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Research Article

On the Exact Analytical and Numerical Solutions of Nano Boundary-Layer Fluid Flows

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The nonlinear boundary value problem describing the nanoboundary-layer flow with linear Navier boundary condition is investigated theoretically and numerically in this paper. The G'/G-expansion method is applied to search for the all possible exact solutions, and its results are then validated by the Chebyshev pseudospectral differentiation matrix (ChPDM) approach which has been recently introduced and successfully used. This numerical technique is firstly applied and, on comparing with the other recent work, it is found that the results are very accurate and effective to deal with the current problem. It is then used to examine and validate the present analytical analysis. Although the G'/G-expansion method has been used widely to solve nonlinear wave equations, its application for nonlinear boundary value problems has not been discussed yet, and the present paper may be the first to address this point. It is clarified that the exact solutions obtained via the G'/G-expansion method cannot be obtained by using some of the other methods. In addition, the domain of the physical parameters involved in the current boundary value problem is also discussed. Furthermore, the convex, vicinity of zero, and asymptotic solutions are deduced.

1. Introduction

At nanoscale diameters, it was shown in an interesting paper by He et al. [1] that a fascinating phenomena arise when the diameter of the electrospun nanofibers is less than 100 nm. The nanoeffect has been demonstrated for unusual strength, high surface energy, surface reactivity, and high thermal and electric conductivity.

The notion of a boundary layer was first introduced by Prandtl [2] over a hundred years ago to explain the discrepancies between the theory of inviscid fluid flow and experiment. In the classical boundary-layer theory, the condition of no-slip near the solid walls

is usually applied, where the fluid velocity component is assumed to be zero relative to the solid boundary. However, this is not true for fluid flows at the micro- and nanoscale. Investigations show that the no-slip condition is no longer valid, and instead, a certain degree of tangential slip must be allowed [3]. In words, nanoboundary-layer fluid flows mean nanoscale flows which have many applications in microelectromechanical systems. Because of the microscale dimensions of these devices, the fluid flow behavior deviates significantly from the traditional no-slip flow. In recent years, some interest has been given to the study of this type of flow, and some useful results have been introduced by many authors, see, for example, [4–12].

In this paper we consider the model proposed by Wang [8] to describe the viscous flow due to a stretching surface with both surface slip and suction (or injection). He considered two geometries situations, namely, the two-dimensional and axisymmetric of a stretching surface. Wang [8] applied a similarity transform to convert the Navier-Stokes equations into a 3rd-order nonlinear ordinary differential equation given by

$$f'''(\eta) + mf(\eta)f''(\eta) - \left[f'(\eta)\right]^2 = 0, \tag{1.1}$$

where m is a parameter describing the type of stretching, where $m = 1$ describes the two-dimensional stretching, while $m = 2$ is for axisymmetric stretching. The flow is subjected to the following boundary conditions:

$$f(0) = s, \qquad f'(0) = 1 + \lambda f''(0), \qquad f'(\infty) = 0, \tag{1.2}$$

where $\lambda > 0$ is a nondimensional slip parameter, and $s < 0$ when injection from the surface occurs and $s > 0$ for suction.

During the past two decades, much effort has been spent on searching for exact solutions of nonlinear equations due to their importance in understanding its phenomena. In order to achieve this goal, various direct methods have been proposed, such as tanh-function [13], Jacobi-elliptic function [14], F-expansion [15], exp-function ([16–20]), the generalized exp-function [21], G'/G-expansion ([22, 23]), generalized G'/G-expansion [24], and simplified G'/G-expansion [25]. However, a little attention was devoted for their applications to solving nonlinear boundary value problems (BVPs). It should be noted that the current work may be the first to indicate the way of applying the G'/G-expansion method to solve BVPs, where the advantages of the current method over some of the other ones mentioned above are clarified later in this paper.

In order to solve the BVP given by (1.1) and (1.2), Van Gorder et al. [11] applied the homotopy analysis method. They have also discussed the effects of the slip parameter λ and the suction parameter s on the fluid velocity and on the tangential stress. As expected, they found that for such fluid flows at nanoscales, the shear stress at the wall decreases (in an absolute sense) with an increase in the slip parameter λ. The existence and uniqueness results for each of the two problems were discussed in [8, 9] along with some numerical results.

It is well known that the exact solutions of nonlinear differential equations are not available in most cases, the reason we sometimes resort to implement accurate numerical methods instead. However, the numerical methods as declared by Rashidi and Erfani [10] gave discontinuous points of a curve, and thus, they are often costly and time consuming to get a complete curve of results. In addition, the stability and convergence of these methods

should be considered to avoid divergence or inappropriate results; the numerical method should be therefore chosen carefully.

Chebyshev pseudospectral differentiation matrix (ChPDM) approach has been introduced successfully and applied by Aly et al. [26] to analyze the two-dimensional MHD boundary-layer flow over a permeable surface, with a power law stretching velocity, in the presence of a magnetic field applied normally to the surface. Under certain circumstances, it is shown that the problem has an infinite number of solutions which have been examined by this technique. Recently, Guedda et al. [27] have applied this method to validate and evidence the analysis of two-dimensional mixed convection boundary-layer flow over a vertical flat plate embedded in a porous medium saturated with a water at 4°C (maximum density) and an applied magnetic field. Both cases of the assisting and opposing flows are considered. Multiple similarity solutions are obtained and investigated by ChPDM under the power law variable wall temperature, or variable heat flux, or variable heat transfer coefficient. Very recently, Aly et al. [28] have investigated the effect of magnetic field on viscoelastic fluid flow in boundary-layer through porous media by applying the ChPDM, where the resulting equations for the similar stream function, velocity, and skin friction coefficient were discussed for various parameters. It is found that the results were more accurate than those in the literature.

The motivation of this paper is therefore to extend the applications of both the G'/G-expansion method and the ChPDM to solve nonlinear BVPs with nonclassical boundary conditions, where the ChPDM is modified to treat both the infinity and the mixed boundary conditions in a direct manner. With this modification, we are able to obtain very accurate numerical results as will be shown later. The procedure we followed in this paper is found effective in studying the current BVP and may be useful for similar nonlinear problems. The suggested procedure is based first on obtaining all the possible exact solutions together with prescribing the domains of the physical parameters. The second step of the suggested procedure is to validate these results numerically to explore the effectiveness and efficiency of the proposed numerical approach. Besides, comparisons with other published results are also presented, where a full agreement is observed. Moreover, various types of solutions such as the convex, vicinity to zero, and asymptotic are also obtained.

2. Previous Results

In this section, we report some previous results obtained for (1.1) and (1.2). At $m = 1$ and $s = \lambda = 0$, Crane [29] gave the exact solution

$$f(\eta) = 1 - e^{-\eta}. \tag{2.1}$$

At arbitrary values of λ and s, Wang [8] obtained a solution in the following form:

$$f(\eta) = \gamma - (\gamma - s)e^{-\gamma\eta}, \tag{2.2}$$

where γ is the positive root of the cubic equation

$$\lambda\gamma^3 + (1 - \lambda s)\gamma^2 - s\gamma - 1 = 0. \tag{2.3}$$

When there is no suction, (2.2) reduces to that of Andersson [30]. Moreover, when there is no slip, it reduces to that of P. S. Gupta and A. S. Gupta [31]. Finally, Crane's solution [29] was recovered when both suction and slip are absent.

