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Research Article

A Strongly Convergent Method for the Split Feasibility Problem

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The purpose of this paper is to introduce and analyze a strongly convergent method which combined regularized method, with extragradient method for solving the split feasibility problem in the setting of infinite-dimensional Hilbert spaces. Note that the strong convergence point is the minimum norm solution of the split feasibility problem.

1. Introduction

In 1994, Censor and Elfving [1] first introduced the split feasibility problem, (SFP) in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. A number of image reconstruction problems can be formulated as the SFP; see, for example, [2] and the references therein. Recently, it is found that the SFP can also be applied to study the intensity-modulated radiation therapy; see, for example, [3–5] and the references therein. Very recently, Xu [6] considered the SFP in the framework of infinite-dimensional Hilbert spaces. In this setting, the SFP is formulated as finding a point x^* with the property

$$x^* \in C, \quad Ax^* \in Q, \tag{1.1}$$

where C and Q are two closed convex subsets of two Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. We use Γ to denote the solution set of the (SFP), that is,

$$\Gamma = \{x \in C : Ax \in Q\}. \tag{1.2}$$

Assume that the (SFP) is consistent. A special case of the (SFP) is the convexly constrained linear inverse problem [7] in the finite-dimensional Hilbert spaces

$$x^* \in C, \quad Ax^* = b, \tag{1.3}$$

which has extensively been investigated by using the Landweber iterative method [8]:

$$\text{with } x_0 \text{ arbitrary and } n = 0, 1, \ldots, \text{ let } x_{n+1} = x_n + \gamma A^T(b - Ax_n). \tag{1.4}$$

Comparatively, the SFP has received much less attention so far, due to the complexity resulted from the set Q. Therefore, whether various versions of the projected Landweber iterative method can be extended to solve the SFP remains an interesting open topic. For example, it is yet not clear if the dual approach to (1.2) of [9] can be extended to the SFP.

The original algorithm introduced in [1] involves the computation of the inverse A^{-1}:

$$x_{k+1} = A^{-1}P_Q\left(P_{A(C)}(Ax_k)\right), \quad k \geq 0, \tag{1.5}$$

where $C, Q \subset R^n$ are closed convex sets, A a full rank $n \times n$ matrix, and $A(C) = \{y \in R^n \mid y = Ax, x \in C\}$, and thus does not become popular. A more popular algorithm that solves the (SFP) seems to be the CQ algorithm of Byrne ([2, 10]). The CQ algorithm only involves the computations of the projections P_C and P_Q onto the sets C and Q, respectively, and is therefore implementable in the case where P_C and P_Q have closed-form expressions (e.g., C and Q are the closed balls or half-spaces). There are a large number of references on the CQ method for the (SFP) in the literature, see, for instance, [11–24]. It remains, however, a challenge how to implement the CQ algorithm in the case where the projections P_C and/or P_Q fail to have closed-form expressions though theoretically we can prove (weak) convergence of the algorithm.

Very recently, Xu [6] gave a continuation of the study on the CQ algorithm and its convergence. He applied Mann's algorithm to the SFP and proposed an averaged CQ algorithm, which was proved to be weakly convergent to a solution of the SFP. He derived a weak convergence result, which shows that for suitable choices of iterative parameters (including the regularization), the sequence of iterative solutions can converge weakly to an exact solution of the SFP.

Note that $x \in \Gamma$ means that there is an $x \in C$ such that $Ax - x^* = 0$ for some $x^* \in Q$. This motivates us to consider the distance function $d(Ax, x^*) = \|Ax - x^*\|$ and the minimization problem

$$\min_{x \in C,\, x^* \in Q} \frac{1}{2}\|Ax - x^*\|^2. \tag{1.6}$$

Minimizing with respect to $x^* \in Q$ first makes us consider the minimization:

$$\min_{x \in C} f(x) := \frac{1}{2}\left\|Ax - P_Q Ax\right\|^2. \tag{1.7}$$

However, (1.7) is, in general, ill-posed. So regularization is needed. We consider Tikhonov's regularization

$$\min_{x\in C} f_\alpha := \frac{1}{2}\left\|(I-P_Q)Ax\right\|^2 + \frac{1}{2}\alpha\|x\|^2, \tag{1.8}$$

where $\alpha > 0$ is the regularization parameter. We can compute the gradient ∇f_α of f_α as

$$\nabla f_\alpha = \nabla f(x) + \alpha I = A^*(I-P_Q)A + \alpha I. \tag{1.9}$$

Define a Picard iteration

$$x_{n+1}^\alpha = P_C\left(I - \gamma\left(A^*(I-P_Q)A + \alpha I\right)\right)x_n^\alpha \tag{1.10}$$

