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*Research Article*

# Strong Convergence of Parallel Iterative Algorithm with Mean Errors for Two Finite Families of Ćirić Quasi-Contractive Operators

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The purpose of this paper is to establish a strong convergence of a new parallel iterative algorithm with mean errors to a common fixed point for two finite families of Ćirić quasi-contractive operators in normed spaces. The results presented in this paper generalize and improve the corresponding results of Berinde, Gu, Rafiq, Rhoades, and Zamfirescu.

## 1. Introduction and Preliminaries

Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is said to be $a$-*contraction*, if $d(Tx, Ty) \leq ad(x, y)$ for all $x, y \in X$, where $a \in (0, 1)$.

The mapping $T : X \to X$ is said to be Kannan mapping [1], if there exists $b \in (0, 1/2)$ such that $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$ for all $x, y \in X$.

A mapping $T : X \to X$ is said to be Chatterjea mapping [2], if there exists $c \in (0, 1/2)$ such that $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$ for all $x, y \in X$.

Combining these three definitions, Zamfirescu [3] proved the following important result.

**Theorem Z** (see [3]). *Let $(X, d)$ be a complete metric space and $T : X \to X$ a mapping for which there exist the real numbers $a, b$, and $c$ satisfying $a \in (0, 1)$, $b, c \in (0, 1/2)$ such that for each pair $x, y \in X$, at least one of the following conditions holds:*

$(z_1)$ $d(Tx, Ty) \leq ad(x, y)$,

$(z_2)$ $d(Tx,Ty) \le b[d(x,Tx)+d(y,Ty)]$,

$(z_3)$ $d(Tx,Ty) \le c[d(x,Ty)+d(y,Tx)]$.

*Then $T$ has a unique fixed point $p$ and the Picard iteration $\{x_n\}$ defined by*

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N} \tag{1.1}$$

*converges to $p$ for any arbitrary but fixed $x_1 \in X$.*

*Remark 1.1.* An operator $T$ satisfying the contractive conditions $(z_1)$–$(z_3)$ in the above theorem is called $Z$-operator.

*Remark 1.2.* The conditions $(z_1)-(z_3)$ can be written in the following equivalent form:

$$d(Tx,Ty) \le h \max\left\{ d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2} \right\}, \tag{1.2}$$

for all $x, y \in X, 0 < h < 1$. Thus, a class of mappings satisfying the contractive conditions $(z_1)-(z_3)$ is a subclass of mappings satisfying the following condition:

$$d(Tx,Ty) \le h \max\left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2} \right\}, \tag{CG}$$

$0 < h < 1$. The class of mappings satisfying (CG) is introduced and investigated by Ćirić [4] in 1971.

*Remark 1.3.* A mapping satisfying (CG) is commonly called Ćirić generalized contraction.

In 2000, Berinde [5] introduced a new class of operators on a normed space $E$ satisfying

$$\|Tx - Ty\| \le \rho \|x - y\| + L\|Tx - x\|, \tag{1.3}$$

for any $x, y \in E$, $0 \le \delta < 1$ and $L \ge 0$.

Note that (1.3) is equivalent to

$$\|Tx - Ty\| \le \rho \|x - y\| + L \min\{\|Tx - x\|, \|Ty - y\|\}, \tag{1.4}$$

for any $x, y \in E$, $0 \le \rho < 1$ and $L \ge 0$.

Berinde [5] proved that this class is wider than the class of Zamfiresu operators and used the Mann [6] iteration process to approximate fixed points of this class of operators in a normed space given in the form of following theorem.

**Theorem B** (see [5]). *Let $C$ be a nonempty closed convex subset of a normed space $E$. Let $T : C \to C$ be an operator satisfying* (1.3) *and $F(T) \neq \emptyset$. For given $x_0 \in C$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \ge 0, \tag{1.5}$$

*where* $\{\alpha_n\}$ *be a real sequence in [0, 1]. If* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point of* $T$.