3. Validation of ChPDM Technique

A new numerical technique, namely, Chebyshev pseudospectral differentiation matrix (ChPDM), introduced by Aly et al. [26, 28] and Guedda et al. [27], has been briefly introduced in this section. On supposing that the domain of the problem is $[0, \eta_\infty]$, then the following algebraic mapping:

$$z = \frac{2\eta}{\eta_\infty} - 1 \tag{3.1}$$

transfers the domain to the Chebyshev one, that is, $[-1, 1]$. It is known that the Chebyshev polynomials are usually taken with their associated collocation points in the interval $[-1, 1]$ given by

$$z_j = \cos\left(\frac{\pi}{N} j\right), \quad j = 0, 1, \ldots, N. \tag{3.2}$$

Therefore, the kth derivative of any function, say $\mathbf{F}(z)$, at these collocation points can be approximated by the equation

$$\mathbf{F}^{(k)} = D^{(k)}\mathbf{F}, \tag{3.3}$$

where $D^{(k)}\mathbf{F}$ is the Chebyshev pseudospectral approximation of $\mathbf{F}^{(k)}$ where $\mathbf{F} = [F(z_0), F(z_1), \ldots, F(z_N)]^T$ and $\mathbf{F}^{(k)} = [F^{(k)}(z_0), F^{(k)}(z_1), \ldots, F^{(k)}(z_N)]^T$. The entries of the matrix $D^{(k)}$ are given by

$$d_{i,j}^{(k)} = \frac{2\theta_j}{N} \sum_{r=k}^{N} \sum_{\substack{n=0 \\ (n+r-k)\text{even}}}^{r-k} \theta_r b_{n,r}^k (-1)^{[(rj+ni)/N]} z_{rj-N[rj/N]} z_{ni-N[ni/N]}, \tag{3.4}$$

where $\theta_j = 1$, except for $\theta_0 = \theta_N = 1/2$ and

$$b_{n,r}^k = \frac{2^k r}{(k-1)! c_n} \frac{(v-n+k-1)!(v+k-1)!}{(v)!(v-n)!}, \tag{3.5}$$

where $2v = r + n - k$ and $c_0 = 2$, $c_j = 1$, $j \geq 1$. The elements $d_{0,1}^{(k)}$ are the major elements concerning its values. Accordingly, they bear the major error responsibility compared to the other elements. It is shown that the error in $d_{0,1}^{(1)}$ is of order $O(N^2 \varepsilon_r)$, where ε_r is the machine precision.

As shown in [26–28], on applying the new ChPDM approach, the derivatives of the function $f(\eta)$ at the points z_i are given by

$$f^{(k)}(z_i) = \sum_{j=0}^{N} d_{i,j}^{(k)} f(z_j), \quad k = 1,2,3,\ i = 1,2,\ldots,N. \tag{3.6}$$

Therefore, (1.1) and (1.2) become

$$\begin{gathered}\sum_{j=0}^{N} d_{i,j}^{(3)} f(z_j) + m f(z_i)\left(\frac{\eta_\infty}{2}\right)\sum_{j=0}^{N} d_{i,j}^{(2)} f(z_j) - \left(\frac{\eta_\infty}{2}\right)\left(\sum_{j=0}^{N} d_{i,j}^{(1)} f(z_j)\right)^2 = 0,\\ f(z_N) = s, \qquad \left(\frac{\eta_\infty}{2}\right)\sum_{j=0}^{N} d_{N,j}^{(1)} f(z_j) = \left(\frac{\eta_\infty}{2}\right)^2 + \lambda \sum_{j=0}^{N} d_{N,j}^{(2)} f(z_j), \qquad \sum_{j=0}^{N} d_{0,j}^{(1)} f(z_j) = 0,\end{gathered} \tag{3.7}$$

respectively.

Before starting the current analysis, ChPDM approach is therefore applied by using the system (3.7). Figures 1(a) and 1(b) show the comparison between ChPDM solutions and homotopy analysis method [11] over the current problem for (a) $f(\eta)$ and (b) $f'(\eta)$ for various values of the investigated parameters, m, λ, and s. As shown, these figures are exactly the same as Figures 1(a) and 1(b) given in [11]. In addition, Figures 2(a) and 2(b) show the results of applying ChPDM approach at $\lambda = 0.5$ and $\lambda = 5$, respectively, in injection ($s < 0$) and suction ($s > 0$) cases. These figures present exactly the same as Figures 2(c), 3(c), 2(d), and 3(d), respectively, in [11] and as Figures 2(b) and 4(b), respectively, for $\lambda = 0.5$, in [10]. Hence, without any hesitation, ChPDM technique may be applied with highly trust in the next sections.

4. The Generalized G'/G-Expansion Method

In the next subsections, we introduce the basic concept of the generalized G'/G-function method. It is then applied to solve the BVP given by (1.1) and (1.2).

4.1. Description of the Method

Consider a given nonlinear ordinary differential equation

$$N\left(f, \frac{df}{d\eta}, \frac{d^2 f}{d\eta^2}, \frac{d^3 f}{d\eta^3}, \ldots\right) = 0. \tag{4.1}$$

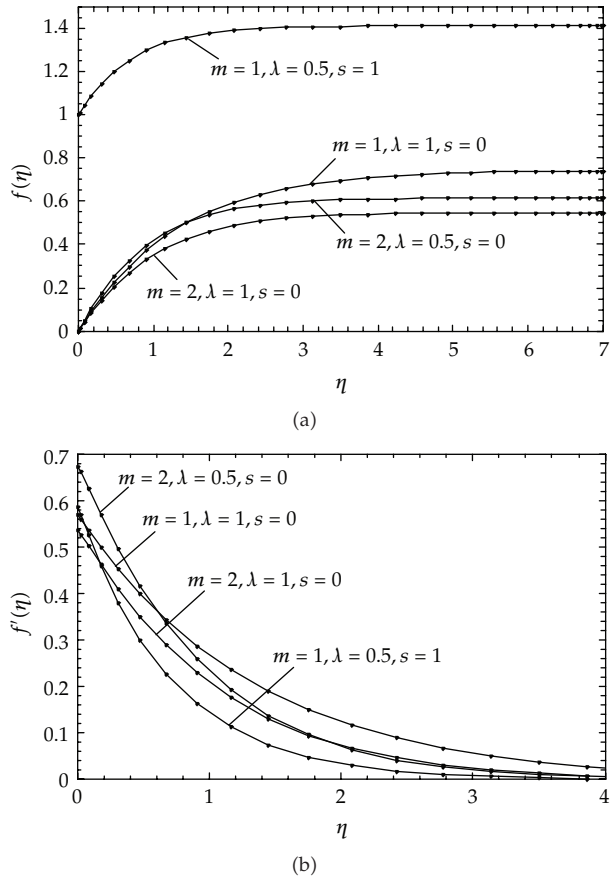


Figure 1: ChPDM solutions (solid line) and homotopy analysis method (circles) [11] for (a) $f(\eta)$ and (b) $f'(\eta)$ for various values of the investigated parameters.

The generalized G'/G-expansion method is then based on the assumption that the exact solution can be expressed in the following form:

$$f(\eta) = \sum_{i=-n}^{n} a_i \left(\frac{G'}{G}\right)^i, \tag{4.2}$$

where $a_i \neq 0$ and $G = G(\eta)$ satisfy the following second-order linear ODE:

$$G'' + \sigma G' + \mu G = 0, \tag{4.3}$$

where σ and μ are constants to be determined. Degree of the polynomial n can be determined by balancing the highest-order derivative with the highest nonlinear terms. Substituting (4.2) into (4.1), using the second-order linear ODE (4.3), and then equating each coefficient of the resulted polynomial to zero yield a set of algebraic equations with respect to a_i, σ, and μ.

