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Research Article

The Zeros of Difference Polynomials of Meromorphic Functions

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We investigate the value distributions of difference polynomials $\Delta f(z) - af(z)^n$ and $f(z)^n f(z+c)$ which related to two well-known differential polynomials, where $f(z)$ is a meromorphic function.

1. Introduction and Main Results

In this paper, we will assume that the reader is familiar with the fundamental results and the standard notation of the Nevanlinna value distribution theory of meromorphic functions (see [1, 2]). The term "meromorphic function" will mean meromorphic in the whole complex plane $\mathbb{C}$. In addition, we will use notations $\rho(f)$ to denote the order of growth of a meromorphic function $f(z)$, $\lambda(f)$ to denote the exponents of convergence of the zero-sequence of a meromorphic function $f(z)$, $\lambda(1/f)$ to denote the exponents of convergence of the sequence of distinct poles of $f(z)$.

Hayman [3] proved the following famous result.

Theorem A. *If $f(z)$ is a transcendental meromorphic function, $n \geq 5$ is an integer, and $a(\neq 0)$ is a constant, then $f'(z) - af(z)^n$ assumes all finite values infinitely often.*

He also conjectured in [1] that the same result holds for $n = 3$ and 4. However, Mues [4] proved that the conjecture is not true for $n = 4$ by providing a counterexample and proved that $f' - af^4$ has infinitely many zeros. If $f(z)$ is a transcendental entire function, $n \geq 3$ holds in Theorem A.

Recently, many papers have focused on complex difference, giving many difference analogues in value distribution theory of meromorphic functions.

It is well known that $\Delta f(z) = f(z+c) - f(z)$, where $c \in \mathbb{C} \setminus \{0\}$ is a constant satisfying $f(z+c) - f(z) \not\equiv 0$) which is regarded as the difference counterpart of $f'(z)$, so that $\Delta f(z) - af(z)^n$ is regarded as the difference counterpart $f'(z) - af(z)^n$, where $a \in \mathbb{C} \setminus \{0\}$.

In 2011, Chen [5] considered the difference counterpart of Theorem A under the condition that f is transcendental entire.

Theorem B. *If $f(z)$ is a transcendental entire function of finite order, and let $a, c \in \mathbb{C} \setminus \{0\}$ be constants, with c such that $f(z+c) \not\equiv f(z)$. Set $\Psi_n(z) = \Delta f(z) - af(z)^n$ and $n \geq 3$ is an integer. Then $\Psi_n(z)$ assumes all finite values infinitely often, and for every $b \in \mathbb{C}$ one has $\lambda(\Psi_n(z) - b) = \rho(f)$.*

Theorem C. *If $f(z)$ is a transcendental entire function of finite order with a Borel exceptional values 0. Let $a, c \in \mathbb{C} \setminus \{0\}$ be constants, with c such that $f(z+c) \not\equiv f(z)$. Then $\Psi_2(z)$ assumes the value b infinitely often, and $\lambda(\Psi_n(z) - b) = \rho(f)$.*

Theorem D. *If $f(z)$ is a transcendental entire function of finite order with a nonzero Borel exceptional values d. Let $a, c \in \mathbb{C} \setminus \{0\}$ be constants, with c such that $f(z+c) \not\equiv f(z)$. Then for $b \in \mathbb{C}$ with $b \neq -ad^2$, $\Psi_2(z)$ assumes the value b infinitely often, and $\lambda(\Psi_n(z) - b) = \rho(f)$.*

In this paper, we will extend and improve the above results from entire functions to meromorphic functions.

Theorem 1.1. *If $f(z)$ is a transcendental meromorphic function with exponent of convergence of poles $\lambda(1/f) < \rho(f) < +\infty$, and let $a, c \in \mathbb{C} \setminus \{0\}$ be constants, with c such that $f(z+c) \not\equiv f(z)$. Set $\Psi_n(z) = \Delta f(z) - af(z)^n$ and $n \geq 3$ is an integer. Then $\Psi_n(z)$ assumes all finite values infinitely often, and for every $b \in \mathbb{C}$ one has $\lambda(\Psi_n(z) - b) = \rho(f)$.*

Remark 1.2. Compared with Theorem 1.2 in [6], our result not only gives a Picard type result but also gives an estimate of numbers of b-points, namely, $\lambda(\Psi_n(z) - b) = \rho(f)$ for every $b \in \mathbb{C}$. Our method bases on [5], which is different from the method in [6, 7].

In the same paper, Chen gave the following example to show the Borel exceptional value may arise in Theorem D.

