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Research Article

Hopf Bifurcation of Limit Cycles in Discontinuous Liénard Systems

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We consider a class of discontinuous Liénard systems and study the number of limit cycles bifurcated from the origin when parameters vary. We establish a method of studying cyclicity of the system at the origin. As an application, we discuss some discontinuous Liénard systems of special form and study the cyclicity near the origin.

1. Introduction and Main Results

As well known, Liénard systems describe the dynamics of systems of one degree of freedom under existence of a linear restoring fore and a nonlinear dumping. In the first half of the last century models based on the Liénard system were important for the development of radio and vacuum tube technology. Nowadays the system is widely used to describe oscillatory processes arising in various studies of mathematical models of physical, biological, chemical, epidemiological, physiological, economical, and many other phenomena (see Glade et al. [1], Llibre [2] and references therein). Further, quadratic systems and some other systems can be transformed into Liénard systems by suitable changes, see for instance Han et al. [3], Cherkas [4], Gasull [5], Giné, and Llibre [6]. As we have seen, the main concern on Liénard systems is the center and focus problem and the number of limit cycles, see [7–17], and references therein. Here, we briefly list some known results related to our study in this paper. Consider a Liénard system of the form

$$\begin{aligned} \dot{x} &= y - F(x), \\ \dot{y} &= -g(x), \end{aligned} \tag{1.1}$$

where we suppose that the functions F and g satisfy two assumptions below.

(I) There exists a positive number ε_0 such that F is continuous for $|x| < \varepsilon_0$ with $F(0) = 0$ and g is continuous for $0 < |x| < \varepsilon_0$ with $xg(x) > 0$, and $\lim_{x\to 0^+} g(x), \lim_{x\to 0^-} g(x)$ existing.

(II) For $|x| < \varepsilon_0$, $|F(x)| < a_1\sqrt{G(x)}$, where $0 < a_1 < \sqrt{8}$ and $G(x) = \int_0^x g(u)du$. Obviously, if $g(0) = 0$, system (1.1) has a singular point at the origin. By Han and Zhang [12], the origin is a generalized singular point if $g(0) \neq 0$. For example, the following system:

$$(\dot{x}, \dot{y}) = \begin{cases} (y, -1), & x \geq 0, \\ (y, 1), & x < 0 \end{cases} \tag{1.2}$$

has a generalized center at the origin.

Under the condition (I), the equation $G(x) = z$ has two solutions $x = x_1(z) > 0$ and $x = x_2(z) < 0$ for $0 < z \ll 1$. Let $F_i(z) = F(x_i(z)), i = 1, 2$. Then in 1952, Filippov proved the following theorem (see Chapter 5 of Ye [15]).

Theorem 1.1. *Let (I) and (II) be satisfied. Further suppose g in* (1.1) *is continuous at $x = 0$. Then system* (1.1) *has a stable focus (resp., center, unstable focus) if*

$$F_2(z) - F_1(z) < 0 \quad (\textit{resp.}, \equiv 0, > 0) \textit{ for } 0 < z \ll 1. \tag{1.3}$$

In 1985, Han [10] obtained the following.

Theorem 1.2. *Consider the system*

$$\begin{aligned} \dot{x} &= h(y) - F(x), \\ \dot{y} &= -g(x), \end{aligned} \tag{1.4}$$

where h, F, and g are continuous functions satisfying

(i) $h(y) = (sgny)|y|^{\overline{m}} + O(y^{\overline{m}+1})$ *with* $\overline{m} > 0$,

(ii) $xg(x) > 0$ *for* $0 < |x| \ll 1$ *and*

$$|F(x)| < c[G(x)]^{\overline{m}/(\overline{m}+1)} \quad \textit{for } |x| \ll 1, \tag{1.5}$$

where $0 < c < (\overline{m}+1)((\overline{m}+1)/\overline{m})^{\overline{m}/(\overline{m}+1)}$. *Let* $\alpha(x) = -x + O(x^2)$ *satisfy* $G(\alpha(x)) = G(x)$ *for* $|x| \ll 1$. *Then system* (1.4) *has a stable focus (resp., center, unstable focus) if*

$$F(\alpha(x)) - F(x) < 0 \quad (\textit{resp.}, \equiv 0, > 0) \tag{1.6}$$

for $|x| \ll 1$.

For center conditions, in 1976, Cherkas [7] proved the following.

Theorem 1.3. *Let the functions F and g in system* (1.1) *be polynomials in x with*

$$F(0) = g(0) = 0, \qquad g'(0) > 0. \tag{1.7}$$

Then the origin is a center of system (1.1) *if and only if the equations*

$$F(x) = F(y), \qquad G(x) = G(y) \tag{1.8}$$

have a unique solution $y = \alpha(x) < 0$ for $x > 0$ sufficiently small.

From Cherkas [7], one can see that the above result is also true for analytic system (1.1). In 1998, Gasull and Torregrosa [9] studied the center problem for analytic systems of form (1.4) using the Cherkas' method, generalized Theorem 1.3 to (1.4) and presented interesting applications to some polynomial systems. Then in 2006, Cherkas and Romanovski [8] gave a necessary and sufficient condition for a Liénard system with nonlinearities of degree six to have a center at the origin. About Theorem 1.2, we have the following two remarks.

Remark 1.4. By the variable change

$$u = (\operatorname{sgn} x)\sqrt{2G(x)} \tag{1.9}$$

and the scaling of the time $d\tau = (g(X(u))/u)dt$, we can obtain from (1.4)

$$\begin{aligned} \dot{u} &= h(y) - F(X(u)), \\ \dot{y} &= -u, \end{aligned} \tag{1.10}$$

where $X(u)$ satisfies $|u| = \sqrt{2G(X(u))}$, $uX(u) > 0$. Thus, Theorem 1.2 is also true if $g(x)$ is not continuous at $x = 0$.

Remark 1.5. If F and g are C^∞ with $g'(0) > 0$. Then $F(\alpha(x)) - F(x)$ in (1.6) has the form

$$F(\alpha(x)) - F(x) = \sum_{j\ge1} B_j x^j. \tag{1.11}$$

In this case, Han [11] proved that if $B_j = 0$ for $j = 1, \ldots, 2k-1$ $(k \ge 1)$, then $B_{2k} = 0$, and the origin is a focus of order k if $B_{2k+1} \neq 0$ in addition. Further, if a vector parameter a appears in F, then $B_j = B_j(a)$ and

$$B_{2k} = O(|B_1, B_3, \ldots, B_{2k-1}|) \tag{1.12}$$

for all a and any $k \ge 1$.