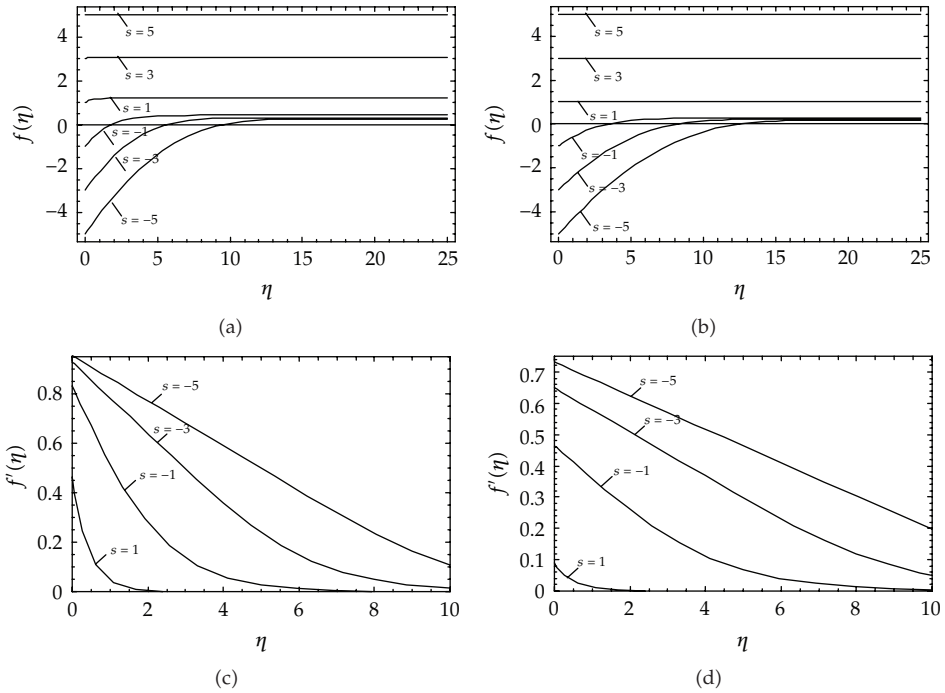


Figure 2: ChPDM solutions when $m = 2$ at (a) $\lambda = 0.5$ (exactly same as Figures 2(c) and 3(c) in [11]) and (b) $\lambda = 5$ (exactly same as Figures 2(d) and 3(d) in [11]) for various values of s, in injection and suction cases.

On solving this algebraic system, we may evaluate the values of unknowns. In addition, the solutions of (4.3) depend on whether $\sigma^2 - 4\mu(>, <$ or $=)0$:

$$\frac{G'}{G} = \begin{cases} \dfrac{\sqrt{\sigma^2-4\mu}}{2}\tanh\left(\dfrac{\sqrt{\sigma^2-4\mu}}{2}\eta+\eta_0\right)-\dfrac{\sigma}{2}, & \sigma^2-4\mu>0,\ \left|\tanh(\eta_0)\right|>1 \\ \dfrac{\sqrt{\sigma^2-4\mu}}{2}\coth\left(\dfrac{\sqrt{\sigma^2-4\mu}}{2}\eta+\eta_0\right)-\dfrac{\sigma}{2}, & \sigma^2-4\mu>0,\ \left|\coth(\eta_0)\right|<1 \\ \dfrac{\sqrt{4\mu-\sigma^2}}{2}\cot\left(\dfrac{\sqrt{4\mu-\sigma^2}}{2}\eta+\eta_0\right)-\dfrac{\sigma}{2}, & \sigma^2-4\mu<0 \\ \dfrac{C_2}{C_1+C_2\eta}-\dfrac{\sigma}{2}, & \sigma^2-4\mu=0 \\ \dfrac{1}{-1/\sigma+be^{\sigma\eta}}, & \mu=0 \\ \dfrac{\sqrt{\mu}\left[C_3\cos(\eta\sqrt{\mu})-\sin(\eta\sqrt{\mu})\right]}{\cos(\eta\sqrt{\mu})+C_3\sin(\eta\sqrt{\mu})}, & \sigma=0, \end{cases} \tag{4.4}$$

where C_1, C_2, and C_3 are constants.

4.2. Application to the Problem

As mentioned before, we discuss here the applicability of the generalized G'/G-expansion method to solve (1.1) and (1.2). On using the ansatz (4.2) and considering the homogeneous balance between $f'''(\eta)$ and $f(\eta)f''(\eta)$ or $[f'(\eta)]^2$ in (1.1), we get $n = 1$, so the solution can be supposed in the form

$$f(\eta) = a_0 + a_1\left(\frac{G'}{G}\right) + a_{-1}\left(\frac{G'}{G}\right)^{-1}. \tag{4.5}$$

By substituting (4.5) and (4.3) into (1.1) and collecting all terms with the same power of G'/G and G/G' together, the left-hand side of (4.3) can be converted into another polynomial in G'/G and G/G'. Equating each coefficient of this polynomial to zero, we obtain the following equations:

$$\begin{gathered}
a_{-1}\left[(2m-1)a_{-1} + 6\mu\right] = 0,\\
\mu a_{-1}\left[(3m-2)\sigma a_1 + 2m\mu a_0 + 12\mu\sigma\right] = 0,\\
a_{-1}\left[(m-1)\left(\sigma^2+2\mu\right)a_{-1} + \mu\left(2(m+1)\mu a_1 + 3m\sigma a_0 + 8\mu + 7\sigma^2\right)\right] = 0,\\
a_{-1}\left[8\sigma\mu + \sigma^3 + (m-2)\sigma a_{-1} + m\left(\sigma^2+2\mu\right)a_0 + 4(m+1)\sigma\mu a_1\right] = 0,\\
a_{-1}\left[m\sigma a_0 + \left(\sigma^2+2\mu\right)(1+2(m+1)a_1)\right] - \mu a_1\left(\sigma^2 + 2\mu - m\sigma a_0 + \mu a_1\right) - a_{-1}^2 = 0,\\
a_1\left[4(m+1)\sigma a_{-1} - \left(\sigma^2+8\mu\right)\sigma + m\left(\sigma^2+2\mu\right)a_0 + (m-2)\sigma\mu a_1\right] = 0,\\
a_1\left[2(m+1)a_{-1} + (m-1)\left(\sigma^2+2\mu\right)a_1 + 3m\sigma a_0 - 7\sigma^2 - 8\mu\right] = 0,\\
a_1[2ma_0 + \sigma((3m-2)a_1 - 12)] = 0,\\
a_1[(2m-1)a_1 - 6] = 0.
\end{gathered} \tag{4.6}$$

Solving the above algebraic equations by using MATHEMATICA 6 yields the following cases:

(1)

$$m = 1, \qquad \mu = 0, \qquad a_1 = 0, \qquad a_{-1} = \sigma(\sigma + a_0), \qquad f(\eta) = -\sigma + \sigma(\sigma + a_0)b\mathrm{e}^{\sigma\eta}, \tag{4.7}$$

(2)

$$\begin{gathered}
m = 0, \qquad a_{-1} = 0, \qquad a_1 = -6, \qquad \sigma^2 = 4\mu,\\
f(\eta) = s + \frac{3\eta}{8\gamma^2 + 2\gamma\eta}, \quad \text{where } 16\gamma^2 - 6\gamma - 3\lambda = 0,
\end{gathered} \tag{4.8}$$