Xu [6] has has shown that if the (SFP) (1.1) is consistent, then as $n \to \infty$, $x_n^\alpha \to x_\alpha$ and consequently the strong $\lim_{\alpha\to 0} x_\alpha$ exists and is the minimum-norm solution of the (SFP). Note that (1.10) is a double-step iteration. Xu [6] further suggested a single-step regularized method:

$$x_{n+1} = P_C\left(I - \gamma_n \nabla f_{\alpha_n}\right)x_n = P_C\left((1-\alpha_n\gamma_n)x_n - \gamma_n A^*(I-P_Q)Ax_n\right). \tag{1.11}$$

Xu proved that the sequence $\{x_n\}$ generated by (1.11) converges in norm to the minimum-norm solution of the (SFP) provided the parameters $\{\alpha_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\alpha_n \to 0$ and $0 < \gamma_n \le (\alpha_n/(\|A\|^2 + \alpha_n))$;

(ii) $\sum_n \alpha_n\gamma_n = \infty$;

(iii) $(|\gamma_{n+1} - \gamma_n| + \gamma_n|\alpha_{n+1} - \alpha_n|)/(\alpha_{n+1}\gamma_{n+1})^2 \to 0$.

Motivated by the ideas of extragradient method and Xu's regularization, Ceng et al. [25] presented the following extragradient method with regularization for finding a common element of the solution set of the split feasibility problem and the set $\mathrm{Fix}(S)$ of fixed points of a nonexpansive mapping S:

$$\begin{gathered} x_0 = x \in H_1 \text{ chosen arbitrarily}, \\ y_n = P_C\left(x_n - \lambda_n\left(\nabla f(x_n) + \alpha_n x_n\right)\right), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)SP_C\left(x_n - \lambda_n\left(\nabla f(y_n) + \alpha_n y_n\right)\right), \quad n \ge 0. \end{gathered} \tag{1.12}$$

Ceng et al. only obtained the weak convergence of the algorithm (1.12).

The purpose of this paper is to further introduce and analyze a strongly convergent method, which combined regularized method with extragradient method for solving the split feasibility problem in the setting of infinite-dimensional Hilbert spaces. Note that the strong convergence point is the minimum norm solution of the split feasibility problem.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{2.1}$$

We will use $\mathrm{Fix}(T)$ to denote the set of fixed points of T, that is, $\mathrm{Fix}(T) = \{x \in C : x = Tx\}$. A mapping $T : C \to C$ is said to be ν-inverse strongly monotone (ν-ism) if there exists a constant $\nu > 0$ such that

$$\langle x - y, Tx - Ty \rangle \geq \nu \|Tx - Ty\|^2, \quad \forall x, y \in C. \tag{2.2}$$

Recall that the (nearest point or metric) projection from H onto C, denoted P_C, assigns, to each $x \in H$, the unique point $P_C(x) \in C$ with the property

$$\|x - P_C(x)\| = \inf\{\|x - y\| : y \in C\}. \tag{2.3}$$

It is well known that the metric projection P_C of H onto C has the following basic properties:

(a) $\|P_C(x) - P_C(y)\| \leq \|x - y\|$ for all $x, y \in H$;

(b) $\langle x - y, P_C(x) - P_C(y) \rangle \geq \|P_C(x) - P_C(y)\|^2$ for every $x, y \in H$;

(c) $\langle x - P_C(x), y - P_C(x) \rangle \leq 0$ for all $x \in H$, $y \in C$.

Especially, $2P_C - I$ is nonexpansive.

Let K be a nonempty closed convex subset of a real Hilbert space H, and let $F : K \to H$ be a monotone mapping. The variational inequality problem (VIP) is to find $x \in K$ such that

$$\langle Fx, y - x \rangle \geq 0, \quad \forall y \in K. \tag{2.4}$$

The solution set of the VIP is denoted by $\mathrm{VIP}(K, F)$. It is well known that

$$x \in \mathrm{VI}(K, F) \Longleftrightarrow x = P_K(x - \lambda Fx), \quad \forall \lambda > 0. \tag{2.5}$$

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$, and $g \in Ty$ imply

$$\langle x - y, f - g \rangle \geq 0. \tag{2.6}$$

A monotone mapping $T : H \to 2^H$ is called maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$

implies $f \in Tx$. Let $F : K \to H$ be a monotone and k-Lipschitz continuous mapping, and let $N_K v$ be the normal cone to K at $v \in K$, that is,

$$N_K v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in K\}. \tag{2.7}$$

Define

$$Tv = \begin{cases} Fv + N_K v, & \text{if } v \in K, \\ \emptyset, & \text{if } v \notin K. \end{cases} \tag{2.8}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(K, F)$; see [21] for more details.

Next we adopt the following notation

(i) $x_n \to x$ means that x_n converges strongly to x;

(ii) $x_n \rightharpoonup x$ means that x_n converges weakly to x;

(iii) $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$ is the weak ω-limit set of the sequence $\{x_n\}$.