In this paper, we consider the following discontinuous Liénard system:

$$\begin{aligned} \dot{x} &= h(y) - F(x,a), \\ \dot{y} &= -g(x,b), \end{aligned} \tag{1.13}$$

where $(a,b) \in R^{n_1}$ with n_1 being an integer, $h(y)$ is a C^∞ function satisfying $h(y) = y + O(y^2)$ and

$$F(x,a) = \begin{cases} F^+(x,a), & x \geq 0, \\ F^-(x,a), & x < 0, \end{cases} \qquad g(x) = \begin{cases} g^+(x,b), & x \geq 0, \\ g^-(x,b), & x < 0, \end{cases} \tag{1.14}$$

where F^+, g^+ and F^-, g^- are C^∞ on $[0,x_0]$, and $[-x_0,0]$, respectively, with $x_0 > 0$. Further, suppose there exist integers $m \geq 1$, $n \geq 1$, $k \geq 0$, $l \geq 0$ such that

$$F^+(x,a) = \sum_{j \geq m} F_j^+(a)x^j, \qquad g^+(x,b) = \sum_{j \geq k} g_j^+(b)x^j, \qquad g_k^+(b) > 0 \tag{1.15}$$

for $0 < x \ll 1$ and

$$F^-(x,a) = \sum_{j \geq n} F_j^-(a)x^j, \quad g^-(x,b) = g_l^-(b)(-x)^l + \sum_{j \geq l+1} g_j^-(b)x^j, \quad g_l^-(b) < 0 \tag{1.16}$$

for $0 < -x \ll 1$.

Let

$$G(x,b) = \int_0^x g(u,b)du = \begin{cases} G^+(x,b), & 0 \leq x \leq x_0, \\ G^-(x,b), & 0 < -x \leq x_0, \end{cases} \tag{1.17}$$

where

$$\begin{aligned} G^+(x,b) &= \frac{g_k^+(b)}{k+1}x^{k+1} + \sum_{j \geq k+1} \frac{g_j^+(b)}{j+1}x^{j+1}, \\ G^-(x,b) &= \frac{-g_l^-(b)}{l+1}(-x)^{l+1} + \sum_{j \geq l+1} \frac{g_j^-(b)}{j+1}x^{j+1} \end{aligned} \tag{1.18}$$

for $0 < x \ll 1$ and $0 < -x \ll 1$, respectively.

This paper is devoted to studying the local property of system (1.13) at the origin and the number of limit cycles bifurcated from the origin as (a,b) varies. The authors [13] considered system (1.13) for the case $m = n = k = l = 1$ and studied the center focus problem.

It is easy to prove that for system (1.13) under (1.15) and (1.16), the condition (ii) of Theorem 1.2 is equivalent to the following:

$$\begin{aligned} &(\mathrm{H}_1)\ m > ((k+1)/2) \text{ or } m = ((k+1)/2), \qquad |F_m^+(a)| < c\left(\left(g_k^+(b)\right)/(k+1)\right)^{1/2}, \\ &(\mathrm{H}_2)\ n > ((l+1)/2) \text{ or } n = ((l+1)/2), \qquad \left|F_n^-(a)\right| < c\left(\left(-g_l^-(b)\right)/(l+1)\right)^{1/2}, \end{aligned} \tag{1.19}$$

where $0 < c < 2\sqrt{2}$. For convenience, introduce

$$r = \frac{k+1}{l+1} = \frac{q}{p}, \tag{1.20}$$

where p, q are relatively prime numbers, that is, $(p, q) = 1$. Then it is easy to see that there must exist an integer $\eta \in [0, p-1]$ such that $(n+\eta)r$ is an integer, and our main results are the following.

Theorem 1.6. *Let* (1.15), (1.16), *(H_1), and (H_2) hold. Then*

(i) *$F^-(\alpha(x), a) - F^+(x, a)$ can be expressed as*

$$F^-(\alpha(x,b), a) - F^+(x, a) = \sum_{j\geq 1} B_j(a,b) x^{k_j}, \tag{1.21}$$

where $\alpha(x,b) = -[((l+1)g_k^+(b))/(-(k+1)g_l^-(b))]^{1/(l+1)} x^r(1+o(1))$ satisfies $G^-(\alpha(x,b), b) = G^+(x,b)$ for $x > 0$ small and $k_\tau \in \{(n+i)r + j \mid 0 \leq i \leq p-1, i \neq \eta, j \geq 0\} \cup \{j \mid j \geq \min\{m, (n+\eta)r\}\}$ for $\tau \geq 1$ with $\min\{nr, m\} = k_1 < k_2 < \cdots$.

(ii) *If there exist $(a_0, b_0) \in R^{n_1}$ and $s(\geq 1)$ such that $B_{s+1}(a_0, b_0) \neq 0$ and*

$$B_j(a_0,b_0) = 0, \quad j = 1,\dots,s, \qquad \operatorname{rank} \frac{\partial(B_1, B_2, \dots, B_s)}{\partial(a,b)}(a_0,b_0) = s, \tag{1.22}$$

where $n_1 \geq s$, then system (1.13) *has cyclicity s near the origin for all (a,b) near (a_0,b_0).*

Theorem 1.7. *Let* (1.15), (1.16), *(H_1), and (H_2) hold. If there exists $s \geq 1$ such that*

$$F^-(\alpha(x,b), a) \equiv F^+(x,a) \quad \text{as } B_1 = B_2 = \cdots = B_s = 0 \tag{1.23}$$

for $x \geq 0$ small and (1.22) *is satisfied, then system* (1.13) *has cyclicity $s-1$ at the origin for $|a - a_0| + |b - b_0|$ small.*

In many cases, the function F in (1.13) is linear in a. Then the coefficients $B_j(a,b)$ ($j \geq 1$) in (1.21) are linear in a. Let there exist $a_0 \in R^{n_2}, b_0 \in R^{n_3}$ ($n_2 + n_3 = n_1$) and integer $s > 0$ ($s \leq n_2$) such that

$$B_j(a_0,b_0) = 0, \quad j = 1,2,\dots,s, \qquad \det \frac{\partial(B_1, B_3, \dots, B_s)}{\partial(a_1, a_2, \dots, a_s)}(a_0,b_0) \neq 0 \tag{1.24}$$

which implies that the linear equations $B_j = 0$, $j = 1, 2, \ldots, s$ of a have a unique solution of the form

$$(a_1, \ldots, a_s) = \varphi(a_{s+1}, \ldots, a_{n_2}, b) \tag{1.25}$$

for b near b_0. Obviously, φ is linear in $a_{s+1}, \ldots, a_{n_2}$. Further, let

$$B_{s+j}\big|_{(a_1,\ldots,a_s)=\varphi(a_{s+1},\ldots,a_{n_2},b)} = L_j(a_{s+1}, \ldots, a_{n_2})\Delta_j(b), \quad j = 1, \ldots, \sigma \tag{1.26}$$

for some integer $\sigma > 0$. Then, we can obtain the following.