(3)

$$m=-1,\qquad a_{-1}=2\mu,\qquad a_0=0,\qquad a_1=-2,\qquad \Gamma=\pm\frac{\sqrt{s^2-s^3\lambda}}{\sqrt{2+s^2(-1+s\lambda)}},$$
$$f(\eta)=\frac{s}{\Gamma}\left(\frac{\Gamma+\tan((s/2\Gamma)\eta)}{1-\Gamma\tan((s/2\Gamma)\eta)}\right). \tag{4.9}$$

This is valid in the case of $\sigma=0$, where the expression under the root of Γ should be a negative to avoid the singularity of the trigonometric function $\tan((s/2\Gamma)\eta)$:

(4)

$$m=-1,\qquad a_0=\sigma,\qquad a_1=0,\qquad a_{-1}=2\mu,$$
$$f(\eta)=\sigma-\frac{4\mu}{\sigma-\sqrt{\sigma^2-4\mu}\tanh\left[\left(\sqrt{(\sigma^2-4\mu)}/2\right)\eta+\eta_0\right]}, \tag{4.10}$$

(5)

$$\sigma^2=4\mu,\qquad a_0=\frac{3\sigma}{1-2m},\qquad a_1=0,\qquad a_{-1}=\frac{6\mu}{1-2m},$$
$$f(\eta)=\frac{6\sigma C_2}{(2m-1)\left(C_1\sigma-2C_2+\sigma C_2\eta\right)}. \tag{4.11}$$

Remark 4.1. It should be noted that Fang et al. [32] studied recently the viscous flow over a shrinking sheet with a second-order slip flow model. It is observed in their paper that the full Navier-Stocks equations reduce to the same nonlinear ordinary differential equation given by (1.1) in the present paper at $m=1$. However, boundary conditions are slightly different at the second one and given in [32] as

$$f(0)=s,\qquad f'(0)=-1+\lambda f''(0)+\delta f'''(0),\qquad f'(\infty)=0. \tag{4.12}$$

In fact, we can easily deduce the exact solution of the model in [32] by using some of our results as follows. In particular, the exact solution of equation (7) in Fang et al. [32] is already obtained in the current paper and given by (4.7) as

$$f(\eta)=-\sigma+\sigma(\sigma+a_0)b\mathrm{e}^{\sigma\eta}. \tag{4.13}$$

The third boundary condition at infinity is satisfied if σ is replaced by $-\beta$, where $\beta>0$. Therefore, the exact solution takes the form

$$f(\eta)=-\beta+\beta(\beta-a_0)b\mathrm{e}^{-\beta\eta}, \tag{4.14}$$

where a_0 and b are unknown parameters determined by applying the boundary conditions (4.12). So, a_0 and b are obtained by solving two resulted algebraic equations. Therefore, by

substituting their values into (4.14), we get the same results given by equations (11) and (12) in [32]. Here, it may be important to refer to the importance of the present result in (4.7) which can be used to obtain many exact solutions for different models with different boundary conditions. We just apply the boundary conditions to (4.7), and the exact solution is immediately resulted.

The advantages of the generalized G'/G-expansion method over some of the other methods are clarified at the end of the next section.

5. Exact and Numerical Solutions

It is important here to mention that the exact solutions in (4.8) and (4.9) are both in the final form as they satisfied all the boundary conditions. For the other cases (4.7), (4.10), and (4.11), the boundary conditions are applied to construct more possible exact analytical and numerical solutions of the current boundary value problem. On the spirit of [9], where m was examined for any $m > 0$, and to study the possibility of a general solution for the BVP system (1.1) and (1.2), it should be noted that the values of m, s, and λ have been considered as any parameters in the following subsections.

Case 1 (at $m = 1$). Applying the first two boundary conditions to the solution given by (4.7), it then follows that

$$-\sigma + \sigma(\sigma + a_0)b = s, \tag{5.1}$$

$$\lambda\sigma^3(\sigma + a_0)b - \sigma^2(\sigma + a_0)b + 1 = 0. \tag{5.2}$$

By solving (5.1) for a_0 and then substituting it into (4.7) and (5.2), respectively, we obtain the closed form solution

$$f(\eta) = -\sigma + (\sigma + s)e^{\sigma\eta}, \tag{5.3}$$

where σ satisfies the cubic equation

$$\lambda\sigma^3 - \sigma^2(1 - \lambda s) - \sigma s + 1 = 0. \tag{5.4}$$

From (5.3), it should be noted that the third boundary condition at infinity is satisfied providing that σ is negative. So, σ has to be a negative root of (5.4). Alternatively, the solution can be written as

$$f(\eta) = \sigma - (\sigma - s)e^{-\sigma\eta}, \tag{5.5}$$

where σ is a positive root of the cubic equation

$$\lambda\sigma^3 + \sigma^2(1 - \lambda s) - \sigma s - 1 = 0. \tag{5.6}$$

The solution given by (5.5) and (5.6) is the same solution reported in [8]. Figures 3(a) and 3(b) show comparison between the solution of ChPDM and G'/G-expansion techniques for

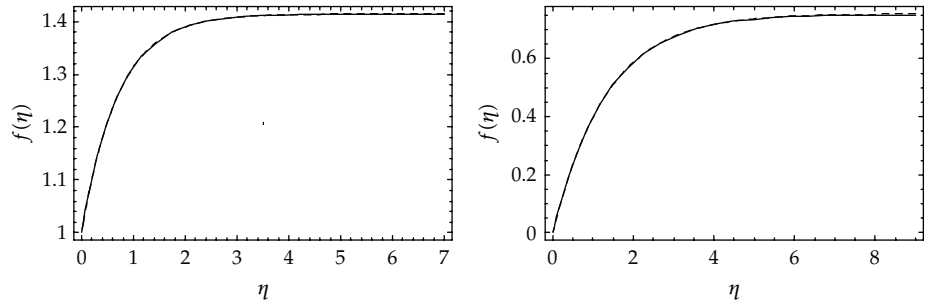


Figure 3: Profiles of the similar stream function $f(\eta)$ as a function of η at $m = 1$ for (a) $s = 1,\ \lambda = 0.5$ and (b) $s = 0,\ \lambda = 1$, solid and dashed lines are for ChPDM and G'/G-expansion solutions, respectively.

$(\eta, f(\eta))$ plane at $m = 1$ for (a) $s = 1$, $\lambda = 0.5$ and (b) $s = 0$, $\lambda = 1$. It can be seen from this figure that the solutions are identical and matching with those results in [11].

Convex Solution. In the current value of m and at $\lambda f''(0) = -2$, we set $f = g + k$ where $k \in \mathbb{R}$, then f is a solution of (1.1) and (1.2) if and only if g satisfies

$$g''' + kg'' = (g')^2 - gg'', \tag{5.7}$$

$$g(0) = s - k, \qquad g'(0) = -1, \qquad g(\infty) = 0. \tag{5.8}$$

We are now looking for functions g such that both hand sides of (5.7) vanish. For $s \geq 2$, we get that the functions $f_1, f_2 : [0, \infty) \to \mathbb{R}$ given by [33]

$$f_i(x) = K_i + \frac{1}{K_i} e^{-K_i x} \quad \text{for } i = 1, 2 \tag{5.9}$$

with $K_{1,2} = (1/2)(s \mp \sqrt{s^2 - 4})$ are convex solutions of (1.1) and (1.2).