Example 1.3. For $f(z) = \exp\{z\} + 1$, $c = \log 3$, $a = 1$, we have $\Psi_2(z) = \Delta f(z) - af(z)^2 = -\exp\{2z\} - 1$. Here $\Psi_2(z) \neq -1$, which shows that the Borel exceptional value $-ad^2 (= -1)$ may arise.

Naturally, it is an interesting question to find the conditions which can remove the Borel exceptional value of $\Psi_2(z)$ when $d \neq 0$.

Example 1.4. For $f(z) = \exp\{z\} + 1$, $c = \pi i$, $a = 1$, we have $\Psi_2(z) = \Delta f(z) - af(z)^2 = -(e^z + 2)^2 + 3$, which assumes all finite values infinitely often. This is, $\Psi_2(z)$ has no Borel exceptional value.

Theorem 1.5. *If $f(z)$ is a transcendental meromorphic function of finite order with two Borel exceptional values d, ∞. Let $a, c \in \mathbb{C} \setminus \{0\}$ be constants, with c such that $f(z+c) \not\equiv f(z)$. Then for $b \in \mathbb{C}$ with $b \neq -ad^2$, $\Psi_2(z)$ assumes the value b infinitely often, and $\lambda(\Psi_n(z) - b) = \rho(f)$.*

Moreover, if $f(z)$ satisfies $f(z+c) \not\equiv (2ad+1)f(z) - 2ad^2$, we can remove the condition $b \neq -ad^2$.

Remark 1.6. By the simple calculation, we can see $f(z+c) \not\equiv (2ad+1)f(z)-2ad^2$ in Example 1.4, $\Psi_2(z)$ has no finite Borel exceptional value. Hence, the conclusion of Theorem 1.5 is sharp.

From the proof of Theorem 1.5, we can obtain the following.

Corollary 1.7. *If $f(z)$ is a transcendental meromorphic function of finite order with two Borel exceptional values $0, \infty$. Let $a, c \in \mathbb{C} \setminus \{0\}$ be constants, with c such that $f(z+c) \not\equiv f(z)$. Then $\Psi_2(z)$ assumes every value b infinitely often, and $\lambda(\Psi_n(z) - b) = \rho(f)$.*

Example 1.8. For $f(z) = (e^z - 1)/(e^z + 1)$, $c = \pi i$, $a = -1$, it is easy to see $0, \infty$ are not Borel exceptional values, and

$$\Psi_2(z) - 1 = \Delta f(z) - af(z)^2 - 1 = \frac{8e^z}{(e^z+1)^2(e^z-1)} \tag{1.1}$$

has no zeros. Thus our condition in Corollary 1.7 is sharp.

Remark 1.9. In fact, by the definition of Borel exceptional value, we know the condition $f(z)$ is a transcendental meromorphic function of finite order with two Borel exceptional values $0, \infty$ equivalent to $\lambda(f) < \rho(f)$, $\lambda(1/f) < \rho(f)$.

Hayman also posed the following conjecture: if f is a transcendental meromorphic function and $n \geq 1$, then $f^n f'$ takes every finite nonzero value infinitely often. This conjecture has been solved by Hayman [1] for $n \geq 3$, by Mues [4] for $n = 2$, by Bergweiler and Eremenko [8] for $n = 1$.

Recently, for an analog of Hayman conjecture for difference, Laine and Yang [9] proved the following.

Theorem E. *Let f be a transcendental entire function with finite order and c be a nonzero complex constant. Then for $n \geq 2$, $f^n(z)f(z+c)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

Liu et al. [10] consider the question when f is a transcendental meromorphic function.

Theorem F. *Let f be a transcendental meromorphic function with finite order and c be a nonzero complex constant. Then for $n \geq 6$, $f^n(z)f(z+c)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

In this paper, we improved the above result by reducing the condition $n \geq 6$.

Theorem 1.10. *Let f be a transcendental meromorphic function with finite order with two Borel exceptional values d, ∞, and c be a nonzero complex constant. Then for $n \geq 1$, $G = f^n(z)f(z+c)$ assumes every value $a(\neq d^{n+1}) \in \mathbb{C}$ infinitely often and $\lambda(G - a) = \rho(f)$.*

From Theorem 1.10, we can obtain the following.

