exists an N-function H such that $Ho\overline{M}^{-1}oM \leq D$ and $\overline{H} \leq D$ near infinity.

Lemma 3.3. *Let (u_n) be a sequence of solutions of the problem*

$$\begin{gathered} A(u_n) = F_n, \\ u_n \in W_0^1 L_M(\Omega), \end{gathered} \tag{3.6}$$

where (F_n) is a sequence of functions bounded in $L^1(\Omega)$. Then (u_n) is bounded in $W_0^1 L_D(\Omega)$ for all N-functions $D \in K$.

For the proof one can see that the technique used in [13] and adapted to the elliptic case gives the result, but for the simplicity we give a sketched proof.

Proof. Let denote by $\mathcal{X}_N =: N C_N^{1/N}$, C_N the measure of the unit ball of $\mathbf{R}^N$, and $\mu(\theta) = |\{|u| > \theta\}|$.

Let φ be a truncation defined by

$$\varphi(\xi) = \begin{cases} 0 & 0 \le \xi \le \theta \\ \dfrac{1}{h}(\xi - t) & \theta < \xi < \theta + h \\ 1 & \xi \ge \theta + h \\ -\varphi(-\xi) & \xi < 0 \end{cases} \tag{3.7}$$

for all $\theta, h > 0$.

Using $v = \varphi(u_n)$ as a test function, we obtain after tending h to zero

$$-\frac{d}{d\theta} \int_{\{|u_n| > \theta\}} M(|\nabla u_n|)dx \le C \int_{\{|u_n| \ge \theta\}} |F_n| dx. \tag{3.8}$$

Following the same way as in [14], we have for $D \in K$,

$$-\frac{d}{d\theta} \int_{\{|u_n| > \theta\}} D(|\nabla u_n|)dx \le (-\mu'(\theta)) D o B^{-1} \left(\left(-\frac{1}{\mathcal{X}_N \mu(\theta)^{1-1/N}} \frac{d}{d\theta} \int_{\{|u_n| > \theta\}} M(|\nabla u_n|)dx \right) \right). \tag{3.9}$$

We obtain

$$-\frac{d}{d\theta} \int_{\{|u_n| > \theta\}} D(|\nabla u_n|)dx \le (-\mu'(\theta)) D o B^{-1} \left(-\frac{C}{\mathcal{X}_N \mu(\theta)^{1-1/N}} \right). \tag{3.10}$$

So,

$$\begin{aligned} \int_\Omega D(|\nabla u_n|)dx &= \int_0^{+\infty} \left(-\frac{d}{d\theta} \int_{\{|u_n| > \theta\}} D(|\nabla u_n|)dx \right) dt \\ &\le \frac{1}{C'} \int_0^{C'|\Omega|} D o B^{-1} \left(\left(\frac{C}{s^{1-1/N}} \right) \right) ds. \end{aligned} \tag{3.11}$$

Then the sequence (u_n) is bounded in $W_0^1 L_D(Q)$. □

Lemma 3.4. *Let μ be a nonnegative Radon measure which is concentrated in a set E of zero p-capacity. Then there exists a sequence (ψ_δ) of $D(\Omega)$ functions such that*

$$\lim_{\delta\to\infty}\int_\Omega |\nabla\psi_\delta|^p dx = 0, \quad 0 \le \psi_\delta \le 1, \quad \lim_{\delta\to\infty}\int_\Omega (1-\psi_\delta)d\mu = 0. \tag{3.12}$$

For the proof see [15].

Remark 3.5. By the above lemma, we have that (ψ_δ) converge to zero both strongly in $W_0^{1,p}(\Omega)$, a.e. in Ω, and in the weak * topology of $L^\infty(\Omega)$.

3.2. Existence Result

In what follows, we suppose that the set K is nonempty.

Theorem 3.6. *Let $x_0 \in \Omega$, under the hypotheses* (1.2)–(1.6) *and $N \ge 2$, there exists at least one positive weak solution of the problem*

$$\begin{gathered} A(u) = g(u)M(|\nabla u|) + \delta_{x_0} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag*{$\left(P_{\delta_{x_0}}\right)$}$$

Remark 3.7. (1) The condition $N \ge 2$ is very important to ensure the existence of cut functions for the measure δ_{x_0}. The case $N = 1$ is easily treated, since we come back to the variational case.

(1.9) The condition $\int_{\cdot}^{+\infty}(\overline{M}(t)/t^{1+N'})dt = +\infty$ is supposed to guarantee that we are not in the variational case and the study has a sense (see [16]).