Case 2 (at $m = 0$). In this case, we obtain the following closed-form solution:

$$f(\eta) = s + \frac{3\eta}{8\gamma^2 + 2\gamma\eta}, \tag{5.10}$$

where γ is a real root of the following cubic equation:

$$16\gamma^3 - 6\gamma - 3\lambda = 0. \tag{5.11}$$

Figures 4(a) and 4(b) show comparison between the solution of ChPDM and G'/G-expansion techniques for $(\eta, f(\eta))$ plane at $m = 0$ for (a) $s = 1$, $\lambda = 0.5$ and (b) $s = 0$, $\lambda = 1$. In this figure, the solutions are very closed for η in the range $[0, 20]$ and slightly closed in the range $[20, \eta_\infty]$. The solution of the present case is physically acceptable at a certain range for λ, which is discussed in the following lemma.

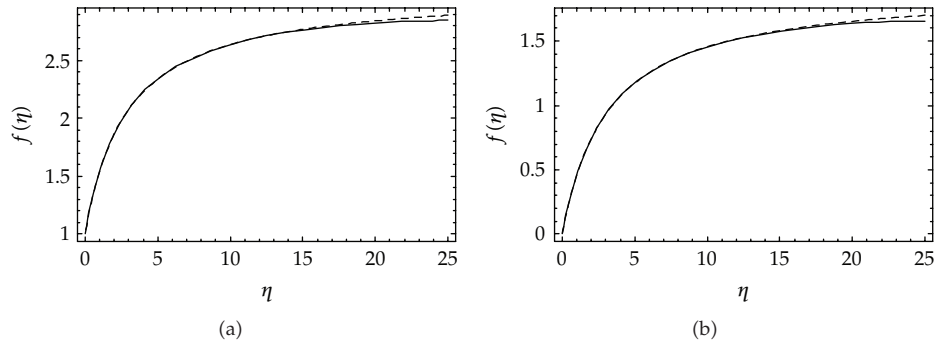


Figure 4: Profiles of the similar stream function $f(\eta)$ as a function of η at $m = 0$ for (a) $s = 1,\ \lambda = 0.5$ and (b) $s = 0,\ \lambda = 1$, solid and dashed lines are for ChPDM and G'/G-expansion solutions, respectively.

Lemma 5.1. *The given solution by* (5.10) *and* (5.11) *is valid and unique only if* $\lambda \in]\sqrt{2}/3, \infty[$.

Proof. To get the range of λ at which the solution (5.10) is unique, γ should satisfy the condition of obtaining only one real root for the cubic equation $a\gamma^3 + b\gamma + c = 0$, which is well known as

$$4b^3 - 27ac^2 < 0. \tag{5.12}$$

Inserting $a = 16$, $b = -6$, and $c = -\lambda$ into (5.12) yields

$$2 - 9\lambda^2 < 0. \tag{5.13}$$

On noting that $\lambda > 0$, it follows from (5.13) that $\lambda > \sqrt{2}/3$, and this completes the proof. □

Case 3 (at $m = -1$).

(1) When $\sigma = 0$.

As mentioned before, the solution of this case is given in the final form by (4.9). However, it should be noted that λ and s are to be chosen so that Γ is complex to avoid the singularities of $\tan((s/2\Gamma)\eta)$. For more specification, we can write the solution in the following simplest form:

$$f(\eta) = \frac{s}{\Lambda}\left[\frac{\Lambda - \tanh\left((s/2\Lambda)\eta\right)}{1 - \Lambda\tanh\left((s/2\Lambda)\eta\right)}\right], \quad \text{where } \Lambda = \sqrt{\frac{\lambda s^3 - s^2}{\lambda s^3 - s^2 + 2}}, \qquad \Gamma = i\Lambda. \tag{5.14}$$

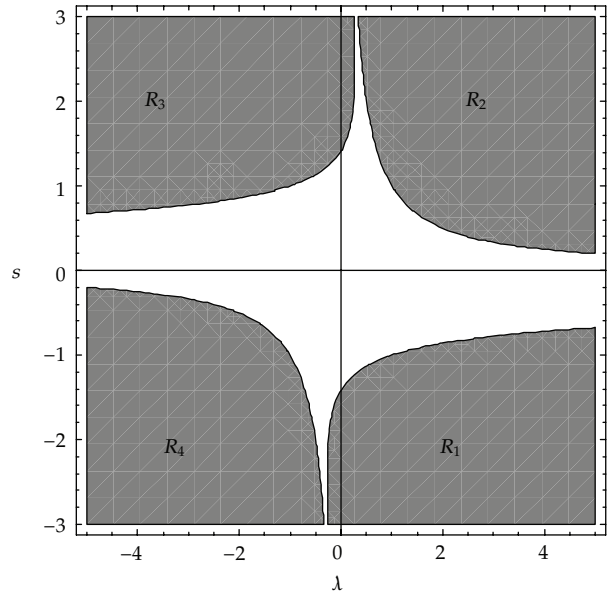


Figure 5: Existence four domains of (5.14) and (5.15) in the parameter plane (λ, s) at $m = -1$ and $\sigma = 0$.

Therefore, λ and s have to be specified such that

$$s^3 - s^2 + 2 < 0, \qquad \lambda s^3 - s^2 < 0, \qquad s^3 - s^2 + 2 > 0, \qquad \lambda s^3 - s^2 > 0. \tag{5.15}$$

Figure 5 presents then existence domain of (5.14) and (5.15) in the parameter plane (λ, s) in the ranges $[-5, 5]$ and $[-3, 3]$ for λ and s, respectively.

It is observed from the solution given above that it posses singular points. These singularities can be obtained by

$$\eta_{\text{singular}} = \frac{2\Lambda}{s} \operatorname{arctanh}\left(\frac{1}{\Lambda}\right). \tag{5.16}$$

For given values of λ and s, we can find the singular points of the solution. However, $f(\eta)$ must be continuous in the domain of definition $[0, \infty[$ of the current BVP. This means that the solution at some given values of the parameters λ and s is physically acceptable if the singularities lie outside the physical domain, that is, η_{singular} should be negative.

Figures 6(a)–6(e) and 7(a)–7(d) present profiles of the similar stream function $f(\eta)$ as a function of η at the case under consideration for some values of (λ, s) in the regions R_1 and R_2, respectively. These figures show that the ChPDM and G'/G-expansion solutions are identical inside the domains with slightly difference as $\eta \to \infty$ near the boundaries of R_1 and R_2. However, it is found that for any chosen values of λ and s in R_3, the singular points lie within the domain of interest $[0, \infty[$, and hence, the solution is discontinuous in the physical domain. Table 1 shows singularities in R_3 for some values of λ and s. It should be also noted that the behaviour of the solutions in R_4, as λ becomes negative, inverses those in R_1, as shown in Figure 8. However, more investigations regarding these regions may be taken into account as a future work.