Theorem 1.8. *Consider system* (1.13)*, where the function F is linear in $a \in R^{n_2}$. Suppose* (1.15)*,* (1.16)*, (H_1), and (H_2) hold. Let there exist integers $s > 0, \sigma > 0$ and points $a_0 = (a_{10}, \ldots, a_{n_2,0}) \in R^{n_2}$ and $b_0 \in R^{n_3}$ such that* (1.24) *and* (1.26) *hold with*

$$L_j(a_{s+1,0}, \ldots, a_{n_2,0}) \neq 0, \quad j = 1, \ldots, \sigma, \qquad \Delta_j(b_0) = 0, \quad j = 1, \ldots, \sigma - 1. \tag{1.27}$$

(i) *If $\Delta_\sigma(b_0) \neq 0$ and*

$$\operatorname{rank} \frac{\partial(\Delta_1 \ldots, \Delta_{\sigma-1})}{\partial(b_1, \ldots, b_{\sigma-1})}(b_0) = \sigma - 1, \tag{1.28}$$

then system (1.13) *has cyclicity $s + \sigma - 1$ limit cycles near the origin for all (a, b) near (a_0, b_0).*

(ii) *If $\Delta_\sigma(b_0) = 0$ and $F^-(\alpha(x), a) \equiv F^+(x, a)$ as $(a_1, \ldots, a_s) = \varphi(a_{s+1}, \ldots, a_{n_2}, b)$, $\Delta_j(b) = 0$, $j = 1, \ldots, \sigma$ with*

$$\operatorname{rank} \frac{\partial(\Delta_1, \ldots, \Delta_\sigma)}{\partial(b_1, \ldots, b_\sigma)}(b_0) = \sigma, \tag{1.29}$$

then system (1.13) *has cyclicity $s+\sigma-1$ limit cycles at the origin for all (a, b) near (a_0, b_0).*

The conclusion (i) of Theorem 1.8 can be proved in a similar manner to the proof of Theorem 2 of [16] and (ii) can be verified by Theorem 1.7. Thus, we do not verify it here. The proof of Theorems 1.6 and 1.7 is presented in Section 2. In Section 3, we give some applications.

2. Proof of Theorems 1.6 and 1.7

In this section, we verify Theorems 1.6 and 1.7. Before proving them, we need to introduce a bifurcation function d of system (1.13) and establish some preliminary lemmas, which will be used in our proof. First, we have the following lemma.

Lemma 2.1. *Suppose* (1.18) *holds. Then the function* $\alpha(x,b)$ *defined in Theorem 1.6 has the form*

$$\alpha(x,b)=\sum_{i=1}^{p}\sum_{j\geq 0}\alpha_{ij}x^{ir+j} \tag{2.1}$$

for $0<x\ll 1$ *with* $\alpha_{10}=-[((l+1)g_k^+(b))/(-(k+1)g_l^-(b))]^{1/(l+1)}$.

Proof. Consider the equation

$$G^-(u,b)=G^+(x,b) \tag{2.2}$$

for $0<-u\ll 1$ and $0<x\ll 1$. By (1.18), we can obtain

$$\begin{aligned}
\left[G^-(u,b)\right]^{1/(l+1)}&=\left(\frac{-g_l^-(b)}{l+1}\right)^{1/(l+1)}(-u)\left[1+\sum_{j\geq l+1}\frac{(l+1)g_j^-(b)}{(-1)^{l+2}(j+1)g_l^-(b)}u^{j-l}\right]^{1/(l+1)},\\
\left[G^+(x,b)\right]^{1/(l+1)}&=\left(\frac{g_k^+(b)}{k+1}\right)^{1/(l+1)}x^{(k+1)/(l+1)}\left[1+\sum_{j\geq k+1}\frac{(k+1)g_j^+(b)}{(j+1)g_k^+(b)}x^{j-k}\right]^{1/(l+1)}\\
&=\left(\frac{g_k^+(b)}{k+1}\right)^{1/(l+1)}x^{(k+1)/(l+1)}\sum_{j\geq 0}\tilde{v}_jx^j=\sum_{j\geq 0}v_jx^{j+r}\equiv v(x),
\end{aligned} \tag{2.3}$$

where $v_0=((g_k^+(b))/(k+1))^{1/(l+1)}$. The implicit function theorem implies that the equation $[G^-(u,b)]^{1/(l+1)}=v$ has a unique solution

$$u=u^*(v)=\sum_{i\geq 1}u_iv^i \tag{2.4}$$

for $v>0$ small with $u_1=((l+1)/(-g_l^-(b)))^{1/(l+1)}$. Let $\alpha(x,b)=u^*(v(x))$. Then, $u=\alpha(x,b)$ for $x>0$ sufficiently small is the solution of (2.2). Combining (2.3) and (2.4), the above formula can be represented as

$$\begin{aligned}
\alpha(x,b)&=\sum_{i\geq 1}u_i\left(\sum_{j\geq 0}v_jx^{j+r}\right)^i=\sum_{i\geq 1}\sum_{j\geq 0}\tilde{\alpha}_{ij}x^{ir+j}\\
&=\sum_{i\geq 1}\sum_{j\geq 0}\tilde{\alpha}_{ij}x^{(qi/p)+j}=\sum_{i=1}^{p}\sum_{j\geq 0}\alpha_{ij}x^{ir+j}
\end{aligned} \tag{2.5}$$

with $\alpha_{10}=v_0u_1$. Thus, the proof is ended. □

Following the idea of Han [11], we have the following lemma.

Lemma 2.2. *Let* (1.15) *and* (1.16) *hold. Then system* (1.13) *is equivalent to*

$$\dot{u} = v - K(v)F^*(u,a,b),\qquad \dot{v} = -u, \tag{2.6}$$

where $K(v) = (1/\sqrt{2}) + O(v) \in C^\infty$, *and*

$$F^*(u,a,b) = \begin{cases} \sum_{j\ge m} f_j^+ u^{2j/(k+1)}, & 0 \le u \ll 1, \\ \sum_{j\ge n} f_j^- |u|^{2j/(l+1)}, & 0 < -u \ll 1 \end{cases} \tag{2.7}$$

with $f_m^+ = F_m^+((k+1)/(2g_k^+(b)))^{m/(k+1)}$ *and* $f_n^- = (-1)^n F_n^-((l+1)/(-2g_l^-(b)))^{n/(l+1)}$.