Lemma 2.1 (see [6]). *We have the following assertions.*

(a) *T is nonexpansive if and only if the complement $I - T$ is $(1/2)$-ism.*

(b) *If S is ν-ism, then for $\gamma > 0$, γS is (ν/γ)-ism.*

(c) *S is averaged if and only if the complement $I - S$ is ν-ism for some $\nu > (1/2)$.*

(d) *If S and T are both averaged, then the product (composite) ST is averaged.*

Lemma 2.2 (see [26]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose that*

$$x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n \tag{2.9}$$

for all $n \geq 0$ and

$$\limsup_{n\to\infty}\left(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\right) \leq 0. \tag{2.10}$$

Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.3 (see [27]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \delta_n, \tag{2.11}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) *$\sum_{n=1}^{\infty} \gamma_n = \infty$;*

(2) *$\limsup_{n\to\infty}(\delta_n/\gamma_n) \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.*

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Let H_1 and H_2 be two infinite-dimensional Hilbert spaces. Let C and Q be two nonempty closed and convex subset of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. In this section, we will devote to solve the SFP (1.1). First, we need the following propositions.

Proposition 3.1 (see [6, 25]). *Given $x^* \in H_1$, then the following statements are equivalent.*

(i) *x^* solves the SFP;*

(ii) *x^* solves the fixed equation $x^* = P_C(I - \lambda \nabla f)x^* = P_C(I - \lambda A^*(I - P_Q)A)x^*$;*

(iii) *x^* solves the variational inequality problem (VIP) of finding $x^* \in C$ such that*

$$\left\langle \nabla f(x^*), x - x^* \right\rangle \geq 0, \quad \forall x \in C, \tag{3.1}$$

where $\nabla f = A^*(I - P_Q)A$ and A^* is the adjoint of A.

Proposition 3.2 (see [28]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $B : C \to H$ be α-inverse strongly monotone and $\gamma > 0$ a constant. Then, we have*

$$\left\| (I - \gamma B)x - (I - \gamma B)y \right\|^2 \leq \left\| x - y \right\|^2 + \gamma(\gamma - 2\alpha) \left\| Bx - By \right\|^2, \quad \forall x, y \in C. \tag{3.2}$$

In particular, if $0 \leq \gamma \leq 2\alpha$, then $I - \gamma B$ is nonexpansive.

Proposition 3.3 (see [6]). *We have the following conclusions:*

(i) *$A^*(I - P_Q)A$ is Lipschitz continuous with Lipschitz constant $\|A\|^2$;*

(ii) *$A^*(I - P_Q)A$ is $(1/\|A\|^2)$-ism,*

(iii) *$I - \gamma A^*(I - P_Q)A$ is nonexpansive for all $\gamma \in (0, 2/\|A\|^2)$.*

Algorithm 3.4. For given $x_0 \in C$ arbitrarily, define a sequence $\{x_n\}$ iteratively by

$$\begin{aligned} y_n &= P_C\left[x_n - \lambda A^*(I - P_Q)Ax_n - \alpha_n x_n\right], \\ x_{n+1} &= P_C\left[x_n - \lambda A^*(I - P_Q)Ay_n + \mu(y_n - x_n)\right], \quad n \geq 0, \end{aligned} \tag{3.3}$$

where $\{\alpha_n\} \subset (0,1)$ is a sequence, $\lambda \in [a, b] \subset (0, 2/\|A\|^2)$ and $\mu \in (0,1)$ are two constants such that $(\lambda/\mu) \leq (2/\|A\|^2)$.

Theorem 3.5. *Suppose that $\Gamma \neq \emptyset$. Assume $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}$ generated by (3.3) converges strongly to $\widetilde{x} \in P_\Gamma(0)$, which is the minimum norm element in Γ.*

Proof. Note that the conditions $\alpha_n \to 0$ and $\lambda \in (0, 2/\|A\|^2)$. We deduce $\alpha_n < 1 - (\lambda\|A\|^2/2)$ for enough large n. Without loss of generality, we may assume that, for all $n \in \mathbb{N}$, $\alpha_n < 1 - (\lambda\|A\|^2/2)$, that is, $\lambda/(1 - \alpha_n) \in (0, 2/\|A\|^2)$.