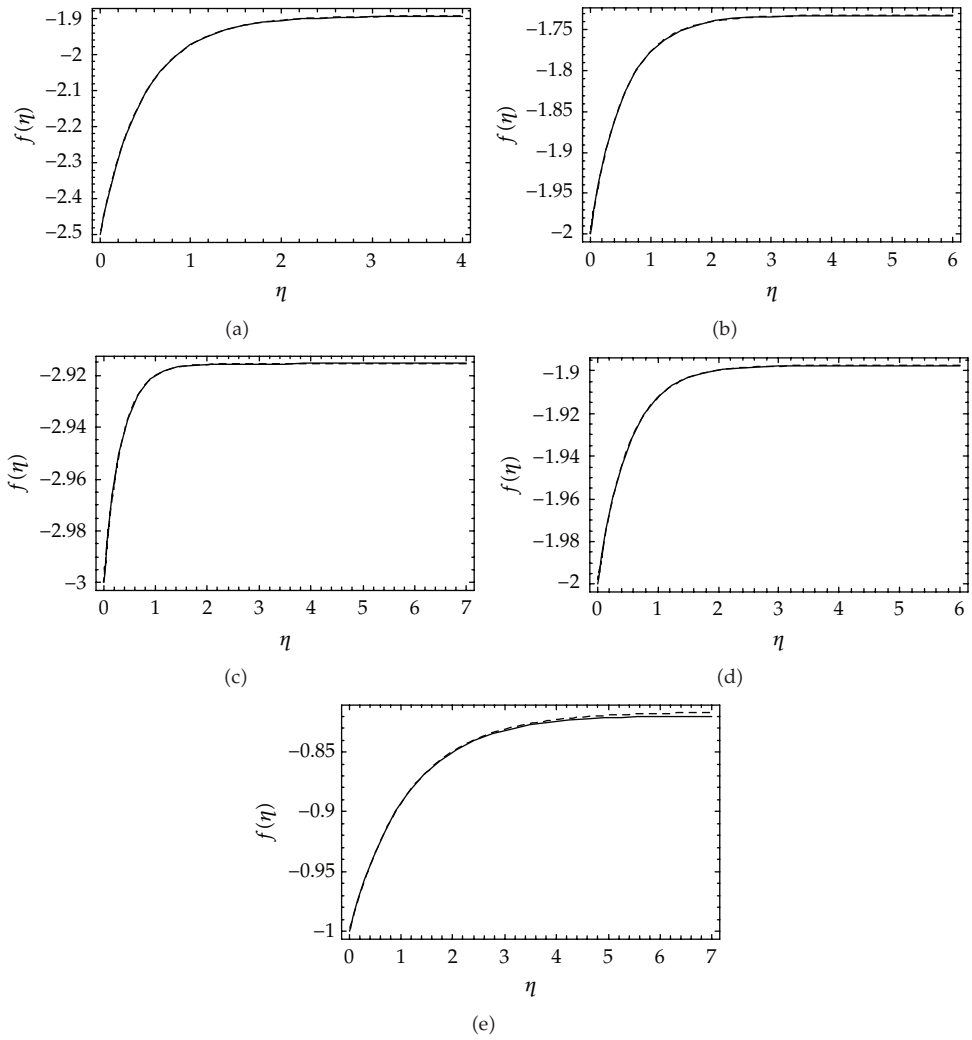


Figure 6: Profiles of the similar stream function $f(\eta)$ as a function of η at $m = -1$, $\sigma = 0$ in R_1 for (a) $\lambda = -0.1$ and $s = -2.5$, (b) $\lambda = 0.5$ and $s = -2$, (c) $\lambda = 1$ and $s = -3$, (d) $\lambda = 2$ and $s = -2$, and (e) $\lambda = 5$ and $s = -1$, where solid and dashed lines are for ChPDM and G'/G-expansion solutions, respectively.

(2) When $\sigma \neq 0$.

Using the first two boundary conditions to the solution given by (4.10) yields

$$\sigma - \frac{4\mu}{\sigma - \sqrt{\sigma^2 - 4\mu}\tanh(\eta_0)} = s, \tag{5.17}$$

$$\begin{aligned}&\left(\sigma\cosh(\eta_0) - \sqrt{\sigma^2 - 4\mu}\sinh(\eta_0)\right)^3 \\ &\quad = 2\mu\left(4\mu - \sigma^2\right)\left[\left(\sigma + 4\lambda\mu - \lambda\sigma^2\right)\cosh(\eta_0) + (\lambda\sigma - 1)\sqrt{\sigma^2 - 4\mu}\sinh(\eta_0)\right].\end{aligned} \tag{5.18}$$

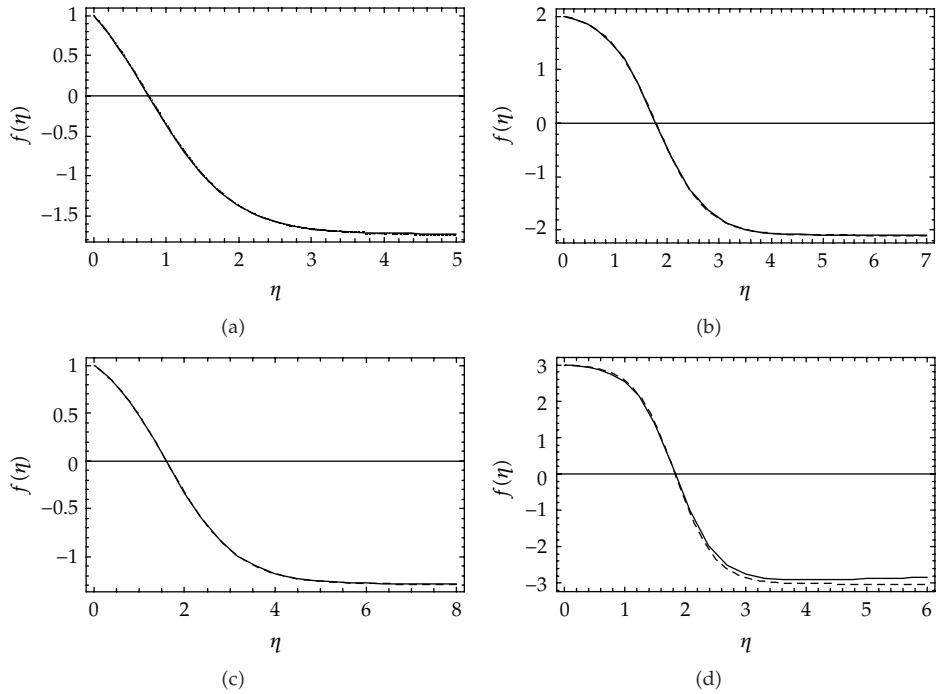


Figure 7: Profiles of the similar stream function $f(\eta)$ as a function of η at $m = -1$, $\sigma = 0$ in R_2 for (a) $\lambda = 2$ and $s = 1$, (b) $\lambda = 3$ and $s = 2$, (c) $\lambda = 4$ and $s = 1$, and (d) $\lambda = 5$ and $s = 3$, where solid and dashed lines are for ChPDM and G'/G-expansion solutions, respectively.

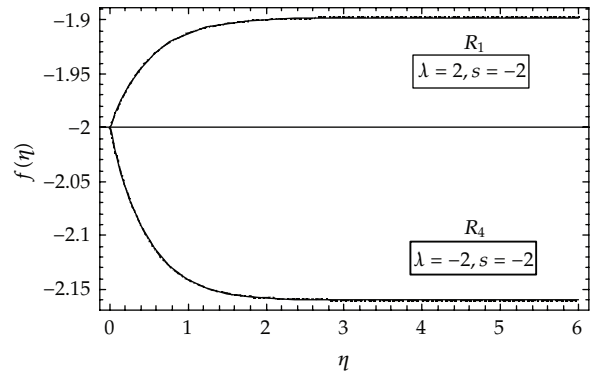


Figure 8: Profiles of the similar stream function $f(\eta)$ as a function of η at $m = -1$, $\sigma = 0$ in R_1, for $\lambda = 2$ and $s = -2$, and R_4, for $\lambda = -2$ and $s = -2$, where solid and dashed lines are for ChPDM and G'/G-expansion solutions, respectively.