Proof. Let $H(y) = \int_0^y h(u)du$ and make the transformation

$$u = \sqrt{G(x,b)}(\operatorname{sgn} x),\qquad v = \sqrt{H(y)}(\operatorname{sgn} y), \tag{2.8}$$

together with the scaling of the time $d\tau = (g(X(u))h(Y(v))/2uv)dt$ to system (1.13) so that (1.13) can be changed into

$$\dot{u} = v - K(v)F^*(u,a,b),\qquad \dot{v} = -u, \tag{2.9}$$

where $K(v) = v/h(Y(v)) = (1/\sqrt{2}) + O(v) \in C^\infty$, $F^*(u,a,b) = F(X(u),a)$ and $X(u), Y(v)$ denote the inverse of the transformation (2.8). In fact, by (1.18) and similar to the proof of Lemma 2.1, we can obtain

$$X(u) = \begin{cases} X^+(u) \\ X^-(u) \end{cases} = \begin{cases} \sum_{j\ge 1} x_j^+ u^{2j/(k+1)},\ x_1^+ = \left(\dfrac{k+1}{2g_k^+(b)}\right)^{1/(k+1)}, & 0 \le u \ll 1, \\ \sum_{j\ge 1} x_j^- |u|^{2j/(l+1)},\ x_1^- = -\left(\dfrac{l+1}{-2g_l^-(b)}\right)^{1/(l+1)}, & 0 < -u \ll 1. \end{cases} \tag{2.10}$$

For $0 \le u \ll 1$, by (1.15) and (2.10), we have

$$F^*(u,a,b) = F^+(X^+(u),a) = \sum_{j\ge m} F_j^+(a)\left(\sum_{i\ge 1} x_i^+ u^{2i/(k+1)}\right)^j = \sum_{j\ge m} f_j^+ u^{2j/(k+1)}, \tag{2.11}$$

where $f_m^+ = F_m^+((k+1)/2g_k^+(b))^{m/(k+1)}$. Similarly, for $0 < -u \ll 1$, by (1.16) and (2.10), we can obtain

$$F^*(u,a,b) = F^-\left(X^-(u),a\right) = \sum_{j\ge n} f_j^- |u|^{2j/(l+1)}, \tag{2.12}$$

where $f_n^- = (-1)^n F_n^-((l+1)/-2g_l^-(b))^{n/(l+1)}$. This ends the proof. □

For a relation between the function $F(x,a)$ in (1.13) and the function $F^*(u,a,b)$ in (2.6) we have

Lemma 2.3. *Suppose* (1.15) *and* (1.16) *hold. Then we have*

(i) $F^-(\alpha(x,b),a) - F^+(x,a)$ *has the form* (1.21) *for* $0 \le x \ll 1$;

(ii) *let* $F_0(u,a,b) = F^*(-u,a,b) - F^*(u,a,b)$. *Then* F_0 *can be expressed as*

$$F_0(u,a,b) = \sum_{j\ge 1} A_j(a,b) u^{2k_j/(k+1)}, \quad \text{for } 0 \le u \ll 1, \tag{2.13}$$

where

$$A_1(a,b) = B_1(a,b)N_1\left(x_1^+\right), \qquad A_j(a,b) = B_j(a,b)N_j\left(x_1^+\right) + O\left(\left|B_1, B_2, \dots, B_{j-1}\right|\right), \quad j \ge 2 \tag{2.14}$$

with N_j $(j \ge 1)$ *are positive* C^∞ *functions in* x_1^+, *and* k_j, $j \ge 1$, *are as appeared in* (1.21).

Proof. By (1.15), (1.16), and Lemma 2.1, we can have

$$\begin{aligned} F^-(\alpha(x,b),a) - F^+(x,a) &= \sum_{s\ge n} F_s^-(a)\left[\sum_{i=1}^{p} x^{ir} \sum_{j\ge 0} \alpha_{ij} x^j\right]^s - \sum_{s\ge m} F_s^+(a) x^s \\ &= \sum_{s\ge n} F_s^-(a) x^{(q/p)s}\left[\sum_{i=0}^{p-1} x^{qi/p} \sum_{j\ge 0} \alpha_{i+1,j} x^j\right]^s - \sum_{s\ge m} F_s^+(a) x^s \\ &= \sum_{i=0}^{p-1} x^{(n+i)r} \sum_{j\ge 0} \overline{B}_{ij}(a,b) x^j - \sum_{s\ge m} F_s^+(a) x^s. \end{aligned} \tag{2.15}$$

Since $r(n+\eta)$ is an integer, the above formula can be written as

$$F^{-}(\alpha(x,b),a) - F^{+}(x,a) = \sum_{i=0,i\neq\eta}^{p-1}\sum_{j\geq 0}\overline{B}_{ij}(a,b)x^{(n+i)r+j} + \sum_{j\geq T}\widetilde{B}_{j}(a,b)x^{j}, \tag{2.16}$$

where $T = \min\{m, r(n+\eta)\}$. Then we express (2.16) with respect to the power of x in ascending order, yielding the form of (1.21). In addition, for k_τ defined in Theorem 1.6 there must exist some integers i_τ and j_τ such that $k_\tau = (n+i_\tau)r + j_\tau$. Hence, we can obtain

$$B_\tau(a,b) = \overline{B}_{i_\tau j_\tau} \quad \text{if } i_\tau \neq \eta, \qquad B_\tau(a,b) = \widetilde{B}_{k_\tau} \quad \text{if } i_\tau = \eta. \tag{2.17}$$

This completes the proof of the conclusion (i).

Now we prove the conclusion (ii). For $u \geq 0$ sufficiently small, (2.8) and (2.10) imply that

$$G^{+}(X^{+}(u),b) = G^{-}\left(X^{-}(-u),b\right) = u^{2}. \tag{2.18}$$

Then noting that $G^{-}(\alpha(x,b),b) = G^{+}(x,b)$, we can get

$$X^{-}(-u) = \alpha(X^{+}(u)). \tag{2.19}$$

By (2.11), (2.12) and the above formula, we have for $u \geq 0$ sufficiently small

$$\begin{aligned} F_0(u,a,b) = F^{*}(-u,a,b) - F^{*}(u,a,b) &= \left[F^{-}\left(X^{-}(-u),a\right) - F^{+}(X^{+}(u),a)\right] \\ &= \left[F^{-}(\alpha(x,b),a) - F^{+}(x,a)\right]\big|_{x=X^{+}(u)}. \end{aligned} \tag{2.20}$$

Let

$$\varphi(\mu) = X^{+}\left(\mu^{(k+1)/2}\right). \tag{2.21}$$

Combining (2.10) and (2.21) gives that

$$\varphi(\mu) = \sum_{j\geq 1} x_j^{+}\mu^{j} = x_1^{+}\mu(1+O(\mu)). \tag{2.22}$$

Substituting the above formula into (1.21) or (2.16) yields that

$$
\begin{aligned}
F^-(\alpha(x,b),a) - F^+(x,a)\big|_{x=\varphi(\mu)} &= \sum_{j\geq 1} B_j(a,b)\left(x_1^+\right)^{k_j}\mu^{k_j}(1+O(\mu)) \\
&= \sum_{i=0,i\neq\eta}^{p-1}\sum_{j\geq 0}\overline{B}_{ij}(a,b)\left(\sum_{s\geq 1}x_s^+\mu^s\right)^{(n+i)r+j} \\
&\quad + \sum_{j\geq T}\widetilde{B}_j(a,b)\left(\sum_{s\geq 1}x_s^+\mu^s\right)^j.
\end{aligned}
\tag{2.23}
$$