Remark 3.8. The conditions (i), (ii), and (iii) permit us to determine the regularity of the solutions of $\left(P_{\delta_{x_0}}\right)$ and improve the one given by Porretta in [5]. If $M(t) = t^p$, one can find a solution u such that $\int_\Omega |\nabla u|^{N(p-1)/(N-1)}/Log^\sigma(e+|\nabla u|) < +\infty$, for some $\sigma > 1$.

Remark 3.9 (see [5]). The condition $g \in L^1$ is optimal in the sense introduced by Porretta in [5].

Indeed, if μ is the Dirac mass, there is no solution which can be obtained by approximation. In particular, in the reaction case $(g(s)s \le 0)$, if μ is approximated by a sequence of smooth functions, the sequence of approximating solutions converges to a solution of (P_μ) if $g \in L^1$, while it blows up everywhere if $g \notin L^1$.

Finally, let us say a few words on how positive constant will be denoted hereafter. If no otherwise specified, we will write C to denote any positive constant (possibly different) which only depends on the data, that is on quantities that are fixed in the assumptions (N, Ω, and so on...); in any case such constants never depend on the different indexes having a limit. In the sequel and throughout the paper, we will omit for simplicity the dependence on x in the function $a(x,s,\xi)$ and denote $\epsilon(n,j,\delta,s,m)$ all quantities (possibly different) such that

$$\lim_{m\to\infty}\lim_{s\to\infty}\lim_{\delta\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\epsilon(n,j,\delta,s,m) = 0, \tag{3.13}$$

and this will be in the order in which the parameters we use will tend to infinity, that is, first n, then j, δ, s, and finally m. Similarly, we will write only $\epsilon(n)$, or $\epsilon(n, j), \ldots$ to mean that the limits are made only on the specified parameters. Moreover, for the sake of simplicity, in what follows, the convergence, even if not explicitly stressed, may be understood to be taken possibly up to a suitable subsequence extraction.

3.2.1. A Sequence of Approximating Problems

Consider the approximate problem

$$\begin{gathered} A(u_n) = g(u_n)M(|\nabla u_n|) + f_n \quad \text{in } \Omega, \\ u_n = 0 \quad \text{in } \partial\Omega, \end{gathered} \tag{P_{f_n}}$$

where (f_n) is a smooth sequence of functions such that $\|f_n\|_1 \leq C$ and $f_n \rightharpoonup \delta_{x_0}$ in $M_b(\Omega)$.

The existence of solutions of the above problem $u_n \in W_0^1 L_M(\Omega)$ was ensured by Lemma 3.2.

3.2.2. A Priori Estimates

Lemma 3.10. *There exists a subsequence of (u_n) (also denoted (u_n)); there exists a measurable function u such that $T_k(u) \in W_0^1 L_M(\Omega)$ and $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^1 L_M(\Omega)$, strongly in $E_M(\Omega)$, and a.e. in Ω.*

Proof. Let $v = T_k(u_n)\exp(\int_0^{u_n} g(s)ds)$ as test function in (P_{f_n}). One has

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n)\exp\left(\int_0^{u_n} g(s)ds\right) \\ &\quad + \int_\Omega a(x, u_n, \nabla u_n)\nabla u_n g(u_n) T_k(u_n)\exp\left(\int_0^{u_n} g(s)ds\right) \\ &\quad = \int_\Omega M(|\nabla u_n|) g(u_n) T_k(u_n)\exp\left(\int_0^{u_n} g(s)ds\right) + \int_\Omega f_n T_k(u_n)\exp\left(\int_0^{u_n} g(s)ds\right), \end{aligned} \tag{3.14}$$

then

$$\int_\Omega M(|\nabla T_k(u_n)|)\exp\left(\int_0^{u_n} g(s)ds\right)dx \leq \int_\Omega f_n T_k(u_n)\exp\left(\int_0^{u_n} g(s)ds\right) \leq Ck, \tag{3.15}$$

so $(T_k(u_n))$ is bounded in $W_0^1 L_M(\Omega)$.