Pick up any $x^* \in \Gamma$. From Proposition 3.1, we have $x^* = P_C[x^* - \delta A^*(I - P_Q)Ax^*]$ for any $\delta > 0$. Thus,

$$\begin{aligned} x^* &= P_C\left[x^* - \frac{\lambda}{1-\alpha_n}A^*(I-P_Q)Ax^*\right] \\ &= P_C\left[\alpha_n x^* + (1-\alpha_n)\left(x^* - \frac{\lambda}{1-\alpha_n}A^*(I-P_Q)Ax^*\right)\right], \quad \forall n \geq 0. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we have

$$\begin{aligned} \|y_n - x^*\| &= \|P_C[(1-\alpha_n)x_n - \lambda A^*(I-P_Q)Ax_n] - x^*\| \\ &= \left\| P_C\left[(1-\alpha_n)\left(x_n - \frac{\lambda}{1-\alpha_n}A^*(I-P_Q)Ax_n\right)\right]\right. \\ &\quad \left. -P_C\left[\alpha_n x^* + (1-\alpha_n)\left(x^* - \frac{\lambda}{1-\alpha_n}A^*(I-P_Q)Ax^*\right)\right]\right\| \\ &\leq \left\|\alpha_n(-x^*) + (1-\alpha_n)\left[\left(x_n - \frac{\lambda}{1-\alpha_n}A^*(I-P_Q)Ax_n\right)\right.\right. \\ &\quad \left.\left. -\left(x^* - \frac{\lambda}{1-\alpha_n}A^*(I-P_Q)Ax^*\right)\right]\right\|. \end{aligned} \tag{3.5}$$

From Propositions 3.1 and 3.2, we get that $I - (\lambda/(1-\alpha_n))A^*(I-P_Q)A$ is nonexpansive. It follows that

$$\begin{aligned} \|y_n - x^*\| &\leq \alpha_n\|x^*\| + (1-\alpha_n)\left\|\left(I - \frac{\lambda}{1-\alpha_n}A^*(I-P_Q)A\right)x_n\right. \\ &\quad \left. -\left(I - \frac{\lambda}{1-\alpha_n}A^*(I-P_Q)A\right)x^*\right\| \\ &\leq \alpha_n\|x^*\| + (1-\alpha_n)\|x_n - x^*\|. \end{aligned} \tag{3.6}$$

Thus,

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|P_C[x_n - \lambda A^*(I-P_Q)Ay_n + \mu(y_n - x_n)] - x^*\| \\ &= \left\|P_C\left[(1-\mu)x_n + \mu\left(y_n - \frac{\lambda}{\mu}A^*(I-P_Q)Ay_n\right)\right]\right. \\ &\quad \left. -P_C\left[(1-\mu)x^* + \mu\left(x^* - \frac{\lambda}{\mu}A^*(I-P_Q)Ax^*\right)\right]\right\| \\ &\leq (1-\mu)\|x_n - x^*\| + \mu\left\|\left(y_n - \frac{\lambda}{\mu}A^*(I-P_Q)Ay_n\right)\right. \\ &\quad \left. -\left(x^* - \frac{\lambda}{\mu}A^*(I-P_Q)Ax^*\right)\right\| \end{aligned}$$

$$
\begin{aligned}
&\le (1-\mu)\|x_n - x^*\| + \mu\left\|y_n - x^*\right\| \\
&\le (1-\mu)\|x_n - x^*\| + \mu\alpha_n\|x^*\| + \mu(1-\alpha_n)\|x_n - x^*\| \\
&= (1-\mu\alpha_n)\|x_n - x^*\| + \mu\alpha_n\|x^*\| \\
&\le \max\{\|x^*\|, \|x_0 - x^*\|\}.
\end{aligned}
\tag{3.7}
$$

Hence, $\{x_n\}$ is bounded.

Set $S = 2P_C - I$. Note that S is nonexpansive. Thus, we can rewrite x_{n+1} in (3.3) as

$$
\begin{aligned}
x_{n+1} &= \frac{I+S}{2}\left[(1-\mu)x_n + \mu\left(y_n - \frac{\lambda}{\mu}A^*(I-P_Q)Ay_n\right)\right] \\
&= \frac{1-\mu}{2}x_n + \frac{\mu}{2}\left(y_n - \frac{\lambda}{\mu}A^*(I-P_Q)Ay_n\right) \\
&\quad + \frac{1}{2}S\left[(1-\mu)x_n + \mu\left(y_n - \frac{\lambda}{\mu}A^*(I-P_Q)Ay_n\right)\right] \\
&= \frac{1-\mu}{2}x_n + \frac{1+\mu}{2}z_n,
\end{aligned}
\tag{3.8}
$$

where

$$
z_n = \frac{\mu(y_n - (\lambda/\mu)A^*(I-P_Q)Ay_n) + S[(1-\mu)x_n + \mu(y_n - (\lambda/\mu)A^*(I-P_Q)Ay_n)]}{1+\mu}.
\tag{3.9}
$$