In 2006, Rafiq [7] considered a class of mappings satisfying the following condition:

$$\|Tx - Ty\| \leq h \max\left\{ \|x - y\|, \frac{\|x - Tx\| + \|y - Ty\|}{2}, \|x - Ty\|, \|y - Tx\| \right\}, \tag{CR}$$

$0 < h < 1$. This class of mappings is a subclass of mappings satisfying the following condition:

$$\|Tx - Ty\| \leq h \max\{ \|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\| \}, \tag{CQ}$$

$0 < h < 1$. The class of mappings satisfying (CQ) was introduced and investigated by Ćirić [8] in 1974 and a mapping satisfying is commonly called Ćirić quasi-contraction.

Rafiq [7] proved the following result.

**Theorem R** (see [7]). *Let* $C$ *be a nonempty closed convex subset of a normed space* $E$. *Let* $T : C \to C$ *be an operator satisfying the condition* (CR). *For given* $x_0 \in C$, *let* $\{x_n\}$ *be generated by the algorithm*

$$x_{n+1} = \alpha_n x_n + \beta_n T x_n + \gamma_n u_n, \quad n \geq 0, \tag{1.6}$$

*where* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *be three real sequences in [0, 1] satisfying* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$, $\{u_n\}$ *is a bounded sequences in* $C$. *If* $\sum_{n=1}^{\infty} \beta_n = \infty$ *and* $\gamma_n = o(\alpha_n)$, *then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point of* $T$.

In 2007, Gu [9] proved the following theorem.

**Theorem G** (see [9]). *Let* $C$ *be a nonempty closed convex subset of a normed space* $E$. *Let* $\{T_i\}_{i=1}^{N} : C \to C$ *be* $N$ *operators satisfying the condition* (CR) *with* $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ *(the set of common fixed points of* $\{T_i\}_{i=1}^{N}$*). Let* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *be three real sequences in [0, 1] satisfying* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \geq 1$, $\{u_n\}$ *a bounded sequences in* $C$ *satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *or* $\gamma_n = o(\beta_n)$.

*Suppose further that* $x_0 \in C$ *is any given point and* $\{x_n\}$ *is generated by the algorithm*

$$x_{n+1} = \alpha_n x_n + \beta_n T_n x_n + \gamma_n u_n, \quad n \geq 0, \tag{1.7}$$

*where* $T_n = T_{n(\bmod N)}$. *Then* $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_i\}_{i=1}^{N}$.

*Remark 1.4.* It should be pointed out that Theorem G extends Theorem R from a Ćirić quasi-contractive operator to a finite family of Ćirić quasi-contractive operators.

Inspired and motivated by the facts said above, we introduced a new two-step parallel iterative algorithm with mean errors for two finite family of operators $\{S_i\}_{i=1}^m$ and $\{T_j\}_{j=1}^k$ as follows:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n-\gamma_n)x_n + \alpha_n \sum_{i=1}^{m} \lambda_i S_i y_n + \gamma_n u_n, \quad n \geq 1, \\ y_n &= (1-\beta_n-\delta_n)x_n + \beta_n \sum_{j=1}^{k} \mu_i T_j x_n + \delta_n v_n, \quad n \geq 1, \end{aligned} \tag{1.8}$$

where $\{\lambda_i\}_{i=1}^m$, $\{\mu_j\}_{j=1}^k$ are two finite sequences of positive number such that $\sum_{i=1}^m \lambda_i = 1$ and $\sum_{j=1}^k \mu_j = 1$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are four real sequences in $[0, 1]$ satisfying $\alpha_n + \gamma_n \leq 1$ and $\beta_n + \delta_n \leq 1$ for all $n \geq 1$, $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in $C$ and $x_0$ is a given point.

Especially, if $\{\alpha_n\}$, $\{\gamma_n\}$ are two sequences in $[0, 1]$ satisfying $\alpha_n + \gamma_n \leq 1$ for all $n \geq 1$, $\{\lambda_i\}_{i=1}^m \subset [0,1]$ satisfying $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$, $\{u_n\}$ is a bounded sequence in $C$ and $x_0$ is a given point in $C$, then the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\alpha_n-\gamma_n)x_n + \alpha_n \sum_{i=1}^{m} \lambda_i S_i x_n + \gamma_n u_n, \quad n \geq 1 \tag{1.9}$$

is called the one-step parallel iterative algorithm with mean errors for a finite family of operators $\{S_i\}_{i=1}^m$.