Corollary 1.11. *If $f(z)$ is a transcendental meromorphic function of finite order with two Borel exceptional values $0, \infty$, and let c be a nonzero complex constant. Then for $n \geq 1$, $G = f^n(z)f(z+c)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.*

Example 1.12. For $f(z) = ((e^z - 1)/(e^z + 1))$, $c = \pi i$ and $a = 1$, it is easy to see $0, \infty$ are not Borel exceptional values, and

$$G - 1 = f(z)^2 f(z+c) - 1 = \frac{-2}{e^z + 1}, \tag{1.2}$$

has no zeros. Thus our condition in Corollary 1.11 is sharp.

Remark 1.13. Theorem 1.10 also improved the result in [11, Theorem 1.2], where they considered the case of entire function and $n = 1$.

For the analog of Hayman conjecture on $f^n f'$, $f(z+c)$ can be replaced by $\Delta f(z) = f(z+c) - f(z)$ in Theorem 1.10. Then a simple modification of the proof of Theorem 1.10 yields the following result.

Theorem 1.14. *Let f be a transcendental meromorphic function with finite order with two Borel exceptional values d, ∞, and c be a nonzero complex constant with c such that $\Delta f(z) \not\equiv 0$. Then for $n \geq 1$, $G = f^n(z)\Delta f(z)$ assumes every value $a \in \mathbb{C}$ infinitely often and $\lambda(G - a) = \rho(f)$.*

Corollary 1.15. *If $f(z)$ is a transcendental entire function of finite order with a Borel exceptional values d, and c be a nonzero complex constant with c such that $\Delta f(z) \not\equiv 0$. Then for $n \geq 1$, $G = f^n(z)\Delta f(z)$ assumes every value $a \in \mathbb{C}$ infinitely often and $\lambda(G - a) = \rho(f)$.*

Remark 1.16. Theorem 1.14 also improved the result in [7, Theorem 1.4], where they consider the case of entire function and $n \geq 2$. the value a can be a polynomial $a(z) \not\equiv 0$ in their result. In fact, our results also can allow the value a to be a polynomial, even be a meromorphic function $a(z) \not\equiv 0$ satisfying $\rho(a) < \rho(f)$.

Example 1.17. For $f(z) = \exp\{z\} + z$, $c = 2\pi i$, $a = cz$, it is easy to see that $f(z)$ has no Borel exceptional value, we have $G = f(z)\Delta f(z) - cz = ce^z$, which has no zeros. Hence, $f(z)$ has a Borel exceptional value necessary in Corollary 1.15.

2. Lemmas

The following lemma, due to Gross [12], is important in the factorization and uniqueness theory of meromorphic functions, playing an important role in this paper as well. We give a slight changed form.

Lemma 2.1 (see [13]). *Suppose that $f_j(z)$ $(j = 1, 2, \ldots, n+1)$ are meromorphic functions and g_j $(j = 1, 2, \ldots, n)$ are entire functions satisfying the following conditions.*

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv f_{n+1}$.

(ii) *If $1 \leq j \leq n+1$, $1 \leq k \leq n$, the order of f_j is less than the order of $e^{g_k(z)}$. If $n \geq 2$, $1 \leq j \leq n+1$, $1 \leq h < k \leq n$, and the order of $f_j(z)$ is less than the order of $e^{g_h - g_k}$.*

Then $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n+1)$.

Lemma 2.2 (see [14, 15]). *Let $f(z)$ be a moermorphic function of finite order, and let $c \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r,f), \tag{2.1}$$

where $S(r,f) = o\{T(r,f)\}$.

Lemma 2.3 (see [14]). *Let $f(z)$ be a meormorphic function of finite order ρ, and let $c \in \mathbb{C} \setminus \{0\}$. Then, for each $\varepsilon > 0$, one has*

$$T(r, f(z+c)) = T(r,f) + O\left(r^{\rho-1+\varepsilon}\right) + O(\log r). \tag{2.2}$$

We are concerned with functions which are polynomials in $f(z+c_j)$, where $c_j \in \mathbb{C}$, with coefficients $a_\lambda(z)$ such that

$$T(r, a_\lambda) = o(T(r,f)), \tag{2.3}$$

except possibly for a set of r having finite logarithmic measure. Such functions will be called *difference polynomials* in $f(z)$. Similarly, we are concerned with functions which are polynomials in $f(z+c_j)$ and the derivatives of f, with coefficients $a_\lambda(z)$ such that (2.3) holds, except possibly for a set of r having finite logarithmic measure. Such functions will be called *differential-difference polynomials* in $f(z)$. We also denote $|c| = \max\{|c_j|\}$.

Halburd and Korhonen proved the following difference analogous to the Clunie Lemma [16], which has numerous applications in the study of complex differential equations, and beyond.