It follows that

$$
\begin{aligned}
\|z_{n+1} - z_n\| &= \left\| \frac{\mu(y_{n+1} - (\lambda/\mu)A^*(I-P_Q)Ay_{n+1})}{1+\mu} \right. \\
&\quad + \frac{S[(1-\mu)x_{n+1} + \mu(y_{n+1} - (\lambda/\mu)A^*(I-P_Q)Ay_{n+1})]}{1+\mu} \\
&\quad - \frac{\mu(y_n - (\lambda/\mu)A^*(I-P_Q)Ay_n)}{1+\mu} \\
&\quad \left. + \frac{S[(1-\mu)x_n + \mu(y_n - (\lambda/\mu)A^*(I-P_Q)Ay_n)]}{1+\mu} \right\| \\
&\le \frac{\mu}{1+\mu}\left\|\left(y_{n+1} - \frac{\lambda}{\mu}A^*(I-P_Q)Ay_{n+1}\right) - \left(y_n - \frac{\lambda}{\mu}A^*(I-P_Q)Ay_n\right)\right\| \\
&\quad + \frac{1}{1+\mu}\left\|S\left[(1-\mu)x_{n+1} + \mu\left(y_{n+1} - \frac{\lambda}{\mu}A^*(I-P_Q)Ay_{n+1}\right)\right]\right. \\
&\qquad \left. -S\left[(1-\mu)x_n + \mu\left(y_n - \frac{\lambda}{\mu}A^*(I-P_Q)Ay_n\right)\right]\right\|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{\mu}{1+\mu}\|y_{n+1}-y_n\| \\
&\quad + \frac{1}{1+\mu}\left\|(1-\mu)(x_{n+1}-x_n)+\mu\left[\left(y_{n+1}-\frac{\lambda}{\mu}A^*(I-P_Q)Ay_{n+1}\right)\right.\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left.\left.-\left(y_n-\frac{\lambda}{\mu}A^*(I-P_Q)Ay_n\right)\right]\right\| \\
&\le \frac{\mu}{1+\mu}\|y_{n+1}-y_n\| + \frac{1-\mu}{1+\mu}\|x_{n+1}-x_n\| + \frac{\mu}{1+\mu}\|y_{n+1}-y_n\| \\
&\le \frac{2\mu}{1+\mu}\|y_{n+1}-y_n\| + \frac{1-\mu}{1+\mu}\|x_{n+1}-x_n\|.
\end{aligned}
\tag{3.10}
$$

By (3.3), we have

$$
\begin{aligned}
\|y_{n+1}-y_n\| &= \left\|P_C\left[(1-\alpha_{n+1})x_{n+1}-\lambda A^*(I-P_Q)Ax_{n+1}\right]-P_C\left[(1-\alpha_n)x_n-\lambda A^*(I-P_Q)Ax_n\right]\right\| \\
&\le \left\|\left[(1-\alpha_{n+1})x_{n+1}-\lambda A^*(I-P_Q)Ax_{n+1}\right]-\left[(1-\alpha_n)x_n-\lambda A^*(I-P_Q)Ax_n\right]\right\| \\
&\le \left\|\left[x_{n+1}-\lambda A^*(I-P_Q)Ax_{n+1}\right]-\left[x_n-\lambda A^*(I-P_Q)Ax_n\right]\right\|+\alpha_{n+1}\|x_{n+1}\|+\alpha_n\|x_n\| \\
&\le \|x_{n+1}-x_n\|+\alpha_{n+1}\|x_{n+1}\|+\alpha_n\|x_n\|.
\end{aligned}
\tag{3.11}
$$

Hence, we deduce

$$
\begin{aligned}
\|z_{n+1}-z_n\| &\le \frac{2\mu}{1+\mu}\|x_{n+1}-x_n\| + \frac{1-\mu}{1+\mu}\|x_{n+1}-x_n\| + \alpha_{n+1}\|x_{n+1}\| + \alpha_n\|x_n\| \\
&= \|x_{n+1}-x_n\| + \alpha_{n+1}\|x_{n+1}\| + \alpha_n\|x_n\|.
\end{aligned}
\tag{3.12}
$$

Therefore,

$$
\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0. \tag{3.13}
$$

By Lemma 2.2, we obtain

$$
\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{3.14}
$$

Hence,

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\frac{1+\mu}{2}\|z_n-x_n\|=0. \tag{3.15}
$$