There exist a subsequence also denoted (u_n) and a measurable function ω_k such that

$$T_k(u_n) \rightharpoonup \omega_k, \quad \text{weakly in } W_0^1 L_M(\Omega), \text{ a.e. in } \Omega, \text{ and strongly in } E_M(\Omega). \tag{3.16}$$

By an easy argument we can see that, there exists a measurable function u such that

$$T_k(u) = \omega_k. \tag{3.17}$$

□

3.2.3. Almost Everywhere Convergence of Gradients

Lemma 3.11. *The subsequence (u_n) obtained in Lemma 3.10 satisfies*

$$\nabla u_n \longrightarrow \nabla u \quad \textit{a.e. in } \Omega. \tag{3.18}$$

Proof. Let us recall that since the N-capacity $C_N(\{x_0\}, \Omega) = 0$, there exists a sequence (ψ_δ) satisfying the Lemma 3.4 with $\mu = \delta_{x_0}$ and $p = N$. □

Step 1. In this step we will show the following:

(i) $\lim_{n\to+\infty} \int_{\Omega_r} (a(\cdot, T_k(u_n), \nabla T_k(u_n)) - a(\cdot, T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))(1 - \psi_\delta) dx = 0$,

(ii) $\lim_{n\to+\infty} \int_{\Omega_r} (a(\cdot, T_k(u_n), \nabla T_k(u_n)) - a(\cdot, T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u)) \psi_\delta dx = 0$.

Let $m > 0$, $k > 0$ such that $m > k$. Let ρ_m be a truncation defined by

$$\rho_m(s) = \begin{cases} 1 & |s| \le m, \\ m + 1 - |s| & m < |s| < m + 1, \\ 0 & |s| \ge m + 1. \end{cases} \tag{3.19}$$

Let $s > 0$, $\Omega^s = \{x \in \Omega : |\nabla T_k(u)| \le s\}$, $\Omega_j^s = \{x \in \Omega : |\nabla T_k(v_j)| \le s\}$.

We denote, respectively, by χ^s, χ_j^s the indicator function of Ω^s, Ω_j^s.

Let $v_j \in D(\Omega)$ such that $v_j \to T_k(u)$ with the modular convergence in $W_0^1 L_M(\Omega)$ (see [11]).

Let $z_{n,m}^{j,\delta} = (T_k(u_n) - T_k(v_j)) \exp(\int_0^{u_n} g(s) ds)(1 - \psi_\delta) \rho_m(u_n)$, as test function in the approximate problem. Then

$$\left\langle A(u_n), z_{n,m}^{j,\delta} \right\rangle = \int_\Omega g(u_n) M(|\nabla u_n|) z_{n,m}^{j,\delta} dx + \int_\Omega f_n z_{n,m}^{j,\delta} dx := J_1 + J_2. \tag{3.20}$$

We have

$$
\begin{aligned}
&\int_{\Omega} a(\cdot, u_n, \nabla u_n)\nabla z_{n,m}^{j,\delta} dx \\
&\quad= \int_{\Omega} a(\cdot, u_n, \nabla u_n)\rho_m(u_n)\left(\nabla T_k(u_n) - \nabla T_k(v_j)\right)\exp\left(\int_0^{u_n} g(s)ds\right)(1-\psi_\delta)dx \\
&\qquad- \int_{\Omega} a(\cdot, u_n, \nabla u_n)\rho_m(u_n)\left(T_k(u_n) - T_k(v_j)\right)\exp\left(\int_0^{u_n} g(s)ds\right)\nabla\psi_\delta dx \\
&\qquad+ \int_{\Omega} a(\cdot, u_n, \nabla u_n)\nabla u_n \rho_m'(u_n)\left(T_k(u_n) - T_k(v_j)\right)\exp\left(\int_0^{u_n} g(s)ds\right)(1-\psi_\delta)dx \\
&\qquad+ \int_{\Omega} a(\cdot, u_n, \nabla u_n)\nabla u_n \rho_m(u_n)\left(T_k(u_n) - T_k(v_j)\right)g(u_n)\exp\left(\int_0^{u_n} g(s)ds\right)(1-\psi_\delta)dx \\
&\quad= \int_{\Omega}\left(a(\cdot, T_k(u_n), \nabla T_k(u_n)) - a\left(\cdot, T_k(u_n), \nabla T_k(v_j)\chi_j^s\right)\right)\left(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right)\rho_m(u_n) \\
&\qquad\quad\times\exp\left(\int_0^{u_n} g(s)ds\right)(1-\psi_\delta)dx \\
&\qquad+ \int_{\Omega} a\left(\cdot, T_k(u_n), \nabla T_k(v_j)\chi_j^s\right)\left(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right)\rho_m(u_n) \\
&\qquad\quad\times\exp\left(\int_0^{u_n} g(s)ds\right)(1-\psi_\delta)dx \\
&\qquad+ \int_{\Omega-\Omega_j^s} a(\cdot, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)\rho_m(u_n)\exp\left(\int_0^{u_n} g(s)ds\right)(1-\psi_\delta)dx \\
&\qquad- \int_{\Omega-\Omega_j^s} a(\cdot, T_k(u_n), \nabla T_k(u_n))\nabla T_k(v_j)\rho_m(u_n)\exp\left(\int_0^{u_n} g(s)ds\right)(1-\psi_\delta)dx \\
&\qquad- \int_{|u_n|>k} a(\cdot, u_n, \nabla u_n)\nabla T_k(v_j)\rho_m(u_n)\exp\left(\int_0^{u_n} g(s)ds\right)(1-\psi_\delta)dx \\
&\qquad+ \int_{\Omega} a(\cdot, u_n, \nabla u_n)\nabla u_n \rho_m'(u_n)\left(T_k(u_n) - T_k(v_j)\right)\exp\left(\int_0^{u_n} g(s)ds\right)(1-\psi_\delta)dx \\
&\qquad+ \int_{\Omega} a(\cdot, u_n, \nabla u_n)\nabla u_n \rho_m(u_n)g(u_n)\exp\left(\int_0^{u_n} g(s)ds\right)(1-\psi_\delta)dx \\
&\quad= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned}
\tag{3.21}
$$