The purpose of this paper is to study the convergence of two-steps parallel iterative algorithm with mean errors defined by (1.8) to a common fixed point for two finite family of Ćirić quasi-contractive operators in normed spaces. The results presented in this paper generalized and extend the corresponding results of Berinde [5], Gu [9], Rafiq [7], Rhoades [10], and Zamfirescu [3]. Even in the case of $\beta_n = \delta_n = 0$ or $\gamma_n = \delta_n = 0$ for all $n \geq 1$ or $m = k = 1$ are also new.

In order to prove the main results of this paper, we need the following Lemma.

**Lemma 1.5** (see [11]). *Suppose that $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are three nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1-t_n)a_n + b_n + c_n, \quad \forall n \geq n_0, \tag{1.10}$$

*where $n_0$ is some nonnegative integer, $t_n \in [0,1]$, $\sum_{n=0}^\infty t_n = \infty$, $b_n = o(t_n)$ and $\sum_{n=0}^\infty c_n < \infty$. Then* $\lim_{n\to\infty} a_n = 0$.

## 2. Main Results

We are now in a position to prove our main results in this paper.

**Theorem 2.1.** *Let $C$ be a nonempty closed convex subset of a normed space $E$. Let $\{S_i\}_{i=1}^m : C \to C$ be $m$ operators satisfying the condition $(CR)$ and $\{T_j\}_{j=1}^k : C \to C$ be $k$ operators satisfying the* *condition (CR) with* $F = (\bigcap_{i=1}^{m} F(S_i)) \cap (\bigcap_{j=1}^{k} F(T_j)) \neq \emptyset$*, where* $F(S_i)$ *and* $F(T_j)$ *are the set of fixed points of* $S_i$ *and* $T_j$ *in* $C$*, respectively. Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\delta_n\}$ *be four real sequences in [0, 1] satisfying* $\alpha_n + \gamma_n \leq 1$ *and* $\beta_n + \delta_n \leq 1$ *for all* $n \geq 1$, $\{\lambda_i\}_{i=1}^{m}$, $\{\mu_j\}_{j=1}^{k}$ *two finite sequences of positive number such that* $\sum_{i=1}^{m} \lambda_i = 1$ *and* $\sum_{j=1}^{k} \mu_j = 1$, $\{u_n\}$ *and* $\{v_n\}$ *two bounded sequences in* $C$ *satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} \delta_n = 0$;

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *or* $\gamma_n = o(\alpha_n)$.

*Suppose further that* $x_0 \in C$ *is any given point and* $\{x_n\}$ *is an iteration sequence with mane errors defined by* (1.8)*, then* $\{x_n\}$ *converges strongly to a common fixed point of* $\{S_i\}_{i=1}^{m}$ *and* $\{T_j\}_{j=1}^{k}$.

*Proof.* Since $\{S_i\}_{i=1}^{m} : C \to C$ is $m$ Ćirić operator satisfying the condition (CR), hence there exists $0 < h_i < 1$ ($i \in I = \{1, 2, \ldots, m\}$) such that

$$\|S_i x - S_i y\| \leq h_i \max\left\{ \|x - y\|, \frac{\|x - S_i x\| + \|y - S_i y\|}{2}, \|x - S_i y\|, \|y - S_i x\| \right\}. \tag{2.1}$$