From (3.5), (3.7), Proposition 3.2, and the convexity of the norm, we deduce

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\quad\le (1-\mu)\|x_n-x^*\|^2+\mu\|y_n-x^*\|^2 \\
&\quad\le (1-\mu)\|x_n-x^*\|^2+\mu\left\|\alpha_n(-x^*)+(1-\alpha_n)\left[\left(x_n-\frac{\lambda}{1-\alpha_n}A^*(I-P_Q)Ax_n\right)\right.\right. \\
&\qquad\qquad\left.\left.-\left(x^*-\frac{\lambda}{1-\alpha_n}A^*(I-P_Q)Ax^*\right)\right]\right\|^2 \\
&\quad\le \mu\left[\alpha_n\|x^*\|^2+(1-\alpha_n)\right. \\
&\qquad\left.\times\left\|\left(I-\frac{\lambda}{1-\alpha_n}A^*(I-P_Q)A\right)x_n-\left(I-\frac{\lambda}{1-\alpha_n}A^*(I-P_Q)A\right)x^*\right\|^2\right] \\
&\qquad+(1-\mu)\|x_n-x^*\|^2 \\
&\quad\le (1-\alpha_n)\mu \\
&\qquad\times\left[\|x_n-x^*\|^2+\frac{\lambda}{1-\alpha_n}\left(\frac{\lambda}{1-\alpha_n}-\frac{2}{\|A\|^2}\right)\|A^*(I-P_Q)Ax_n-A^*(I-P_Q)Ax^*\|^2\right] \\
&\qquad+(1-\mu)\|x_n-x^*\|^2+\alpha_n\mu\|x^*\|^2 \\
&\quad\le \alpha_n\|x^*\|^2+\|x_n-x^*\|^2 \\
&\qquad+\mu a\left(\frac{b}{1-\alpha_n}-\frac{2}{\|A\|^2}\right)\|A^*(I-P_Q)Ax_n-A^*(I-P_Q)Ax^*\|^2.
\end{aligned}
\tag{3.16}
$$

Therefore, we have

$$
\begin{aligned}
\mu a&\left(\frac{2}{\|A\|^2}-\frac{b}{1-\alpha_n}\right)\|A^*(I-P_Q)Ax_n-A^*(I-P_Q)Ax^*\|^2 \\
&\le \alpha_n\|x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2 \\
&\le \alpha_n\|x^*\|^2+(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\times\|x_n-x_{n+1}\|.
\end{aligned}
\tag{3.17}
$$

Since $\alpha_n \to 0$ and $\|x_n-x_{n+1}\| \to 0$ as $n\to\infty$, we obtain $\liminf_{n\to\infty}\mu a((2/\|A\|^2)-(b/(1-\alpha_n)))>0$. Thus, we have

$$
\lim_{n\to\infty}\|A^*(I-P_Q)Ax_n-A^*(I-P_Q)Ax^*\|=0. \tag{3.18}
$$

By the property (b) of the metric projection P_C, we have

$$
\begin{aligned}
&\|y_n - x^*\|^2 \\
&\quad = \left\|P_C\left[(1-\alpha_n)x_n - \lambda A^*(I-P_Q)Ax_n\right] - P_C\left[x^* - \lambda A^*(I-P_Q)Ax^*\right]\right\|^2 \\
&\quad \le \left\langle (1-\alpha_n)x_n - \lambda A^*(I-P_Q)Ax_n - (x^* - \lambda A^*(I-P_Q)Ax^*), y_n - x^*\right\rangle \\
&\quad = \frac{1}{2}\Big\{\left\|x_n - \lambda A^*(I-P_Q)Ax_n - (x^* - \lambda A^*(I-P_Q)Ax^*) - \alpha_n x_n\right\|^2 + \|y_n - x^*\|^2 \\
&\qquad\quad - \left\|(1-\alpha_n)x_n - \lambda A^*(I-P_Q)Ax_n - (x^* - \lambda A^*(I-P_Q)Ax^*) - (y_n - x^*)\right\|^2\Big\} \\
&\quad \le \frac{1}{2}\Big\{\left\|(x_n - \lambda A^*(I-P_Q)Ax_n) - (x^* - \lambda A^*(I-P_Q)Ax^*)\right\|^2 \\
&\qquad\quad + 2\alpha_n\|x_n\|\left\|x_n - \lambda A^*(I-P_Q)Ax_n - (x^* - \lambda A^*(I-P_Q)Ax^*) - \alpha_n x_n\right\| \\
&\qquad\quad + \|y_n - x^*\|^2 - \left\|(x_n - y_n) - \lambda(A^*(I-P_Q)Ax_n - A^*(I-P_Q)Ax^*) - \alpha_n x_n\right\|^2\Big\} \\
&\quad \le \frac{1}{2}\Big\{\left\|(x_n - \lambda A^*(I-P_Q)Ax_n) - (x^* - \lambda A^*(I-P_Q)Ax^*)\right\|^2 + \alpha_n M + \|y_n - x^*\|^2 \\
&\qquad\quad - \left\|(x_n - y_n) - \lambda(A^*(I-P_Q)Ax_n - A^*(I-P_Q)Ax^*) - \alpha_n x_n\right\|^2\Big\} \\
&\quad \le \frac{1}{2}\Big\{\|x_n - x^*\|^2 + \alpha_n M + \|y_n - x^*\|^2 - \|x_n - y_n\|^2 + 2\alpha_n\langle x_n, x_n - y_n\rangle \\
&\qquad\quad + 2\lambda\langle x_n - y_n, A^*(I-P_Q)Ax_n - A^*(I-P_Q)Ax^*\rangle \\
&\qquad\quad - \left\|\lambda(A^*(I-P_Q)Ax_n - A^*(I-P_Q)Ax^*) + \alpha_n x_n\right\|^2\Big\} \\
&\quad \le \frac{1}{2}\Big\{\|x_n - x^*\|^2 + \alpha_n M + \|y_n - x^*\|^2 - \|x_n - y_n\|^2 + 2\alpha_n\|x_n\|\|x_n - y_n\| \\
&\qquad\quad + 2\lambda\|x_n - y_n\|\left\|A^*(I-P_Q)Ax_n - A^*(I-P_Q)Ax^*\right\|\Big\},
\end{aligned}
\tag{3.19}
$$