Then,

$$
\left\langle A(u_n), z_{n,m}^{j,\delta}\right\rangle = \sum_{i=1}^{7} I_i = J_1 + J_2. \tag{3.22}
$$

About I_2: it is obvious since

$$a\left(\cdot, T_k(u_n), \nabla T_k(v_j)\chi_j^s\right) \longrightarrow a\left(\cdot, T_k(u), \nabla T_k(v_j)\chi_j^s\right) \text{ strongly,} \tag{3.23}$$

that is

$$\int_\Omega a\left(\cdot, T_k(u_n), \nabla T_k(v_j)\chi_j^s\right)\left(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right)\rho_m(u_n)(1-\psi_\delta)\,dx = \epsilon(n,j). \tag{3.24}$$

About I_4: since $(T_k(u_n))$ is bounded in $W_0^1 L_M(\Omega)$ and $\nabla T_k(v_j)\chi_{\Omega-\Omega_j^s} \in (E_M(\Omega))^N$, there exists a measurable function h_k such that (up to a subsequence also denoted $T_k(u_n)$)

$$a(\cdot, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k \quad \text{weakly in } \left(L_{\overline{M}}(\Omega)\right)^N. \tag{3.25}$$

Then $-\int_{\Omega-\Omega_j^s} a(\cdot, T_k(u_n), \nabla T_k(u_n))\nabla T_k(v_j)(1-\psi_\delta)dx = -\int_{\Omega-\Omega_j^s} h_k \nabla T_k(v_j)(1-\psi_\delta)dx + \epsilon(n)$.

Since $v_j \to T_k(u)$ in modular convergence, then

$$-\int_{\Omega-\Omega_j^s} h_k \nabla T_k(v_j)(1-\psi_\delta)\,dx = -\int_{\Omega-\Omega^s} h_k \nabla T_k(u)(1-\psi_\delta)\,dx + \epsilon(j). \tag{3.26}$$

So,

$$-\int_{\Omega-\Omega_j^s} a(\cdot, T_k(u_n), \nabla T_k(u_n))\nabla T_k(v_j)(1-\psi_\delta)\,dx = -\int_{\Omega-\Omega^s} h_k \nabla T_k(u)(1-\psi_\delta)\,dx + \epsilon(n,j). \tag{3.27}$$

About I_5: since $\rho(u_n) = 0$ on the set $\{|u_n| \geq m+1\}$, then

$$\begin{aligned} &-\int_{|u_n|>k} a(\cdot, u_n, \nabla u_n)\nabla T_k(v_j)\rho_m(u_n)(1-\psi_\delta)\,dx \\ &\quad = -\int_{|u_n|>k} a(\cdot, T_{m+1}(u_n), \nabla T_{m+1}(u_n))\nabla T_k(v_j)\rho_m(u_n)(1-\psi_\delta)\,dx \end{aligned} \tag{3.28}$$

and as for I_4, one has

$$\begin{aligned} &-\int_{|u_n|>k} a(\cdot, u_n, \nabla u_n)\nabla T_k(v_j)\rho_m(u_n)(1-\psi_\delta)\,dx \\ &\quad = -\int_{|u|>k} h_{m+1}\nabla T_k(u)\rho_m(u)(1-\psi_\delta)\,dx + \epsilon(n,j) = \epsilon(n,j) \end{aligned} \tag{3.29}$$

since $\nabla T_k(u) = 0$ when $|u| \geq k$.