Lemma 2.4 (see [15]). *Let $f(z)$ be a nonconstant meromorphic solution of*

$$f^n P(z,f) = Q(z,f), \tag{2.4}$$

where $P(z,f)$ and $Q(z,f)$ are difference polynomials in $f(z)$, and let $\delta < 1$ and $\varepsilon > 0$. If the degree of $Q(z,f)$ as a polynomial in $f(z)$ and its shifts is at most n, then

$$m(r, P(z,f)) = o\left(\frac{T(r+|c|, f)}{r^\delta}\right) + o(T(r,f)) \tag{2.5}$$

for all r outside of a possible exceptional set with finite logarithmic measure.

We can obtain the following differential-difference analogous to the Clunie lemma by the same method as Lemma 2.4.

Lemma 2.5. *Let $f(z)$ be a non-constant meromorphic solution of* (2.4), *where $P(z,f)$ and $Q(z,f)$ are differential-difference polynomials in $f(z)$, and let $\delta < 1$ and $\varepsilon > 0$. If the degree of $Q(z,f)$ as a polynomial in $f(z)$ and its shifts is at most n, then* (2.5) *holds.*

Remark 2.6. If the coefficients of $P(z,f)$ and $Q(z,f)$ are $a_\lambda(z)$ satisfying the order less than $\rho(f)$, from the proof of Lemma 2.5, we can obtain

$$m(r,P(z,f)) = o\left(\frac{T(r+|c|,f)}{r^\delta}\right) + o(T(r,f)) + O(m(r,a_\lambda(z))). \tag{2.6}$$

Lemma 2.7. *If $f(z)$ is a transcendental meromorphic function with exponent of convergence of poles $\lambda(1/f) = \lambda < +\infty$, and let $c \in \mathbb{C} \setminus \{0\}$. Then, for each $\varepsilon > 0$, one has*

$$N(r,f(z+c)) = N(r,f) + O\left(r^{\lambda-1+\varepsilon}\right) + O(\log r). \tag{2.7}$$

From the above lemma, we can obtain the following important result.

Lemma 2.8. *If $f(z)$ is a transcendental meromorphic function with exponent of convergence of poles $\lambda(1/f) = \lambda < +\infty$, and let $c \in \mathbb{C} \setminus \{0\}$. Then, for each $\varepsilon > 0$, one has*

$$\lambda\left(\frac{1}{f(z+c)}\right) = \lambda\left(\frac{1}{f(z)}\right) = \lambda, \qquad \lambda\left(\frac{1}{\Delta f}\right) \le \lambda. \tag{2.8}$$

Proof. By the definition of exponent of convergence of poles, we can easily prove it by Lemma 2.7. □

Remark 2.9. The second result is similar with $\lambda(1/f') \le \lambda(1/f)$.

Lemma 2.10 (see [17]). *Let f be a nonconstant meromorphic function, n be a positive integer. $P(f) = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f$ where a_i is a meromorphic function satisfying $T(r,a_i) = S(r,f)$ $(i = 1,2,\ldots,n)$. Then*

$$T(r,P(f)) = nT(r,f) + S(r,f). \tag{2.9}$$

Lemma 2.11. *If $f(z)$ is a transcendental meromorphic function with exponent of convergence of poles $\lambda(1/f) = \lambda < \rho(f) < +\infty$, and let $c \in \mathbb{C} \setminus \{0\}$ and $n \ge 1$ be an integer. Set $G(z) = f^n(z)f(z+c)$, then $\rho(G) = \rho(f)$.*

Proof. We can rewrite $G(z)$ as the form

$$G(z) = f(z)^{n+1}\frac{f(z+c)}{f(z)}. \tag{2.10}$$

For each $\varepsilon > 0$, by Lemma 2.2 and (2.10), we get that

$$m(r,G) \le (n+1)m(r,f) + m\left(r,\frac{f(z+c)}{f(z)}\right) = (n+1)m(r,f) + O\left(r^{\rho-1+\varepsilon}\right). \tag{2.11}$$

From Lemma 2.10, we have

$$N(r,G) \le nN(r,f) + N(r,f(z+c)) \le (n+1)N(r,f) + O\left(r^{\lambda-1+\varepsilon}\right) + O(\log r). \tag{2.12}$$

By (2.11) and (2.12), we have

$$T(r,G) \le (n+1)T(r,f) + O\left(r^{\rho-1+\varepsilon}\right) + O(\log r). \tag{2.13}$$

This is, $\rho(G) \le \rho(f)$. Next we prove $\rho(G) \ge \rho(f)$.