where $M > 0$ is some constant such that

$$
\sup_n\left\{2\|x_n\|\left\|x_n - \lambda A^*(I-P_Q)Ax_n - (x^* - \lambda A^*(I-P_Q)Ax^*) - \alpha_n x_n\right\|\right\} \le M. \tag{3.20}
$$

It follows that

$$
\begin{aligned}
\|y_n - x^*\|^2 \le{}& \|x_n - x^*\|^2 + \alpha_n M - \|x_n - y_n\|^2 + 2\alpha_n\|x_n\|\|x_n - y_n\| \\
&+ 2\lambda\|x_n - y_n\|\left\|A^*(I-P_Q)Ax_n - A^*(I-P_Q)Ax^*\right\|,
\end{aligned}
\tag{3.21}
$$

and hence

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1-\mu)\|x_n-x^*\|^2+\mu\|y_n-x^*\|^2\\ &\le \|x_n-x^*\|^2+\alpha_n M-\mu\|x_n-y_n\|^2+2\alpha_n\|x_n\|\|x_n-y_n\|\\ &\quad+2\lambda\|x_n-y_n\|\|A^*(I-P_Q)Ax_n-A^*(I-P_Q)Ax^*\|,\end{aligned}\tag{3.22}$$

which implies that

$$\begin{aligned}&\mu\|x_n-y_n\|^2\\ &\quad\le (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|+\alpha_n M+2\alpha_n\|x_n\|\|x_n-y_n\|\\ &\qquad+2\lambda\|x_n-y_n\|\|A^*(I-P_Q)Ax_n-A^*(I-P_Q)Ax^*\|.\end{aligned}\tag{3.23}$$

Since $\alpha_n \to 0$, $\|x_n-x_{n+1}\| \to 0$, and $\|A^*(I-P_Q)Ax_n-A^*(I-P_Q)Ax^*\| \to 0$, we derive

$$\lim_{n\to\infty}\|x_n-y_n\|=0.\tag{3.24}$$

Next we show that

$$\limsup_{n\to\infty}\langle\tilde{x},\tilde{x}-y_n\rangle\le 0,\tag{3.25}$$

where $\tilde{x}=P_\Gamma(0)$. To show it, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle\tilde{x},\tilde{x}-y_n\rangle=\lim_{i\to\infty}\langle\tilde{x},\tilde{x}-y_{n_i}\rangle.\tag{3.26}$$

Since $\{x_n\}$ is bounded, we have that $\{y_n\}$ is also bounded. As $\{y_{n_i}\}$ is bounded, we have that a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ converges weakly to z.

Next we show that $z\in\Gamma$. We define a mapping T by

$$Tv=\begin{cases}A^*(I-P_Q)Av+N_C v, & v\in C,\\ \emptyset, & v\notin C.\end{cases}\tag{3.27}$$

Then T is maximal monotone. Let $(v,w)\in G(T)$. Since $w-A^*(I-P_Q)Av\in N_C v$ and $y_n\in C$, we have $\langle v-y_n, w-A^*(I-P_Q)Av\rangle\ge 0$. On the other hand, from $y_n=P_C[(1-\alpha_n)x_n-\lambda A^*(I-P_Q)Ax_n]$, we have

$$\langle v-y_n, y_n-(1-\alpha_n)x_n+\lambda A^*(I-P_Q)Ax_n\rangle\ge 0,\tag{3.28}$$

that is,

$$\left\langle v-y_n,\frac{y_n-x_n}{\lambda}+Ax_n+\frac{\alpha_n}{\lambda}x_n\right\rangle\ge 0.\tag{3.29}$$