Note that

$$
\begin{aligned}
&\sum_{i=0,i\neq\eta}^{p-1}\sum_{j\geq 0}\overline{B}_{ij}(a,b)\left(\sum_{s\geq 1}x_s^+\mu^s\right)^{(n+i)r+j} \\
&\quad = \sum_{i=0,i\neq\eta}^{p-1}\sum_{j\geq 0}\overline{B}_{ij}(a,b)\left(x_1^+\right)^{(n+i)r+j}\mu^{(n+i)r+j}\left(1+\sum_{s\geq 1}\frac{x_{s+1}^+}{x_1^+}u^s\right)^{(n+i)r+j} \\
&\quad = \sum_{i=0,i\neq\eta}^{p-1}\sum_{j\geq 0}\overline{B}_{ij}(a,b)\left(x_1^+\right)^{(n+i)r+j}\mu^{(n+i)r+j}\left(1+\sum_{s\geq 1}\overline{x}_s\mu^s\right) \\
&\quad = \sum_{i=0,i\neq\eta}^{p-1}\sum_{j\geq 0}B_{ij}^*(a,b)\mu^{(n+i)r+j},
\end{aligned}
\tag{2.24}
$$

where

$$
B_{i0}^* = \left(x_1^+\right)^{(n+i)r}\overline{B}_{i0}, \qquad B_{ij}^* = \left(x_1^+\right)^{(n+i)r+j}\overline{B}_{ij} + O\left(\left|\overline{B}_{i0},\overline{B}_{i1},\ldots,\overline{B}_{i,j-1}\right|\right) \quad ,j\geq 1 \tag{2.25}
$$

with $i = 0,\ldots,p-1$ and $i\neq\eta$. Similarly,

$$
\sum_{j\geq T}\widetilde{B}_j(a,b)\left(\sum_{s\geq 1}x_s^+\mu^s\right)^j = \sum_{j\geq T}B_j^*(a,b)\mu^j, \tag{2.26}
$$

where

$$
B_T^* = \left(x_1^+\right)^T\overline{B}_T, \qquad B_j^* = \left(x_1^+\right)^j\overline{B}_j + O\left(\left|\overline{B}_T,\overline{B}_{T+1},\ldots,\overline{B}_{j-1}\right|\right), \quad j\geq T+1. \tag{2.27}
$$

Combining (2.23), (2.24), and (2.26), we can obtain

$$
F^-(\alpha(x,b),a) - F^+(x,a)\big|_{x=\varphi(\mu)} = \sum_{i=0,i\neq\eta}^{p-1}\sum_{j\geq 0}B_{ij}^*(a,b)\mu^{(n+i)r+j} + \sum_{j\geq T}B_j^*(a,b)\mu^j. \tag{2.28}
$$

Similar to (1.21) and (2.16), the above formula can be rewritten as

$$F^{-}(\alpha(x,b),a) - F^{+}(x,a)|_{x=\varphi(\mu)} = \sum_{j\geq 1} A_j(a,b)\mu^{k_j}, \tag{2.29}$$

where from (2.17)

$$A_\tau(a,b) = B^*_{i_\tau j_\tau} \quad \text{if } i_\tau \neq \eta, \qquad A_\tau(a,b) = B^*_{k_\tau} \quad \text{if } i_\tau = \eta. \tag{2.30}$$

From (2.25), (2.27), and the above equation, A_j in (2.29) can be rewritten as the form of (2.14). Further, combining (2.17), (2.20), and (2.29) yields (2.13). Thus, the proof is ended. □

Next, we establish the bifurcation function d of system (1.13). For the purpose, let $u = r\cos\theta, v = r\sin\theta$. Then (2.6) can be translated into

$$\frac{dr}{d\theta} = \frac{\cos\theta K(r\sin\theta)F^*(r\cos\theta,a,b)}{1 - \sin\theta K(r\sin\theta)F^*(r\cos\theta,a,b)/r} \equiv R^*(\theta,r,a,b). \tag{2.31}$$

In a similar way to the proof of Lemma 2.3 of [13], it is easy to prove that

$$1 - \frac{\sin\theta K(r\sin\theta)F^*(r\cos\theta,a,b)}{r} > 0 \tag{2.32}$$

for $r > 0$ small under (H_1) and (H_2), which shows that $R^*(\theta,r,a,b)$ in (2.31) is well defined. Introduce

$$\beta = (k+1)(l+1), \qquad \rho = r^{1/\beta}. \tag{2.33}$$

Then by (2.31) and (2.33), we can obtain

$$\begin{aligned}\frac{d\rho}{d\theta} &= \frac{\rho^{1-\beta}}{\beta} \times \frac{\cos\theta K(\rho^\beta\sin\theta)F^*(\rho^\beta\cos\theta,a,b)}{1 - \sin\theta K(\rho^\beta\sin\theta)F^*(\rho^\beta\cos\theta,a,b)/\rho^\beta} \\ &= \frac{\rho}{\beta} \times \frac{\cos\theta K(\rho^\beta\sin\theta)F_1(\theta,\rho,a,b)}{1 - \sin\theta K(\rho^\beta\sin\theta)F_1(\theta,\rho,a,b)} \equiv R(\theta,\rho,a,b),\end{aligned} \tag{2.34}$$

where $F_1(\theta,\rho,a,b) = \rho^{-\beta}F^*(\rho^\beta\cos\theta,a,b)$. Clearly, (2.34) is a 2π-periodic equation in θ. By (2.7), we have

$$F_1(\theta,\rho,a,b) = \begin{cases} \sum_{j\geq m} f_j^{+}(\cos\theta)^{2j/(k+1)}\rho^{(2j/(k+1))\beta-\beta}, & \cos\theta \geq 0, \\ \sum_{j\geq n} f_j^{-}|\cos\theta|^{2j/(l+1)}\rho^{(2j/(l+1))\beta-\beta}, & \cos\theta < 0. \end{cases} \tag{2.35}$$

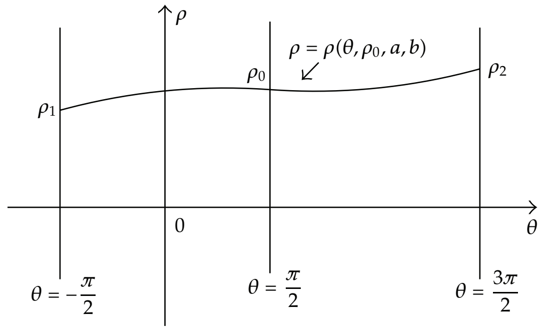

Figure 1

From (2.33) and (2.35), it is easy to see that F_1 is C^∞ in ρ for any given θ. Let $\rho(\theta, \rho_0, a, b)$ denote the solution of (2.34) satisfying $\rho(\pi/2, \rho_0, a, b) = \rho_0$. For convenience, let

$$\rho\left(-\frac{\pi}{2}, \rho_0, a, b\right) = \rho_1, \qquad \rho\left(\frac{3\pi}{2}, \rho_0, a, b\right) = \rho_2. \tag{2.36}$$

Define a function as follows:

$$d(\rho_0, a, b) = \rho\left(-\frac{\pi}{2}, \rho_0, a, b\right) - \rho\left(\frac{3\pi}{2}, \rho_0, a, b\right) = \rho_1 - \rho_2. \tag{2.37}$$

It is obvious that system (1.13), (2.6), (2.31), or (2.34) has a periodic orbit near the origin if and only if $d(\rho_0, a, b)$ in (2.37) has a zero in ρ_0 for $\rho_0 > 0$ sufficiently small. See Figure 1.