By solving (5.17) for η_0, we get

$$\eta_0 = \tanh^{-1}\left(\frac{\sigma(s-\sigma)+4\mu}{(s-\sigma)\sqrt{\sigma^2-4\mu}}\right). \tag{5.19}$$

Table 1: Some values of λ and s with their singularities in R_3.

λ	s	η_{singular}
0.2	2	1.16
0.1	3	0.95
−0.1	2	1.32
−0.1	3	1.12
−1	2	1.69
−3	2	2.07
−4	1	2.66

By substituting η_0 from (5.19) into (5.18) and solving the resulted equation for σ, we obtain

$$\sigma = \pm\sqrt{\frac{\lambda s^3 - s^2 + 2 + 4\mu(\lambda s - 1)}{\lambda s - 1}}. \tag{5.20}$$

Therefore, the exact solution in this case, $m = -1$, is given by $f(\eta)$ in (4.10), where η_0 and σ are defined by (5.19) and (5.20), respectively. This solution can be easily checked by a direct substitution.

Case 4 (a fractional m). Proceeding as above, we obtain the following equations from applying the first two boundary conditions to the solution $f(\eta)$ in (4.11):

$$\frac{6\sigma C_2}{(2m-1)(C_1\sigma - 2C_2)} = s, \tag{5.21}$$

$$(2m-1)(C_1\sigma - 2C_2)^3 = 6\sigma^2 C_2^2[\sigma C_1 + 2(\lambda\sigma - 1)C_2]. \tag{5.22}$$

By solving (5.21) for C_1 and substituting the result into $f(\eta)$ in (4.11) and (5.22), then the following exact solution can be derived:

$$f(\eta) = \frac{6s}{6 + (2m-1)s\eta}, \tag{5.23}$$

provided that m, λ, and s are related by the equation

$$\lambda(2m-1)^2 s^3 + 3(2m-1)s^2 + 18 = 0. \tag{5.24}$$

The solution given by (5.23) and (5.24) can be also checked by a direct substitution.

Lemma 5.2. *From* (5.24), *it is observed that* $\lambda > 0$ *at the following cases:*

(i) *at* $0 < m < 1/2$ *when* $s \in]\sqrt{-6/(2m-1)}, \infty[$,

(ii) *at* $m > 1/2$ *when* $s < 0$,

(iii) *at* $m < 0$ *when* $s \in]\sqrt{6/(1-2m)}, \infty[\cup]-\sqrt{6/(1-2m)}, 0[$.

Proof. On solving (5.24) for λ, we then get

$$\lambda = -\frac{3(2m-1)s^2+18}{(2m-1)^2 s^3}. \tag{5.25}$$

(i) At $0 < m < 1/2$, this leads to $3(2m-1)s^2 < 0$. Now setting $3(2m-1)s^2 + 18 < 0$, we obtain $s^2 > -6/(2m-1)$, that is, $s \in]-\infty, -\sqrt{-6/(2m-1)}[\bigcup]\sqrt{-6/(2m-1)}, \infty[$. However, $s \in]-\infty, -\sqrt{-6/(2m-1)}[\Rightarrow \lambda < 0$ and $s \in]\sqrt{-6/(2m-1)}, \infty[\Rightarrow \lambda > 0$.

(ii) When $m > 1/2$, this means that $3(2m-1)s^2 + 18 >$, for all $s \in \Re$. In this case, the denominator in (5.25) should be negative so that $\lambda > 0$. Hence, s must be negative real number, that is, $s < 0$.

(iii) Here, we may rewrite λ in (5.25) as

$$\lambda = \frac{3(1-2m)s^2-18}{(2m-1)^2 s^3}. \tag{5.26}$$

At $m = 0$, we note that $3(1-2m)s^2 > 0$. Therefore, λ is positive if the numerator and denominator have the same sign, that is, the following conditions have to be held:

$$3(1-2m)s^2 - 18 > 0, \quad s > 0, \qquad 3(1-2m)s^2 - 18 < 0, \quad s < 0. \tag{5.27}$$

On solving the inequalities in (5.27), we obtain the range of s as

$$s \in \left]\sqrt{\frac{6}{1-2m}}, \infty\right[\bigcup\left]-\sqrt{\frac{6}{1-2m}}, 0\right[. \tag{5.28}$$

□

5.1. Important Remark: G'/G-Expansion Method Over the Others

Here, we aim to show that the exact solutions obtained in the previous subsections cannot be achieved by using some other methods. Ebaid [18] pointed out that the Jacobi-elliptic function and F-expansion methods cannot be used to search the exact solutions for nonlinear differential equations that include both odd and even-order derivative terms. However, He indicated the applicability of the standard exp-function method to solve such kind of equations. Despite this ability, on applying the exp-function method [17] to solve (1.1) and (1.2), it does not provide any of the exact solutions obtained by the current method. To clarify this point, this method is applied to the present BVP when $m = 1$, and the results are found as

$$f(\eta) = 1 - (1-s)e^{-\eta}, \tag{5.29}$$

where λ and s are governed by the following equation:

$$\lambda(1-s)-s=0 \quad \text{or} \quad \lambda=\frac{s}{1-s}, \quad s\neq 1. \tag{5.30}$$

In view of (5.29), the solution is trivial at $s=1$. In the case of $\lambda>0$, this exact solution is physically acceptable in a very short range for the parameter ($s\in[0,1[$). The solution given by (5.29) and (5.30) is therefore just a special case of that obtained by generalized G'/G-expansion method and given by (5.5) and (5.6). The same conclusion can be also deduced when applying the standard exp-function method at $m=-1$. Moreover, the rest of solutions given by (5.10) and (5.11) (at $m=0$) and (5.23) and (5.24) cannot be recovered by using the exp-function method. In view of this discussion, it may be concluded that the generalized G'/G-expansion method has many advantages over the exp-function method for the present BVP. Further, in order to make this point as clear as possible, an appendix containing the mathematical details of applying the exp-function method to the current problem is added.

6. Solution for $0<\lambda\ll 1$

In the view of Aly et al. [34], the main attention is paid to the construction of solutions for (1.1) and (1.2), when $\lambda\to 0$. We look for a solution which has the form

$$f(\eta)=f_0+\lambda f_1+\cdots. \tag{6.1}$$

Substituting (6.1) into (1.1) and (1.2), we obtain the equations

$$\begin{gathered}
f_0'''+\lambda f_1'''+m\left(f_0+\lambda f_1\right)\left(f_0''+\lambda f_1''\right)-\left(f_0'+\lambda f_1'\right)^2=0,\\
f_0(0)+\lambda f_1(0)+\mathcal{O}\left(\lambda^2\right)=s,\\
f_0'(0)+\lambda f_1'(0)+\mathcal{O}\left(\lambda^2\right)=1+\lambda\left[f_0''(0)+\lambda f_1''(0)+\mathcal{O}\left(\lambda^2\right)\right],\\
f_0'(\infty)+\lambda f_1'(\infty)+\mathcal{O}\left(\lambda^2\right)=0.
\end{gathered} \tag{6.2}$$

We obtain at λ^0

$$\begin{gathered}
f_0'''+mf_0f_0''-\left(f_0'\right)^2=0,\\
f_0(0)=s, \qquad f_0'(0)=1, \qquad f_0'(\infty)=0,
\end{gathered} \tag{6.3}$$

where at λ^1, we get

$$\begin{gathered}
f_1'''+m\left(f_0f_1''+f_1f_0''\right)-2f_0'f_1'=0,\\
f_1(0)=0, \qquad f_1'(0)=f_0''(0), \qquad f_1'(\infty)=0.
\end{gathered} \tag{6.4}$$

Equation (6.3) appears in the study of similarity solutions to problems of boundary-layer theory in some contexts of fluid mechanics, see, for example, [26, 27, 33, 34]. Exact solutions for these equations can be easily found, for different values of m, by applying the technique in Section 5. We substitute then this solution in the system (6.4) to get the construction of f_1. This means that the form of $f(\eta)$ can be therefore evaluated by (6.1).