Therefore, we have

$$
\begin{aligned}
\langle v - y_{n_i}, w \rangle &\geq \langle v - y_{n_i}, Av \rangle \\
&\geq \langle v - y_{n_i}, A^*(I - P_Q)Av \rangle \\
&\quad - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda} + A^*(I - P_Q)Ax_{n_i} + \frac{\alpha_{n_i}}{\lambda} x_{n_i} \right\rangle \\
&= \left\langle v - y_{n_i}, A^*(I - P_Q)Av - A^*(I - P_Q)Ax_{n_i} - \frac{y_{n_i} - x_{n_i}}{\lambda} - \frac{\alpha_{n_i}}{\lambda} x_{n_i} \right\rangle \\
&= \langle v - y_{n_i}, A^*(I - P_Q)Av - A^*(I - P_Q)Ay_{n_i} \rangle \\
&\quad - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda} + \frac{\alpha_{n_i}}{\lambda} x_{n_i} \right\rangle \\
&\quad + \langle v - y_{n_i}, A^*(I - P_Q)Ay_{n_i} - A^*(I - P_Q)Ax_{n_i} \rangle \\
&\geq \langle v - y_{n_i}, A^*(I - P_Q)Ay_{n_i} - A^*(I - P_Q)Ax_{n_i} \rangle \\
&\quad - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda} + \frac{\alpha_{n_i}}{\lambda} x_{n_i} \right\rangle .
\end{aligned} \tag{3.30}
$$

Noting that $\alpha_{n_i} \to 0$, $\|y_{n_i} - x_{n_i}\| \to 0$, and $A^*(I - P_Q)A$ is Lipschitz continuous, we deduce from above

$$
\langle v - z, w \rangle \geq 0. \tag{3.31}
$$

Since T is maximal monotone, we have $z \in T^{-1}(0)$ and hence $z \in \mathrm{VI}(C, A^*(I - P_Q)A) = \Gamma$. Therefore,

$$
\limsup_{n \to \infty} \langle \tilde{x}, \tilde{x} - y_n \rangle = \lim_{i \to \infty} \langle \tilde{x}, \tilde{x} - y_{n_i} \rangle = \langle \tilde{x}, \tilde{x} - z \rangle \leq 0. \tag{3.32}
$$

By the property (b) of metric projection P_C, we have

$$
\begin{aligned}
&\|y_n - \tilde{x}\|^2 \\
&\quad = \Bigg\| P_C \left[(1 - \alpha_n) \left(x_n - \frac{\lambda}{1 - \alpha_n} A^*(I - P_Q)Ax_n \right) \right] \\
&\qquad - P_C \left[\alpha_n \tilde{x} + (1 - \alpha_n) \left(\tilde{x} - \frac{\lambda}{1 - \alpha_n} A^*(I - P_Q)A\tilde{x} \right) \right] \Bigg\|^2 \\
&\quad \leq \left\langle \alpha_n(-\tilde{x}) + (1 - \alpha_n) \left[\left(x_n - \frac{\lambda}{1 - \alpha_n} A^*(I - P_Q)Ax_n \right) - \left(\tilde{x} - \frac{\lambda}{1 - \alpha_n} A^*(I - P_Q)A\tilde{x} \right) \right], y_n - \tilde{x} \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\leq \alpha_n \langle \tilde{x}, \tilde{x} - y_n \rangle + (1 - \alpha_n) \\
&\quad \times \left\| \left(x_n - \frac{\lambda}{1 - \alpha_n} A^*(I - P_Q) A x_n \right) - \left(\tilde{x} - \frac{\lambda}{1 - \alpha_n} A^*(I - P_Q) A \tilde{x} \right) \right\| \left\| y_n - \tilde{x} \right\| \\
&\leq \alpha_n \langle \tilde{x}, \tilde{x} - y_n \rangle + (1 - \alpha_n) \| x_n - \tilde{x} \| \left\| y_n - \tilde{x} \right\| \\
&\leq \alpha_n \langle \tilde{x}, \tilde{x} - y_n \rangle + \frac{1 - \alpha_n}{2} \left(\| x_n - \tilde{x} \|^2 + \left\| y_n - \tilde{x} \right\|^2 \right).
\end{aligned}
\tag{3.33}
$$

Hence

$$
\left\| y_n - \tilde{x} \right\|^2 \leq (1 - \alpha_n) \| x_n - \tilde{x} \|^2 + 2\alpha_n \langle \tilde{x}, \tilde{x} - y_n \rangle. \tag{3.34}
$$

Therefore,

$$
\begin{aligned}
\| x_{n+1} - \tilde{x} \|^2 &\leq (1 - \mu) \| x_n - \tilde{x} \|^2 + \mu \left\| y_n - \tilde{x} \right\|^2 \\
&\leq (1 - \mu\alpha_n) \| x_n - \tilde{x} \|^2 + 2\mu\alpha_n \langle \tilde{x}, \tilde{x} - y_n \rangle.
\end{aligned}
\tag{3.35}
$$

We apply Lemma 2.3 to the last inequality to deduce that $x_n \to \tilde{x}$. This completes the proof. □

Remark 3.6. Now it is wellknown that the Korpelevich's extragradient method has only weak convergence in the setting of infinite-dimensional Hilbert spaces. But our algorithm (3.3) which is similar to the Korpelevich's method, has strong convergence in the setting of infinite-dimensional Hilbert spaces.

Remark 3.7. Algorithm (1.12) has only weak convergence for solving SFP. Our algorithm with strong convergence solves SFP under some weaker assumptions.

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