About I_6: let $v = \rho_m(u_n)(1-\psi_\delta)\exp(\int_0^{u_n} g(s)ds)$ as test function, in one hand we obtain

$$\begin{aligned}
&\int_{\{m\le u_n\le m+1\}} a(\cdot,u_n,\nabla u_n)\nabla u_n \exp\left(\int_0^{u_n} g(s)ds\right)(1-\psi_\delta)dx \\
&\quad + \int_\Omega a(\cdot,u_n,\nabla u_n)\nabla u_n g(u_n)\exp\left(\int_0^{u_n} g(s)ds\right)\rho_m(u_n)(1-\psi_\delta)dx \\
&\quad - \int_\Omega a(\cdot,u_n,\nabla u_n)\nabla\psi_\delta \exp\left(\int_0^{u_n} g(s)ds\right)\rho_m(u_n)dx \\
&= \int_\Omega g(u_n)M(|\nabla u_n|)\exp\left(\int_0^{u_n} g(s)ds\right)\rho_m(u_n)(1-\psi_\delta)dx \\
&\quad + \int_\Omega f_n \exp\left(\int_0^{u_n} g(s)ds\right)\rho_m(u_n)(1-\psi_\delta)dx.
\end{aligned} \tag{3.30}$$

So,

$$\begin{aligned}
&\int_{\{m\le u_n\le m+1\}} a(\cdot,u_n,\nabla u_n)\nabla u_n \exp\left(\int_0^{u_n} g(s)ds\right)(1-\psi_\delta)dx \\
&\quad \le \int_\Omega a(\cdot,u_n,\nabla u_n)\nabla\psi_\delta \exp\left(\int_0^{u_n} g(s)ds\right)\rho_m(u_n)dx \\
&\qquad + \int_\Omega f_n \exp\left(\int_0^{u_n} g(s)ds\right)\rho_m(u_n)(1-\psi_\delta)dx \\
&\quad \le C\left\|\nabla\psi_\delta\right\|_N \|a(\cdot,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\|_{N'} + C\int_{|u_n|\ge m} f_n(1-\psi_\delta)dx.
\end{aligned} \tag{3.31}$$

On the other hand

$$\begin{aligned}
I_6 &= \int_\Omega a(\cdot,u_n,\nabla u_n)\nabla u_n \rho_m'(u_n)(T_k(u_n)-T_k(v_j))\exp\left(\int_0^{u_n} g(s)ds\right)(1-\psi_\delta)dx \\
&= \int_{\{m\le u_n\le m+1\}} a(\cdot,u_n,\nabla u_n)\nabla u_n(T_k(u_n)-T_k(v_j))\exp\left(\int_0^{u_n} g(s)ds\right)(1-\psi_\delta)dx \\
&\le 2k\left[\left\|\nabla\psi_\delta\right\|_N \|a(\cdot,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\|_{N'} + C\int_{|u_n|\ge m} f_n(1-\psi_\delta)dx\right] \\
&= \epsilon(n,\delta,m).
\end{aligned} \tag{3.32}$$

Let us come back to our main estimation (i). We have

$$\begin{aligned}
&\int_\Omega (a(\cdot,T_k(u_n),\nabla T_k(u_n)) - a(\cdot,T_k(u_n),\nabla T_k(u)\chi^s))(\nabla T_k(u_n)-\nabla T_k(u)\chi^s)\rho_m(u_n)(1-\psi_\delta)dx \\
&= \int_\Omega \left(a(\cdot,T_k(u_n),\nabla T_k(u_n)) - a\left(\cdot,T_k(u_n),\nabla T_k(v_j)\chi_j^s\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big)\rho_m(u_n)(1-\psi_\delta)\,dx \\
&+ \int_\Omega a(\cdot, T_k(u_n), \nabla T_k(u_n))\Big(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s\Big)\rho_m(u_n)(1-\psi_\delta)\,dx (= \epsilon(n,j)) \\
&- \int_\Omega a(\cdot, T_k(u_n), \nabla T_k(u)\chi^s)(\nabla T_k(u_n) - \nabla T_k(u)\chi^s)\rho_m(u_n)(1-\psi_\delta)\,dx (= \epsilon(n,j)) \\
&+ \int_\Omega a\Big(\cdot, T_k(u_n), \nabla T_k(v_j)\chi_j^s\Big)\Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big)\rho_m(u_n)(1-\psi_\delta)\,dx (= \epsilon(n,j)).
\end{aligned}
\tag{3.33}
$$