Hence, the function d in (2.37) is called a displacement function or bifurcation function of system (1.13), (2.6), (2.31), or (2.34). About the bifurcation function d, we have.

Lemma 2.4. *Let* (1.15), (1.16), *(H_1), and (H_2) hold. Then the bifurcation function d in* (2.37) *has the form*

$$d(\rho_0, a, b) = \sum_{j\geq 1} C_j(a, b)\rho_0^{r_j}\left(1 + d_j(\rho_0)\right), \tag{2.38}$$

where $r_j = 2(l+1)k_j - kl - k - l$ are positive integers satisfying $1 \leq r_1 < r_2 < \cdots$,

$$\begin{gathered} C_1(a, b) = B_1(a, b)N_1^*\left(f_n^-, f_m^+\right), \\ C_j(a, b) = B_j(a, b)N_j^*\left(f_n^-, f_m^+\right) + O\left(\left|B_1, B_2, \ldots, B_{j-1}\right|\right), \quad j \geq 2 \end{gathered} \tag{2.39}$$

with $N_j^* \in C^\infty$, $N_j(0,0) > 0$ for $j \geq 1$ and d_j $(j \geq 1)$ are C^∞ functions in ρ_0 satisfying $d_i(0) = 0$.

Proof. Denote $\overline{\rho}$ by $\overline{\rho}(\theta, \rho_0, a, b) = \rho(\pi - \theta, \rho_0, a, b)$. Then

$$\overline{\rho}\left(\frac{\pi}{2}, \rho_0, a, b\right) = \rho\left(\frac{\pi}{2}, \rho_0, a, b\right), \qquad \overline{\rho}\left(-\frac{\pi}{2}, \rho_0, a, b\right) = \rho\left(\frac{3\pi}{2}, \rho_0, a, b\right) = \rho_2 \tag{2.40}$$

and it satisfies

$$\frac{d\overline{\rho}}{d\theta} = -R(\pi - \theta, \rho, a, b). \tag{2.41}$$

Let $\overline{R}(\theta, \rho, a, b) = -R(\pi - \theta, \rho, a, b)$. Then we can obtain that $\overline{\rho}(\theta, \rho_0, a, b)$ is a solution of the equation

$$\frac{d\rho}{d\theta} = \overline{R}(\theta, \rho, a, b) \tag{2.42}$$

satisfying $\overline{\rho}(\pi/2, \rho_0, a, b) = \rho_0$. Let $\theta \in [-\pi/2, \pi/2]$. Then $\pi - \theta \in [\pi/2, 3\pi/2]$ and

$$\begin{aligned} R(\theta, \rho, a, b) &= R_1(\theta, a, b)\rho^{((2m/(k+1))-1)\beta+1}(1 + O(\rho)), \\ \overline{R}(\theta, \rho, a, b) &= \overline{R}_1(\theta, a, b)\rho^{((2n/(l+1))-1)\beta+1}(1 + O(\rho)), \end{aligned} \tag{2.43}$$

where

$$\begin{aligned} R_1(\theta, a, b) &= \begin{cases} \dfrac{K(0) f_m^+}{\beta}(\cos\theta)^{(2m/(k+1))+1}, & m > \dfrac{k+1}{2}, \\ \dfrac{K(0) f_m^+ \cos^2\theta}{\beta(1 - \sin\theta\cos\theta K(0) f_m^+)}, & m = \dfrac{k+1}{2}, \end{cases} \\ \overline{R}_1(\theta, a, b) &= \begin{cases} -\dfrac{K(0) f_n^-}{\beta}(\cos\theta)^{(2m/(l+1))+1}, & n > \dfrac{l+1}{2}, \\ -\dfrac{K(0) f_n^- \cos^2\theta}{\beta(1 - \sin\theta\cos\theta K(0) f_n^-)}, & n = \dfrac{l+1}{2}. \end{cases} \end{aligned} \tag{2.44}$$

Thus, by (2.43), (2.44) we have easily

$$\rho(\theta, \rho_0, a, b) = \begin{cases} \rho_0 + O\left(\rho_0^{((2m/(k+1))-1)\beta+1}\right), & m > \dfrac{k+1}{2}, \\ \rho_0 \exp\int_{\pi/2}^{\theta} R_1(u, a, b)du + O(\rho_0^2), & m = \dfrac{k+1}{2}, \end{cases} \tag{2.45}$$

$$\overline{R}(\theta, \rho(\theta, \rho_0, a, b), a, b) - \overline{R}(\theta, \overline{\rho}(\theta, \rho_0, a, b), a, b) = [\rho(\theta, \rho_0, a, b) - \overline{\rho}(\theta, \rho_0, a, b)]\Delta(\theta, \rho_0) \tag{2.46}$$

by the mean value theorem, where $\Delta(\theta, \rho_0) = O(\rho_0^{((2n/(l+1))-1)\beta})$. Further by (2.34) and (2.42), we can obtain

$$\begin{aligned} &R(\theta, \rho, a, b) - \overline{R}(\theta, \rho, a, b) \\ &\quad = \frac{-\cos\theta K(\rho^\beta \sin\theta)\rho^{1-\beta}[F^*(-\rho^\beta\cos\theta, a, b) - F^*(\rho^\beta\cos\theta, a, b)]}{\beta[1 - \sin\theta K(\rho^\beta\sin\theta)\rho^{-\beta}F^*(\rho^\beta\cos\theta, a, b)][1 - \sin\theta K(\rho^\beta\sin\theta)\rho^{-\beta}F^*(-\rho^\beta\cos\theta, a, b)]} \end{aligned} \tag{2.47}$$

which follows that by Lemma 2.3

$$R(\theta,\rho,a,b)-\overline{R}(\theta,\rho,a,b)=-\cos\theta\rho^{1-\beta}F_0\left(\rho^{\beta}\cos\theta,a,b\right)S(\theta,\rho), \tag{2.48}$$