By Lemma 2.2 and (2.10), we have

$$(n+1)m(r,f) = m\left(r,f^{n+1}\right) \le m(r,G) + m\left(r,\frac{f(z)}{f(z+c)}\right) = m(r,G) + O\left(r^{\rho-1+\varepsilon}\right). \tag{2.14}$$

Note that $\lambda(1/f) < \rho(f)$, we have

$$N(r,f) = O\left(r^{\rho-1+\varepsilon}\right). \tag{2.15}$$

Thus, from (2.14) and (2.15), we have

$$T(r,f) \le m(r,G) + O\left(r^{\rho-1+\varepsilon}\right). \tag{2.16}$$

Hence, we prove $\rho(G) \ge \rho(f)$. Therefore, $\rho(G) = \rho(f)$. □

Remark 2.12. If $n \ge 2$, we can prove $\rho(G) \ge \rho(f)$ by the inequality $T(r,G) \ge (n-1)T(r,f) + S(r,f)$, without the condition $\lambda(1/f) < \rho(f)$.

3. Proof of Theorem 1.1

We only prove the case $\rho(f) = \rho > 0$. For the case $\rho(f) = 0$, we can use the same method in the proof. Suppose that $b \in \mathbb{C}$ and $\lambda(\Psi_n(z) - b) < \rho(f)$. First, we claim that $\Psi_n(z) - b$ is transcendental meromorphic. Suppose that $\Psi_n(z) - b = r(z)$, where $r(z)$ is a rational function. Then

$$-af(z)^n = b - \Delta f(z) + r(z). \tag{3.1}$$

By Lemma 2.4, for each $\varepsilon > 0$, we have

$$T(r,\Delta f(z)) \le 2T(r,f) + S(r,f) + O\left(r^{\rho-1+\varepsilon}\right). \tag{3.2}$$

By Lemma 2.10, we have

$$\begin{aligned} T(r, af(z)^n) &= nT(r,f) + S(r,f) \\ &= T(r, b - \Delta f(z) + r(z)) \\ &\leq 2T(r,f) + S(r,f) + O\left(r^{\rho-1+\varepsilon}\right). \end{aligned} \tag{3.3}$$

This is a contradiction. Hence, the claim holds. Thus, $\Psi_n(z) - b$ can be written as

$$\Psi_n(z) - b = \frac{p_1(z)}{p_2(z)} \exp\{q(z)\} := r(z)\exp\{q(z)\}, \tag{3.4}$$

where $q(z) \not\equiv 0$ is a polynomial, $p_1(z)$ is an entire function with $\rho(p_1) < \rho(f)$, and $p_2(z)$ is the canonical product formed with the poles of $\Psi_n(z) - b$. Hence, $\rho(p_2) = \lambda(p_2) = \lambda(1/r) \leq \max\{\lambda(1/f(z)), \lambda(1/f(z+c))\} = \lambda(1/f) < \rho(f)$. Obviously, $\rho(r) \leq \max\{\rho(p_1), \rho(p_2)\} < \rho(f)$.

Differentiating (3.4) and eliminating $\exp\{q(z)\}$, we obtain

$$f^{n-1}P(z,f) = Q(z,f), \tag{3.5}$$

where

$$\begin{gathered} P(z,f) = anr(z)f'(z) - a(r'(z) + q'(z)p(z))f(z), \\ Q(z,f) = r(z)f'(z) - r(z)f'(z+c) + \Delta f(z)(r'(z) + q'(z)r(z)) - b(r'(z) + q'(z)p(z)). \end{gathered} \tag{3.6}$$

We claim that $P(z,f) \not\equiv 0$. Suppose that

$$anr(z)f'(z) - a(r'(z) + q'(z)r(z))f(z) \equiv 0. \tag{3.7}$$

Integrating (3.7), we have

$$f(z)^n = dr(z)\exp\{q(z)\}, \tag{3.8}$$

where $d(\neq 0)$ is a constant. Therefore, by (3.4) and (3.8), and the definition of $\Psi_n(z)$, we obtain

$$\Psi_n(z) - b = f(z+c) - f(z) - af(z)^n - b = \frac{1}{d}f(z)^n; \tag{3.9}$$

therefore,

$$d(f(z+c) - f(z)) = (ad+1)f(z)^n + bd. \tag{3.10}$$

We can prove that $ad + 1 \neq 0$ by the similar to the proof in [5]. We omit it here.