Then we deduce that

$$
\begin{aligned}
&\int_\Omega (a(\cdot, T_k(u_n), \nabla T_k(u_n)) - a(\cdot, T_k(u_n), \nabla T_k(u)\chi^s))(\nabla T_k(u_n) - \nabla T_k(u)\chi^s)\rho_m(u_n)(1-\psi_\delta)\,dx \\
&\quad = \int_\Omega \Big(a(\cdot, T_k(u_n), \nabla T_k(u_n)) - a\Big(\cdot, T_k(u_n), \nabla T_k(v_j)\chi_j^s\Big)\Big) \\
&\qquad \times \Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big)\rho_m(u_n)(1-\psi_\delta)\,dx + \epsilon(n,j) \\
&\quad \le \epsilon(n,j) + C\int_{\Omega-\Omega^s} h_k \nabla T_k(u)(1-\psi_\delta)\,dx + C\int_{|u_n|\ge m} f_n(1-\psi_\delta)\,dx.
\end{aligned}
\tag{3.34}
$$

Since we have $\sum_{i=1}^{7} I_i = J_1 + J_2$, $I_3 \ge 0$ and using (1.2), we deduce that

$$
\begin{aligned}
I_1 &\le J_2 - I_2 - I_3 - I_4 - I_5 - I_6 \\
&\le \int_\Omega f_n z_{n,m}^{j,\delta} + C\int_{\Omega-\Omega^s} h_k \nabla T_k(u)(1-\psi_\delta)\,dx \\
&\quad + Ck\|\nabla\psi_\delta\|_N \|a(\cdot, T_{m+1}(u_n), \nabla T_{m+1}(u_n))\|_{N'} \\
&\quad + C\int_{|u_n|\ge m} f_n(1-\psi_\delta)\,dx.
\end{aligned}
\tag{3.35}
$$

For $r \le s$, one has

$$
\begin{aligned}
0 \le I_1(r) \le I_1(s) \le{}& C\int_{\Omega-\Omega^s} h_k \nabla T_k(u)(1-\psi_\delta)\,dx \\
&+ Ck\|\nabla\psi_\delta\|_N \|a(\cdot, T_{m+1}(u_n), \nabla T_{m+1}(u_n))\|_{N'} \\
&+ C\int_{|u_n| \ge m} f_n(1-\psi_\delta)\,dx + \epsilon(n,j).
\end{aligned}
\tag{3.36}
$$

We have $(T_{m+1}(u_n))$ is bounded in $W_0^1L_M(\Omega)$, $(a(\cdot,T_{m+1}(u_n),\nabla T_{m+1}(u_n)))$ in $(L_{\overline{M}}(\Omega))^N)$ and so in $(L^{N'}(\Omega))^N$ since $M(t) \ll t^N$.

Since we have

$$\begin{gathered}
\lim_{\delta\to+\infty}\int_\Omega |\nabla\psi_\delta|^N dx = 0,\\
\lim_{\delta\to+\infty}\int_\Omega (1-\psi_\delta)d\delta_{x_0} = 0,\\
\lim_{s\to+\infty}|\Omega-\Omega^s| = 0, \qquad \lim_{m\to+\infty}|\{|u|\geq m\}| = 0,
\end{gathered}\tag{3.37}$$

we easily deduce (i).

For (ii) we proceed as in (i) by using the fact that $\int_{|u_n|\geq m} f_n\psi_\delta dx = \epsilon(n,\delta,m)$.

Then we conclude that

$$\lim_{n\to+\infty}\int_{\Omega_r}(a(\cdot,T_k(u_n),\nabla T_k(u_n)) - a(\cdot,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))dx = 0. \tag{3.38}$$

Step 2. In this step we prove that $\nabla u_n \to \nabla u$ a.e. in Ω.

Since $(T_k(u_n))$ is bounded in $W_0^1L_M(\Omega)$, then $(|\nabla T_k(u_n)|)$ is finite a.e. in Ω.

Hence there exists a measurable set E such that $|E| = 0$ and $|\nabla T_k(u_n)| < +\infty$ in $\Omega - E$.