For each fixed $i \in I = \{1, 2, \ldots, m\}$. Denote $h = \max\{h_1, h_2, \ldots, h_m\}$, then $0 < h < 1$ and

$$\|S_i x - S_i y\| \leq h \max\left\{ \|x - y\|, \frac{\|x - S_i x\| + \|y - S_i y\|}{2}, \|x - S_i y\|, \|y - S_i x\| \right\} \tag{2.2}$$

hold for each fixed $i \in I = \{1, 2, \ldots, m\}$. If from (2.2) we have

$$\|S_i x - S_i y\| \leq \frac{h}{2}\left[\|x - S_i x\| + \|y - S_i y\|\right], \tag{2.3}$$

then

$$\begin{aligned} \|S_i x - S_i y\| &\leq \frac{h}{2}\left[\|x - S_i x\| + \|y - S_i y\|\right] \\ &\leq \frac{h}{2}\left[\|x - S_i x\| + \|y - x\| + \|x - S_i x\| + \|S_i x - S_i y\|\right]. \end{aligned} \tag{2.4}$$

Hence

$$\left(1 - \frac{h}{2}\right)\|S_i x - S_i y\| \leq \frac{h}{2}\|x - y\| + h\|x - S_i x\|, \tag{2.5}$$

which yields (using the fact that $0 < h < 1$)

$$\|S_i x - S_i y\| \leq \frac{h/2}{1 - h/2}\|x - y\| + \frac{h}{1 - h/2}\|x - S_i x\|. \tag{2.6}$$

Also, from (2.2), if

$$\|S_ix - S_iy\| \leq h \max\{\|x - S_iy\|, \|y - S_ix\|\} \tag{2.7}$$

holds, then

(a) $\|S_ix - S_iy\| \leq h\|x - S_iy\|$, which implies $\|S_ix - S_iy\| \leq h\|x - S_ix\| + h\|S_ix - S_iy\|$ and hence, as $h < 1$,

$$\|S_ix - S_iy\| \leq \frac{h}{1-h}\|x - S_ix\|, \tag{2.8}$$

or

(b) $\|S_ix - S_iy\| \leq h\|y - S_ix\|$, which implies

$$\|S_ix - S_iy\| \leq h\|y - x\| + h\|x - S_ix\|. \tag{2.9}$$

Thus, if (2.7) holds, then from (2.8) and (2.9) we have

$$\|S_ix - S_iy\| \leq h\|y - x\| + \frac{h}{1-h}\|x - S_ix\|. \tag{2.10}$$

Denote

$$\rho_1 = \max\left\{h, \frac{h/2}{1-h/2}\right\} = h, \qquad L_1 = \max\left\{h, \frac{h}{1-h/2}, \frac{h}{1-h}\right\} = \frac{h}{1-h}. \tag{2.11}$$

Then we have $0 < \rho_1 < 1$ and $L_1 \geq 0$. Combining (2.2),(2.6), and (2.10) we get

$$\|S_ix - S_iy\| \leq \rho_1\|x - y\| + L_1\|x - S_ix\| \tag{2.12}$$

holds for all $x, y \in C$ and $i \in I$.

On the other hand, since $\{T_j\}_{j=1}^{k} : C \to C$ is $k$ Ćirić operator satisfying the condition (CR), similarly, we can prove

$$\|T_jx - T_jy\| \leq \rho_2\|x - y\| + L_2\|x - S_ix\|, \tag{2.13}$$

for all $x, y \in C$ and $j \in J = \{1, 2, \ldots, k\}$, where $0 < \rho_2 < 1$ and $L_2 \geq 0$.

Let $p \in F = (\bigcap_{i=1}^{m} F(S_i)) \cap (\bigcap_{j=1}^{k} F(T_j))$; using (1.8) we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\|(1-\alpha_n-\gamma_n)(x_n-p) + \alpha_n \sum_{i=1}^{m} \lambda_i (S_i y_n - p) + \gamma_n(u_n - p)\right\| \\
&\le (1-\alpha_n-\gamma_n)\|x_n-p\| + \alpha_n \sum_{i=1}^{m} \lambda_i \|S_i y_n - p\| + \gamma_n \|u_n - p\| \\
&\le (1-\alpha_n)\|x_n-p\| + \alpha_n \sum_{i=1}^{m} \lambda_i \|S_i y_n - p\| + \gamma_n M,
\end{aligned} \tag{2.14}$$

where $M = \sup_{n\ge1}\{\|u_n - p\|, \|v_n - p\|\}$. Now for $y = y_n$ and $x = p$, (2.12) gives

$$\|S_i y_n - p\| = \|S_i y_n - S_i p\| \le \rho_1 \|y_n - p\|. \tag{2.15}$$

Substituting (2.15) into (2.14), we obtain that

$$\|x_{n+1} - p\| \le (1-\alpha_n)\|x_n - p\| + \alpha_n \rho_1 \|y_n - p\| + \gamma_n M. \tag{2.16}$$