Differentiating (3.9), and then dividing by $f'(z)$, we have

$$d\left(\frac{f'(z+c)}{f'(z)}-1\right)=n(ad+1)f(z)^{n-1}. \tag{3.11}$$

Therefore, by Lemma 2.2, we obtain that

$$(n-1)m(r,f)=S(r,f')=S(r,f). \tag{3.12}$$

Note that $n \geq 2$, we have

$$m(r,f)=S(r,f). \tag{3.13}$$

From (3.8), we know that the poles of $f(z)$ come from the poles of $r(z)$, we have

$$N(r,f) \leq O(N(r,r)) \leq O(T(r,r)). \tag{3.14}$$

From (3.13) and (3.14), we have

$$T(r,f) \leq O(T(r,r)). \tag{3.15}$$

We can obtain $\rho(f) \leq \rho(r)$. It is a contradiction. Hence, the claim $P(z,f) \not\equiv 0$ holds.

Since $n \geq 3$ and the total of $Q(z,f)$ as a differential-difference polynomial in $f(z)$, its shift and its derivatives, $\deg_f Q(z,f)=1$, by (3.5), Lemma 2.5 and Remark 2.6, we obtain that for $\delta < 1$,

$$\begin{aligned} m(r,P(z,f)) &= o\left(\frac{T(r+|c|,f)}{r^{\delta}}\right)+o(T(r,f))+O(m(r,r(z))), \\ m(r,fP(z,f)) &= o\left(\frac{T(r+|c|,f)}{r^{\delta}}\right)+o(T(r,f))+O(m(r,r(z))), \end{aligned} \tag{3.16}$$

for all r outside of an exceptional set of finite logarithmic measure. Form (3.16), we have

$$m(r,f)=o\left(\frac{T(r+|c|,f)}{r^{\delta}}\right)+o(T(r,f))+O(m(r,r(z))) \tag{3.17}$$

for all r outside of an exceptional set of finite logarithmic measure. By (3.14) and (3.15), we can get a contradiction with $\rho(r) < \rho(f)$. Hence, $\Psi_n(z)-b$ has infinitely many zeros and $\lambda(\Psi_n(z)-b)=\rho(f)$. The proof of Theorem 1.1 is complete.

4. Proof of Theorem 1.5

Since $f(z)$ has a Borel exceptional value d, we can write $f(z)$ as

$$f(z) = d + \frac{g(z)}{p(z)} \exp\left\{\alpha z^k\right\}, \qquad f(z+c) = d + \frac{g(z+c)}{p(z+c)} h_1(z) \exp\left\{\alpha z^k\right\}, \tag{4.1}$$

where $\alpha \not= 0$ is a constant, $k(\geq 1)$ is an integer satisfying $\rho(f) = k$, and $g(z), h_1(z)$ are entire functions such that $g(z)h_1(z) \not\equiv 0$, $\rho(g) < k$, $\rho(h_1) = k-1$, and $p(z)$ is the canonical product formed with the poles of $f(z)$ satisfying $\rho(p) = \lambda(p) = \lambda(1/f) < \rho(f)$. Set $H(z) = g(z)/p(z)$, it is easy to see that $\rho(H) < \rho(f)$.

First, we prove that $\Psi_2(z) - b = \Delta f(z) - af(z)^2 - b$ is transcendental. If $\Psi_2(z) - b = r(z)$, where $r(z)$ is a rational function, then

$$af(z)^2 = \Delta f(z) - b + r(z). \tag{4.2}$$

Thus, by Lemma 2.10, we have

$$T\left(r, af^2\right) = 2T(r,f) + S(r,f), \tag{4.3}$$

$$\begin{aligned} m(r, \Delta f(z) - b + r(z)) &\leq m(r,f) + m\left(r, \frac{f(z+c)}{f(z)} - 1\right) + O(\log r) \\ &\leq m(r,f) + S(r,f). \end{aligned} \tag{4.4}$$

By Lemma 2.7, we have

$$\begin{aligned} N(r, \Delta f(z) - b + r(z)) &\leq N(r, f(z)) + N(r, f(z+c)) + O(\log r) \\ &\leq 2N(r,f) + O\left(r^{\lambda-1+\varepsilon}\right) + O(\log r). \end{aligned} \tag{4.5}$$

From (4.4) and (4.5), we can get

$$T(r, \Delta f(z) - b + r(z)) \leq T(r,f) + N(r,f) + O\left(r^{\lambda-1+\varepsilon}\right) + O(\log r). \tag{4.6}$$

By (4.2), (4.3), and (4.6), we have

$$T(r,f) \leq N(r,f) + O\left(r^{\lambda-1+\varepsilon}\right) + O(\log r). \tag{4.7}$$

It is contradiction with $\lambda(1/f) < \rho(f)$.