where $S(\theta,\rho)$ is a C^∞ function in ρ with

$$S(\theta,0)=\begin{cases}\dfrac{K(0)}{\beta}, & m>\dfrac{k+1}{2},\ n>\dfrac{l+1}{2},\\ \dfrac{K(0)}{\beta(1-\sin\theta\cos\theta K(0)f_m^+)}, & m=\dfrac{k+1}{2},\ n>\dfrac{l+1}{2},\\ \dfrac{K(0)}{\beta(1-\sin\theta\cos\theta K(0)f_n^-)}, & m>\dfrac{k+1}{2},\ n=\dfrac{l+1}{2},\\ \dfrac{K(0)}{\beta(1-\sin\theta\cos\theta K(0)f_m^+)(1-\sin\theta\cos\theta K(0)f_n^-)}, & m=\dfrac{k+1}{2},\ n=\dfrac{l+1}{2},\end{cases} \tag{2.49}$$

which implies $S(\theta,0)>0$ by the above discussion. Then it follows from (2.13), (2.45), and (2.48) that

$$\begin{aligned}&R(\theta,\rho(\theta,\rho_0,a,b),a,b)-\overline{R}(\theta,\rho(\theta,\rho_0,a,b),a,b)\\&\quad=-\sum_{j\ge1}A_j(a,b)(\cos\theta)^{((2k_j/(k+1))+1)}\rho_0^{(2k_j/(k+1))\beta+1-\beta}q_j(\theta,\rho_0),\end{aligned} \tag{2.50}$$

where q_j $(j\ge1)$ are C^∞ functions in ρ_0 with

$$q_j(\theta,0)=\begin{cases}S(\theta,0), & m>\dfrac{k+1}{2},\\ S(\theta,0)\exp\left(\dfrac{2k_j}{k+1}\beta\int_{\pi/2}^{\theta}R_1(u,a,b)du\right), & m=\dfrac{k+1}{2},\end{cases} \tag{2.51}$$

which shows that $q_j(\theta,0)>0$ for all $j\ge1$. Since $\overline{\rho}(\theta,\rho,a,b)$ and $\rho(\theta,\rho,a,b)$ satisfy (2.42) and (2.34), respectively, using (2.46) we have

$$\frac{d}{d\theta}(\rho-\overline{\rho})=(\rho-\overline{\rho})\Delta(\theta,\rho_0)+R(\theta,\rho,a,b)-\overline{R}(\theta,\rho,a,b), \tag{2.52}$$

where $\overline{\rho}=\overline{\rho}(\theta,\rho_0,a,b)$ and $\rho=\rho(\theta,\rho_0,a,b)$. Then noting (2.40), applying the formula of variation of constants to the above equation yields

$$\begin{aligned}\rho(\theta,\rho_0,a,b)-\overline{\rho}(\theta,\rho_0,a,b)=&\int_{\pi/2}^{\theta}\Big(R(\delta,\rho(\delta,\rho,a,b),a,b)-\overline{R}(\delta,\rho(\delta,\rho_0,a,b),a,b)\Big)\\&\times\exp\left(\int_{\delta}^{\theta}\Delta(\tau,\rho_0)d\tau\right)d\delta.\end{aligned} \tag{2.53}$$

Inserting (2.50) into the above formula and taking $\theta = -\pi/2$ give by (2.37) that

$$d(\rho_0, a, b) = \sum_{j\geq 1} A_j(a,b)\rho_0^{(2k_j/(k+1))\beta+1-\beta} \int_{-\pi/2}^{\pi/2} (\cos\theta)^{(2k_j/(k+1))+1} q_j\ (\theta, 0) \times \exp\left(\int_\delta^\theta \Delta(\tau, 0)d\tau\right)(1+O(\rho_0))\ d\theta, \tag{2.54}$$

which implies (2.38) by (2.33). Hence, The proof is completed. □

In the following part, we verify Theorems 1.6 and 1.7 using the above lemmas.

Proof of Theorem 1.6. One can see that the conclusion (i) is true by Lemmas 2.1 and 2.3. Now, we prove the conclusion (ii). Obviously, it suffices to prove the following two points.

(1) There are at most s limit cycles near the origin for all (a, b) near (a_0, b_0).

(2) There can appear s limit cycles in any given neighborhood of the origin for some (a, b) arbitrarily close to (a_0, b_0).

In fact, noting that

$$C_{s+1}(a_0, b_0) = B_{s+1}(a_0, b_0)N^*_{s+1} \neq 0. \tag{2.55}$$

By our assumption, we have from (2.38) and (2.55)

$$\begin{aligned} d(\rho_0, a, b) &= \sum_{j=1}^{s+1} C_j(a,b)\rho_0^{r_j}(1 + d_j(\rho_0)) + \rho_0^{r_{s+2}} Q(\rho_0, a, b) \\ &= \sum_{j=1}^{s+1} C_j(a,b)\rho_0^{r_j}\left(1 + \overline{d}_j(\rho_0)\right) \equiv D_s, \end{aligned} \tag{2.56}$$

where $Q(\rho_0, a, b) = \sum_{j\geq s+2} C_j(a,b)\rho_0^{r_j - r_{s+1}}$,

$$\overline{d}_j = d_j, \quad j = 1, \ldots, s, \qquad \overline{d}_{s+1} = d_{s+1} + \frac{Q(\rho_0, a, b)}{C_{s+1}(a,b)}\rho_0^{r_{s+2}-r_{s+1}}, \tag{2.57}$$

which implies $\overline{d}_j$ in (2.56) are C^∞ in ρ_0 satisfying $\overline{d}_j(0) = 0$ for (a, b) sufficiently close to (a_0, b_0).

We claim that D_s in (2.56) has at most s zeros in $\rho_0 > 0$ small for $|a - a_0| + |b - b_0| \ll 1$. We can proceed with the proof by induction on s.

First, when $s = 1$, from (2.56), we can obtain

$$\begin{aligned} D_1 &= \left(1 + \overline{d}_1(\rho_0)\right)\rho_0^{r_1}\left[C_1(a,b) + C_2(a,b)\rho_0^{r_2-r_1}\left(1 + \tilde{d}_2(\rho_0)\right)\right] \\ &= \left(1 + \overline{d}_1(\rho_0)\right)\rho_0^{r_1} d_1(\rho_0, a, b). \end{aligned} \tag{2.58}$$

Note that

$$\frac{dd_1(\rho_0,a,b)}{d\rho_0} = C_2(a,b)\rho_0^{r_2-r_1-1}\left[(r_2-r_1)\left(1+\tilde{d}_2(\rho_0)\right)+\rho_0\tilde{d}_2'(\rho_0)\right], \tag{2.59}$$

which implies

$$\frac{dd_1(\rho_0,a,b)}{d\rho_0} \neq 0 \tag{2.60}$$

for $0<\rho_0\ll 1$ and (a,b) near (a_0,b_0) since $C_2(a_0,b_0)=B_2(a_0,b_0)N_2^*\neq 0$ by (2.55). Hence, by Rolle's theorem, the conclusion (1) is true for $s=1$.