Again it follows from (1.8) that

$$\begin{aligned}
\|y_n - p\| &= \left\|(1-\beta_n-\delta_n)(x_n-p) + \beta_n \sum_{j=1}^{k} \mu_j (T_j x_n - p) + \delta_n(v_n - p)\right\| \\
&\le (1-\beta_n-\delta_n)\|x_n-p\| + \beta_n \sum_{j=1}^{k} \mu_j \|T_j x_n - p\| + \delta_n \|v_n - p\| \\
&\le (1-\beta_n)\|x_n-p\| + \beta_n \sum_{j=1}^{k} \mu_j \|T_j x_n - p\| + \delta_n M.
\end{aligned} \tag{2.17}$$

Now for $y = x_n$ and $x = p$, (2.13) gives

$$\|T_j x_n - p\| = \|T_j x_n - T_j p\| \le \rho_2 \|x_n - p\|. \tag{2.18}$$

Combining (2.17) and (2.18) we get

$$\|y_n - p\| \le [1 - \beta_n(1-\rho_2)]\|x_n - p\| + \delta_n M \le \|x_n - p\| + \delta_n M. \tag{2.19}$$

Substituting (2.19) into (2.16), we obtain that

$$\begin{aligned}
\|x_{n+1} - p\| &\le (1-\alpha_n)\|x_n - p\| + \alpha_n \rho_1 (\|x_n - p\| + \delta_n M) + \gamma_n M \\
&= [1 - \alpha_n(1-\rho_1)]\|x_n - p\| + \alpha_n \delta_n \rho_1 M + \gamma_n M \\
&= (1 - t_n)\|x_n - p\| + b_n + c_n,
\end{aligned} \tag{2.20}$$

where

$$t_n = \alpha_n(1-\rho_1), \quad b_n = \alpha_n\delta_n\rho_1 M, \quad c_n = \gamma_n M \tag{2.21}$$

or

$$t_n = \alpha_n(1-\rho_1), \quad b_n = \alpha_n\delta_n\rho_1 M + \gamma_n M, \quad c_n = 0. \tag{2.22}$$

From the conditions (i)–(iii) it is easy to see that $t_n \in [0,1]$, $\sum_{n=1}^{\infty} t_n = \infty$, $b_n = o(t_n)$, and $\sum_{n=1}^{\infty} c_n < \infty$. Thus using (2.20) and Lemma 1.5 we have $\lim_{n\to\infty}\|x_n - p\| = 0$, and so $\lim_{n\to\infty} x_n = p$. This completes the proof of Theorem 2.1. □

**Theorem 2.2.** *Let $C$ be a nonempty closed convex subset of a normed space $E$. Let $\{S_i\}_{i=1}^{m} : C \to C$ be $m$ operators satisfying the condition* (2.12) *and let $\{T_j\}_{j=1}^{k} : C \to C$ be $k$ operators satisfying the condition* (2.13) *with $F = (\bigcap_{i=1}^{m} F(S_i)) \cap (\bigcap_{j=1}^{k} F(T_j)) \neq \emptyset$, where $F(S_i)$ and $F(T_j)$ are the set of fixed points of $S_i$ and $T_j$ in $C$, respectively. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ be four real sequences in [0, 1] satisfying $\alpha_n + \gamma_n \leq 1$ and $\beta_n + \delta_n \leq 1$ for all $n \geq 1$, $\{\lambda_i\}_{i=1}^{m}$, $\{\mu_j\}_{j=1}^{k}$ two finite sequences of positive number such that $\sum_{i=1}^{m} \lambda_i = 1$, and $\sum_{j=1}^{k} \mu_j = 1$, $\{u_n\}$ and $\{v_n\}$ two bounded sequences in $C$ satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty}\delta_n = 0$;

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *or* $\gamma_n = o(\alpha_n)$.