Let $\eta(x) = \lim_n(\nabla T_k(u_n))$, then by using Fatou lemma, we get

$$\int_\Omega |\eta(x)|dx \leq \underline{\lim}\int_\Omega |\nabla T_k(u_n)|dx \leq C. \tag{3.39}$$

So, $|\eta(x)| < +\infty$ a.e. in Ω and $\nabla T_k(u_n) \to \eta$ a.e. in Ω, and for a subsequence still denoted by $T_k(u_n)$, we obtain from (3.38) that

$$\begin{aligned}
0 &= \lim_{n\to\infty}(a(\cdot,T_k(u_n),\nabla T_k(u_n)) - a(\cdot,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u)) \quad \text{in } \Omega\\
&= \big(a(\cdot,u,\eta) - a(\cdot,u,\nabla T_k(u))\big)\big(\eta-\nabla T_k(u)\big).
\end{aligned}\tag{3.40}$$

Combining with (1.3), we get

$$\nabla T_k(u) = \eta \quad \text{a.e. in } \Omega. \tag{3.41}$$

Therefore, we have

$$\nabla T_k(u_n) \longrightarrow \nabla T_k(u) \quad \text{a.e. in } \Omega. \tag{3.42}$$

Lemma 3.12. *For all $k > 0$,*

$$\nabla T_k(u_n) \longrightarrow \nabla T_k(u) \quad \textit{for the modular convergence in } (L_M(\Omega))^N. \tag{3.43}$$

For the proof we can adopt the same way as in [13].

3.2.4. The Convergence of the Problems (P_{f_n}) and the Completion of the Proof of Theorem 3.6

In one hand by using as test function $\varphi = \int_0^{u_n} g(s)\chi_{\{s>h\}} \exp(\int_0^{u_n} g(s)ds)$ in (P_{f_n}), we can deduce as in [5] that $g(u_n)M(|\nabla u_n|)$ converge strongly in $L^1(\Omega)$ to $g(u)M(|\nabla u|)$.

In the other hand, by Lemma 3.3, we deduce that (u_n) is bounded in $W_0^1 L_D(\Omega)$ for every $D \in K$. Then $(a(\cdot, u_n, \nabla u_n))$ is bounded in $L_H(\Omega)$ and the passage to the limit is an easy task.

Corollary 3.13. *Under the hypotheses* (1.2)–(1.7), *μ is a convex combination of Dirac measures ($\mu = \sum_{i=1}^J \alpha_j \delta_{x_j}$ $(x_j \in \Omega)$ and $\sum_{i=1}^J \alpha_j = 1$) and the problem (P_μ) admits at least one weak solution.*

The proof is a simple adaptation of the one of Theorem 3.6 by taking in $z_{n,m}^{j,\delta}$, $\sum_{i=1}^J \alpha_j (1-\varphi_{\delta_{x_j}})$ in the place of $(1-\varphi_{\delta_{x_j}})$, where $\varphi_{\delta_{x_j}}$ is the cut functions corresponding to the measure δ_{x_j}.

Remark 3.14. The technique used in the proof of Theorem 3.6 allows us to prove that the problem

$$\begin{gathered} A(u) = g(u)a(\cdot, u, \nabla u)\nabla u + \delta_{x_0} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.44}$$

has a weak solution.

3.3. General Case

Before giving the general case of Theorem 3.6, let's recall that the set of finite convex combination of Dirac measures is dense in the set of measures probability.

In the following theorem we denote by μ_n the sequence defined by $\mu_n = \sum_{i=1}^n \alpha_i \delta_{x_i}$ for some $x_i \in \Omega$, and $\alpha_i \in \mathbf{R}^+$ with $\sum_{i=1}^n \alpha_i = 1$.

Theorem 3.15. *Let (μ_n) be a sequence of Radon measures convergent to a Radon measure μ, and let (u_n), a sequence of weak solutions of (P_{μ_n}). Then there exists $M > 0$ such that*

$$\int_\Omega M(|\nabla T_k(u_n)|)dx \le Mk \tag{3.45}$$

for every n and every $k > 0$. Moreover, there exists a measurable function u such that $T_k(u) \in W_0^1 L_M(\Omega)$ and u is a weak solution of (P_μ).

Proof. A priori estimate: Let J_η be a sequence of mollifiers functions. Let $v = T_k(u_n) \exp(\int_0^{u_n} g(s)\ ds) * J_\eta$ as test function in (P_{μ_n})

$$\int_\Omega a(x, u_n, \nabla u_n) \left[\nabla T_k(u_n) \exp\left(\int_0^{u_n} g(s)ds\right) * J_\eta\right] + \int_\Omega a(x, u_n, \nabla u_n) \left[\nabla u_n g(u_n) T_k(u_n) \exp\left(\int_0^{u_n} g(s)ds\right) * J_\eta\right]$$

$$= \int_{\Omega} M(|\nabla u_n|) g(u_n) T_k(u_n) \exp\left(\int_0^{u_n} g(s)ds\right) * J_\eta$$
$$+ \int_{\Omega} T_k(u_n) \exp\left(\int_0^{u_n} g(s)ds\right) * J_\eta d\mu_n, \tag{3.46}$$

then, by using an easy argument, the properties of convolution, and tending η to zero, we get

$$\int_{\Omega} M(|\nabla T_k(u_n)|) \exp\left(\int_0^{u_n} g(s)ds\right) dx \leq Ck. \tag{3.47}$$