Secondly, we prove that $\rho(\Psi_2(z) - b) = \rho(f) = k \geq 1$. By the expression of $\Psi_2(z)$, we have $\rho(\Psi_2(z) - b) \leq k$. Set $G(z) = \Psi_2(z) - b$. Suppose that $\rho(G) = k_1 < k$, then by (4.1), we have

$$\left[\frac{g(z+c)}{p(z+c)}h_1(z) - \frac{g(z)}{p(z)}\right] \exp\left\{\alpha z^k\right\} - a\frac{g^2(z)}{p^2(z)} \exp\left\{2\alpha z^k\right\} = b + G(z). \tag{4.8}$$

this is,

$$[H(z+c)h_1(z) - H(z)] \exp\left\{\alpha z^k\right\} - aH(z)^2 \exp\left\{2\alpha z^k\right\} = b + G(z). \tag{4.9}$$

Since $H(z) \not\equiv 0$, we see the order of the left-hand side of (4.9) is k. Obviously, it contradicts the order of the right side of (4.9) is less than k for $\rho(G) < k$. Hence, $\rho(G) = k$.

Thirdly, we prove $\lambda(G) = k(\geq 1)$. If $\lambda(G) < k$, then $G(z)$ can be written as

$$G(z) = \frac{g^*(z)}{p^*(z)} \exp\left\{\beta z^k\right\} = H^*(z) \exp\left\{\beta z^k\right\}, \tag{4.10}$$

where $\beta(\neq 0)$ is a constant, $g^*(z)(\not\equiv 0)$ is an entire function satisfying $\rho(g^*) < k$, and $\rho(p^*) = \lambda(p^*) = \max\{\lambda(1/f(z)), \lambda(1/f(z+c))\} = \lambda(1/f) < \rho(f) = k$ by Lemma 2.8. Hence, $\rho(H^*) < k$.

By (4.1), (4.10), and $G(z) = \Psi_2(z) - b$, we have

$$[H(z+c)h_1(z) - H(z) - 2adH(z)] \exp\left\{\alpha z^k\right\} - aH(z)^2 \exp\left\{2\alpha z^k\right\} - H^*(z) \exp\left\{\beta z^k\right\} = ad^2 + b. \tag{4.11}$$

In fact, (4.11) can be rewritten into

$$f_1(z) \exp\left\{\alpha z^k\right\} + f_2(z) \exp\left\{2\alpha z^k\right\} + f_3(z) \exp\left\{\beta z^k\right\} = f_4(z), \tag{4.12}$$

where $\rho(f_j)$ $(j = 1,2,3,4) < k$. Obviously, $\alpha \neq 2\alpha$. We just consider three cases.

Case 1 ($\beta \neq \alpha, 2\alpha$). By Lemma 2.1, we have $f_2(z) \equiv 0$, this is, $H(z) \equiv 0$, a contradiction.

Case 2 ($\beta = \alpha, \beta \neq 2\alpha$). By Lemma 2.1, we still have $f_2(z) \equiv 0$, this is, $H(z) \equiv 0$, a contradiction.

Case 3 ($\beta = 2\alpha, \beta \neq \alpha$). By Lemma 2.1, we have $f_1(z) \equiv 0$ and $f_4(z) \equiv 0$. In order to complete our proof, we need to get a contradiction.

Subcase 1. If $d = 0$, then $f_1(z) = H(z+c)h_1(z) - H(z)$. Since $f(z+c) \not\equiv f(z)$, we know $f_1(z) \not\equiv 0$. We get a contradiction.

Subcase 2. If $d \neq 0$, then $f_4(z) = ad^2 + b$. By the assumption $ad^2 + b \neq 0$. We can get $f_4(z) \not\equiv 0$, a contradiction.

If the condition $f(z+c) \not\equiv f(z)$ is replaced by $f(z+c) \not\equiv (2ad+1)f(z) - 2ad^2$, from $f_1(z) = H(z+c)h_1(z) - H(z) - 2adH(z)$, we know $f_1(z) \not\equiv 0$. We get a contradiction.

This completes the proof of the Theorem.