*Suppose further that $x_0 \in C$ is any given point and $\{x_n\}$ is an iteration sequence defined by* (1.8), *then $\{x_n\}$ converges strongly to a common fixed point of $\{S_i\}_{i=1}^{m}$ and $\{T_j\}_{j=1}^{k}$.*

**Theorem 2.3.** *Let $C$ be a nonempty closed convex subset of a normed space $E$. Let $\{S_i\}_{i=1}^{m} : C \to C$ be $m$ operators satisfying the condition* (CR) *with $F = \bigcap_{i=1}^{m} F(S_i) \neq \emptyset$ (the set of common fixed points of $\{S_i\}_{i=1}^{m}$). Let $\{\alpha_n\}$ and $\{\gamma_n\}$ be two real sequences in [0, 1] satisfying $\alpha_n + \gamma_n \leq 1$ for all $n \geq 1$, $\{\lambda_i\}_{i=1}^{m}$ a finite sequence of positive number such that $\sum_{i=1}^{m} \lambda_i = 1$, and $\{u_n\}$ a bounded sequence in $C$ satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *or* $\gamma_n = o(\alpha_n)$.

*Suppose further that $x_0 \in C$ is any given point and $\{x_n\}$ is an iteration sequence with mane errors defined by* (1.9), *then $\{x_n\}$ converges strongly to a common fixed point of $\{S_i\}_{i=1}^{m}$.*

**Theorem 2.4.** *Let $C$ be a nonempty closed convex subset of a normed space $E$. Let $\{S_i\}_{i=1}^{m} : C \to C$ be $m$ operators satisfying the condition* (2.12) *with $F = \bigcap_{i=1}^{m} F(S_i) \neq \emptyset$ (the set of common fixed points of $\{S_i\}_{i=1}^{m}$). Let $\{\alpha_n\}$ and $\{\gamma_n\}$ be two real sequences in [0, 1] satisfying $\alpha_n + \gamma_n \leq 1$ for all $n \geq 1$, $\{\lambda_i\}_{i=1}^{m}$ a finite sequence of positive number such that $\sum_{i=1}^{m} \lambda_i = 1$, and $\{u_n\}$ a bounded sequence in $C$ satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *or* $\gamma_n = o(\alpha_n)$.

*Suppose further that $x_0 \in C$ is any given point and $\{x_n\}$ is an iteration sequence defined by* (1.9)*, then $\{x_n\}$ converges strongly to a common fixed point of $\{S_i\}_{i=1}^{m}$.*

**Corollary 2.5** (see [7]). *Let $C$ be a nonempty closed convex subset of a normed space $E$. Let $T : C \to C$ be an operators satisfying the condition* $(CR)$*. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three real sequences in [0, 1] satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$ and $\{u_n\}$ a bounded sequences in $C$ satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *or* $\gamma_n = o(\beta_n)$.

*Suppose further that $x_0 \in C$ is any given point and $\{x_n\}$ is an explicit iteration sequence as follows:*

$$x_{n+1} = \alpha_n x_n + \beta_n T x_n + \gamma_n u_n, \quad n \geq 1, \tag{2.23}$$

*then $\{x_n\}$ converges strongly to the unique fixed point of $T$.*

*Proof.* By Ćirić [8], we know that $T$ has a unique fixed point in $C$. Taking $m = 1$ in Theorem 2.3, then the conclusion of Corollary 2.5 can be obtained from Theorem 2.3 immediately. This completes the proof of Corollary 2.5. □

*Remark 2.6.* Theorems 2.2–2.4 and Corollary 2.5 improve and extend the corresponding results of Berinde [5], Gu [9], Rafiq [7], Rhoades [10], and Zamfirescu [3].

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