7. Asymptotic Solution ($\lambda \gg 1$)

As in [26] and by means of a shooting method, the boundary condition at infinity is replaced by the condition

$$f''(0) = \alpha, \tag{7.1}$$

where α is the shooting parameter which has to be determined. Regarding the difficulties of obtaining the numerical solution of the system (1.1) and (1.2) in the case of $\lambda \gg 1$ and as in [26, 27, 34], we now seek a new set of full equations which do not contain λ on using the following transformation:

$$f(\eta) = \eta + \epsilon^{\vartheta} H(\zeta) \quad \text{where } \zeta = \epsilon^{\gamma}\eta, \tag{7.2}$$

where $\epsilon = \lambda\alpha$; ϑ and γ are constants to be determined. On substituting expressions (7.2) into (1.1), we obtain

$$\epsilon^{2\gamma} H''' + \epsilon^{\gamma} m\eta H'' + \epsilon^{\vartheta+\gamma}\left[mHH'' - H'^2\right] - 2H' - \epsilon^{-\vartheta} = 0. \tag{7.3}$$

In order to ensure that the highest derivative remains present in the resulting equation, so avoiding the need to disregard any of the boundary conditions, we look for a balance within the equation of this term. Hence, we obtain $\vartheta = \gamma = 1/2$. Therefore, when $\epsilon \to \infty$ (i.e., $\lambda \to \infty$), we obtain the following new set of full equations:

$$H''' + mHH'' - H'^2 = 0, \tag{7.4}$$

$$H(0) = 0, \qquad H'(0) = 1, \qquad H''(0) = 0. \tag{7.5}$$

The boundary conditions (7.5) do not contain ϵ. As in the last section, (7.4) and (7.5) can be solved by meaning of Section 5.

8. Conclusion

Third-order nonlinear differential equations describing the nano boundary-layer flow have been investigated theoretically, using G'/G-expansion method, and numerically, applying ChPDM approach. The present results are itemized as follows.

(i) It is found that the ChPDM results are very accurate in an excellent manner on comparing to those published in the literature using the homotopy analysis method and the modified differential transform-Padé method.

(ii) For the first time, we have showed the way of applying G'/G-expansion method to solve nonlinear BVPs. In addition, it has been proven that it has many advantages over some of the other methods on solving the present BVP, where four certain domains for the physical parameters have been discussed.

(iii) ChPDM technique has been successfully applied to validate and evidence the resulted exact solutions, for different positive and negative values of the investigated parameters, m, s, and λ.

(iv) Convex solutions have been obtained at $m = 1$ in the special case of $\lambda f''(0) = -2$.

(v) Vicinity of zero and asymptotic solutions when $0 < \lambda \ll 1$ and $\lambda \gg 1$, respectively, are also deduced. It should be noted that, in both cases, λ does not exist. Therefore, the resulting equations are easy to deal with analytically and numerically.

Appendix

In this section, details of applying the exp-function method to the investigated problem (1.1) and (1.2) are introduced. The aim is to confirm that the solution obtained through this method given by (5.26) and (5.27) is in fact a special case of the exact one obtained by using the G'/G-expansion method. Very recently, Ebaid [35] proved that on searching for exact solution by using the exp-function method, one can go directly by assuming the solution in the form:

$$f = \frac{a_0 + a_1 \exp[\eta] + a_{-1} \exp[-\eta]}{b_0 + b_1 \exp[\eta] + b_{-1} \exp[-\eta]}, \tag{A.1}$$

where the tedious calculations of the balancing procedure are not required. On substituting (A.1) into (1.1), multiplying by $(b_1 e^{\eta} + e^{-\eta} b_{-1} + b_0)^4$, and then equating the coefficients of each exp-function to zero, we obtain the following system of algebraic equations:

$$\begin{aligned}
&2\Big(2(-1+m)a_{-1}^2 b_1^2 + b_{-1}\Big(2(-1+m)a_1^2 b_{-1} + (1-3m)a_0^2 b_1 + a_1(2+(-1+m)a_0 - 16 b_{-1} b_1)\Big)\Big) \\
&\quad + 2(a_{-1}(b_1(-2+(-1+m)a_0 + 16 b_{-1} b_1) + a_1(1+m-4(-1+m)b_{-1} b_1))) = 0, \\
&(-4+3m)a_1^2 b_{-1} + b_1(-a_0(1+m a_0) + (a_{-1}(-5+(-4+5m)a_0) + 23 a_0 b_{-1}) b_1) \\
&\quad + a_1(1 + m a_0 + 2((2+m)a_{-1} + (-9+(2-5m)a_0)b_{-1}) b_1) = 0, \\
&(-4+3m)a_{-1}^2 b_1 + a_{-1}(-1 + 2b_{-1}((2+m)a_1 + 9 b_1) + a_0(m + 2(2-5m)b_{-1} b_1)) \\
&\quad + b_{-1}(5 a_1 b_{-1} - a_0(-1 + m a_0 + b_{-1}((4-5m)a_1 + 23 b_1))) = 0, \\
&\quad b_1^2(a_0(4+(-1+m)a_0) - 8 a_{-1} b_1) + a_1^2(-1+m-4m b_{-1} b_1) \\
&\quad + 2a_1 b_1(-2-(-1+m)a_0 + 2(m a_{-1} + 2b_{-1}) b_1) b_{-1}^2(a_0(-4+(-1+m)a_0) + 8 a_1 b_{-1}) \\
&\quad + 2a_{-1} b_{-1}(2-(-1+m)a_0 + 2b_{-1}(m a_1 - 2b_1)) + a_{-1}^2(-1+m-4m b_{-1} b_1) = 0, \\
&\qquad b_1(m a_1 - b_1)(a_1 - a_0 b_1) = 0, \\
&\qquad b_{-1}(m a_{-1} + b_{-1})(a_{-1} - a_0 b_{-1}) = 0.
\end{aligned} \tag{A.2}$$

On solving this system, we obtain a nontrivial solution at $m = 1$ as

$$a_0 = a_{-1}b_1 + 1, \qquad a_1 = b_1, \qquad b_{-1} = 0, \qquad f = 1 + e^{-\eta}a_{-1}. \tag{A.3}$$

On applying the first boundary condition, we obtain $a_{-1} = s - 1$. The exact solution hence becomes

$$f = 1 - (1 - s)e^{-\eta}, \tag{A.4}$$

which is equivalent to (5.29). Now on applying the second boundary condition, this leads directly to (5.30).

In conclusion, the above discussion shows that application of the exp-function method to the present problem gives the same solution in (5.29) and (5.30) which is already shown in Section 5.1. Exp-function method came therefore as a special case from one of the solutions provided by the G'/G-expansion method.

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