So $(T_k(u_n))$ is bounded in $W_0^1 L_M(\Omega)$ and as above there exists a measurable function u such that

$$T_k(u_n) \rightharpoonup T_k(u), \text{ weakly in } W_0^1 L_M(\Omega), \text{ a.e. in } \Omega, \text{ and strongly in } E_M(\Omega). \tag{3.48}$$

Almost everywhere convergence of gradients: Let us consider $z_{n,m}^{j,\delta} * J_\eta$ as test function in (P_{μ_n}), we have

$$\left\langle A(u_n), z_{n,m}^{j,\delta} * J_\eta \right\rangle = \int_{\Omega} g(u_n) M(|\nabla u_n|) z_{n,m}^{j,\delta} * J_\eta dx + \int_{\Omega} z_{n,m}^{j,\delta} * J_\eta d\mu_n. \tag{3.49}$$

We prove easily, since $z_{n,m}^{j,\delta} \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$, that

$$\begin{gathered}
\left\langle A(u_n), z_{n,m}^{j,\delta} * J_\eta \right\rangle = \left\langle A(u_n), z_{n,m}^{j,\delta} \right\rangle + \epsilon(\eta), \\
\int_{\Omega} g(u_n) M(|\nabla u_n|) z_{n,m}^{j,\delta} * J_\eta dx = \int_{\Omega} g(u_n) M(|\nabla u_n|) z_{n,m}^{j,\delta} dx + \epsilon(\eta), \\
\int_{\Omega} z_{n,m}^{j,\delta} * J_\eta d\mu_n = \sum_{i=1}^{n} \alpha_i z_{n,m}^{j,\delta}(x_i) + \epsilon(\eta), \\
\sum_{i=1}^{n} \alpha_i z_{n,m}^{j,\delta}(x_i) = \epsilon(\eta, n, \delta, m),
\end{gathered} \tag{3.50}$$

Also, $\int_{\Omega} \rho_m(u_n)(1-\varphi_\delta) \exp(\int_0^{u_n} g(s)ds) * J_\eta d\mu_n = \epsilon(\eta, n, \delta, m)$.

By using the above estimation, we obtain a similar equation to (3.20) and we follow the same technique used in the step of almost everywhere convergence of gradients in Theorem 3.6 to prove the existence result. □

Corollary 3.16. *Under the hypotheses* (1.2)–(1.7), *μ is positive Radon measure and the problem* (P_μ) *admits at least one weak solution.*

Since Ω is bounded and by using the approximation of $\mu/\mu(\Omega)$ which is a measure of probability, the existence of solutions can be obtained as consequence of the last theorem.

Remark 3.17. Let us recall that if we suppose that the N-functions M and $\overline{M}$ satisfy the Δ_2 condition, we can prove with the same way as in Theorem 3.6 that the problem (P_μ) has a weak solution for all singular measure μ in the sense that it is concentrated in some Borel set with zero M-capacity.

Remark 3.18. One can see that the technique used in this paper can be adopted to prove the existence of solutions of the following problem

$$\begin{gathered} -\operatorname{div}(a(\cdot,u,\nabla u)) = g(u)|\nabla u|^{p(x)} + \mu \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.51}$$

where a satisfies, for almost every x in Ω, for every s in $\mathbf{R}$, for every ξ and ξ^* in $\mathbf{R}^N$,

$$\begin{gathered} a(x,s,\xi)\xi \geq |\xi|^{p(x)}, \\ [a(x,s,\xi) - a(x,s,\xi^*)][\xi - \xi^*] > 0, \quad \xi \neq \xi^*, \\ |a(x,s,\xi)| \leq C\Big(k(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}\Big). \end{gathered} \tag{3.52}$$

Here $p : \Omega \to]1,+\infty[$ is a measurable function such that

$$1 < \operatorname*{ess\,inf}_{x\in\Omega} p(x) \leq \operatorname*{ess\,sup}_{x\in\Omega} p(x) < N, \tag{3.53}$$

where k is a nonnegative function in $L^{p'(\cdot)}(\Omega)$, $p'(x) = p(x)/(p(x)-1)$, and C is a positive real.

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