5. Proof of Theorem 1.10

Since $f(z)$ has a Borel exceptional value d, we can write $f(z)$ as

$$f(z) = d + \frac{g(z)}{p(z)} \exp\left\{\alpha z^k\right\}, \qquad f(z+c) = d + \frac{g(z+c)}{p(z+c)} h_1(z) \exp\left\{\alpha z^k\right\}, \tag{5.1}$$

where $\alpha \not= 0$ is a constant, $k(\geq 1)$ is an integer satisfying $\rho(f) = k$, and $g(z), h_1(z)$ are entire functions such that $g(z)h_1(z) \not\equiv 0$, $\rho(g) < k$, $\rho(h_1) = k-1$, and $p(z)$ is the canonical product formed with the poles of $f(z)$ satisfying $\rho(p) = \lambda(p) = \lambda(1/f) < \rho(f)$. Set $H(z) = g(z)/p(z) \not\equiv 0$, it is to see that $\rho(H) < \rho(f)$.

Now we suppose that $\lambda(G-a) < \rho(f)$. By Lemma 2.11, we have $\rho(G) = \rho(f) = \rho(G-a)$, so that $\lambda(G-a) < \rho(G-a) = \rho(f) = k$ and $G(z) - a$ can be rewritten into the form

$$G(z) - a = \frac{g^*(z)}{p^*(z)} \exp\left\{\beta z^k\right\} = H^*(z) \exp\left\{\beta z^k\right\}, \tag{5.2}$$

where $\beta(\not= 0)$ is a constant, $g^*(z)(\not\equiv 0)$ is an entire function satisfying $\rho(g^*) < k$, and $\rho(p^*) = \lambda(p^*) = \max\{\lambda(1/f(z)), \lambda(1/f(z+c))\} = \lambda(1/f) < \rho(f) = k$ by Lemma 2.8. Hence, $\rho(H^*) < k$.

By (5.1) and (5.2), we get

$$f_{n+1}(z)e^{(n+1)\alpha z^k} + f_n(z)e^{n\alpha z^k} + \cdots + f_1(z)e^{\alpha z^k} + d^{n+1} - a = H^*(z)e^{\beta z^k}, \tag{5.3}$$

where

$$\begin{gathered} f_j(z) = C_n^j d^{n-j+1} H^j(z) + C_n^{n-j+1} d^{n-j+1} H^{j-1}(z) H(z+c) h_1(z), \quad j = 1, 2, \ldots, n, \\ f_{n+1}(z) = H(z)^n H(z+c) h_1(z). \end{gathered} \tag{5.4}$$

Note that $f_{n+1}(z) \not\equiv 0$ and $H^*(z) \not\equiv 0$, by comparing growths of both sides of (5.3), we see that $\beta = (n+1)\alpha$. Thus, by (5.3), we have

$$\left[f_{n+1}(z) - H^*(z)\right] e^{(n+1)\alpha z^k} + f_n(z)e^{n\alpha z^k} + \cdots + f_1(z)e^{\alpha z^k} = a - d^{n+1}. \tag{5.5}$$

By Lemma 2.1, we get $a = d^{n+1}$. This is a contradiction with our assumption $a \not= d^{n+1}$. Hence, $\lambda(G-a) = \rho(f)$.

6. Proof of Theorem 1.14

Similar to the proof of Theorem 1.10, we can obtain (5.1) and (5.2).

By (5.1) and (5.2), we get

$$f_{n+1}(z)e^{(n+1)\alpha z^k} + f_n(z)e^{n\alpha z^k} + \cdots + f_1(z)e^{\alpha z^k} + d^{n+1} - a = H^*(z)e^{\beta z^k}, \tag{6.1}$$

where

$$f_{j+1}(z) = C_n^j d^{n-j} H^j(z)h(z), \quad j = 1, 2, \ldots, n,$$
$$f_1(z) = d^n h(z), \quad h(z) = H(z+c)h_1(z) - H(z) \not\equiv 0. \tag{6.2}$$

Note that $f_{n+1}(z) \not\equiv 0$ and $H^*(z) \not\equiv 0$, by comparing growths of both sides of (5.3), we see that $\beta = (n+1)\alpha$. Thus, by (5.3), we have

$$\left[f_{n+1}(z) - H^*(z)\right] e^{(n+1)\alpha z^k} + f_n(z)e^{n\alpha z^k} + \cdots + f_1(z)e^{\alpha z^k} = a - d^{n+1}. \tag{6.3}$$

By Lemma 2.1, we get $f_1(z) \equiv 0$. This is a contradiction with $h(z) \not\equiv 0$. Hence, $\lambda(G-a) = \rho(f)$.

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