Assume that the conclusion (1) is true for $s=i$. That is by (2.56)

$$D_i = \sum_{j=1}^{i+1} C_j(a,b)\rho_0^{r_j}\left(1+\overline{d}_j(\rho_0)\right) \tag{2.61}$$

has at most i zeros for $\rho_0>0$ and $|a-a_0|+|b-b_0|$ sufficiently small with $C_{i+1}(a_0,b_0)\neq 0$. Let us prove that the conclusion (1) is also true for $s=i+1$. Then from (2.56), we can obtain

$$\begin{aligned} D_{i+1} &= \sum_{j=1}^{i+2} C_j(a,b)\rho_0^{r_j}\left(1+\overline{d}_j(\rho_0)\right) \\ &= \rho_0^{r_1}\left(1+\overline{d}_1(\rho_0)\right)\sum_{j=1}^{i+2} C_j(a,b)\rho_0^{r_j-r_1}d_j^* \equiv \rho_0^{r_1}\left(1+\overline{d}_1(\rho_0)\right)\overline{D}_{i+1}, \end{aligned} \tag{2.62}$$

where $d_1^*=1$, $d_j^*=(1+\overline{d}_j(\rho_0))/(1+\overline{d}_1(\rho_0))=1+O(\rho_0)$, $j=2,3,\ldots,i+2$. Then,

$$\begin{aligned} \frac{d\overline{D}_{i+1}}{d\rho_0} &= \sum_{j=2}^{i+2}(r_j-r_1)C_j(a,b)\rho_0^{r_j-r_1-1}\left(d_j^*+\frac{\rho_0}{r_j-r_1}\frac{dd_j^*}{d\rho_0}\right) \\ &= \sum_{j=1}^{i+1}\overline{C}_j(a,b)\rho_0^{\tilde{r}_j}\left(1+\hat{d}_j(\rho_0)\right), \end{aligned} \tag{2.63}$$

where $\overline{C}_j(a,b)=(r_{j+1}-r_1)C_{j+1}(a,b)$, $\tilde{r}_j=r_{j+1}-r_1-1$, $j=1,\ldots,i+1$ satisfying $1\leq\tilde{r}_1<\cdots<\tilde{r}_{i+1}$ and $\hat{d}_j(\rho_0)$ is C^∞ in ρ_0 with $\hat{d}_j(0)=0$, which implies that $d\overline{D}_{i+1}/d\rho_0$ has at most i zeros in $\rho_0>0$ small for (a,b) near (a_0,b_0) by induction assumption since $\overline{C}_{i+1}(a_0,b_0)=(r_{i+2}-r_1)C_{i+2}(a_0,b_0)\neq 0$. Hence, by Rolle's theorem, it follows that D_{i+1} in (2.62) has at most $i+1$ zeros in $0<\rho_0\ll 1$ for (a,b) near (a_0,b_0). This ends the proof of conclusion (1).

Now, we verify the conclusion (2). For simplicity, we can assume $(a,b)=\overline{a}=(\overline{a}_1,\overline{a}_2,\ldots,\overline{a}_{n_1})$. Further, by (1.22), without loss of generality, suppose

$$\det\frac{\partial(B_1,\ldots,B_s)}{\partial(\overline{a}_1,\ldots,\overline{a}_s)}\neq 0. \tag{2.64}$$

Then we can fix $\overline{a}_j = \overline{a}_{j0}$ for $j = s+1, \ldots, n_1$ since $n_1 \geq s$. Consider the equations

$$B_j = B_j(\overline{a}), \quad j = 1, \ldots, s. \tag{2.65}$$

By the implicit function theorem, the equations have a unique set of solutions

$$\overline{a}_j = \overline{a}_j^*(B_1, \ldots, B_s, \overline{a}_{s+1}, \ldots, \overline{a}_{n_1}), \quad j = 1, 2, \ldots, s \tag{2.66}$$

for $\overline{a}$ near $\overline{a}_0$, which means $B_j (j = 1, \ldots, s)$ can be taken as free parameters. Hence, by changing the sign of $B_s, B_{s-1}, \ldots, B_1$ in turn such that

$$B_{j-1}B_j < 0, \quad j = s+1, \ldots, 1, 0 < |B_1| \ll |B_2| \ll \cdots = |B_s| \ll 1, \tag{2.67}$$

which follows that C_j in (2.56) satisfy

$$C_{j-1}C_j < 0, \quad j = s+1, \ldots, 1, 0 < |C_1| \ll |C_2| \ll \cdots \ll |C_s| \ll 1. \tag{2.68}$$

This ensures that the bifurcation function d has s zeros in ρ_0 for some (a, b) near (a_0, b_0), which implies the conclusion (2). This ends the proof. □

Proof of Theorem 1.7. From the proof of Theorem 1.6, it is easy to see $B_j (j = 1, \ldots, s)$ can be taken as free parameters by (1.22), which implies $C_j (j = 1, \ldots, s)$ can also be taken as free parameters by the definition of C_j in Lemma 2.4. From (1.23) and Lemma 2.4, we have

$$d(\rho_0, a, b) = 0 \quad \text{when } C_j = 0, \; j = 1, \ldots, s \tag{2.69}$$

for $|a - a_0| + |b - b_0|$ sufficiently small since $B_j = 0$ if and only if $C_j = 0$, which means that the coefficients $C_j (j \geq s+1)$ in (2.38) satisfy

$$C_j(a, b) = 0, \quad j = s+1, s+2, \ldots \quad \text{as long as} \quad C_1 = \cdots = C_s = 0. \tag{2.70}$$

Since C_j can be taken as free parameters, we can write

$$C_j(a, b) = \sum_{i=1}^{s} C_i Q_{ij}(a, b), \tag{2.71}$$

where $Q_{ij} \in C^\infty, j \geq s+1$. Inserting (2.71) into (2.38) gives that

$$d(\rho_0, a, b) = \sum_{j=1}^{s} C_j \rho_0^{r_j} (1 + e_j(\rho_0)), \tag{2.72}$$

where

$$e_j(\rho_0) = 1 + d_j(\rho_0) + \sum_{i \geq s+1} Q_{ji} \rho_0^{r_i - r_j} (1 + d_i(\rho_0)) \quad \text{for } j = 1, \ldots, s. \tag{2.73}$$

We can easily prove d in (2.72) has at most $s-1$ zeros for (a,b) near (a_0,b_0) by induction s in a similar way to the above proof. Further, we can choose C_j satisfying

$$C_{j-1}C_j < 0, \quad j = s,\ldots,1, 0 < |C_1| \ll |C_2| \ll \cdots \ll |C_s| \ll 1, \tag{2.74}$$

which ensures that d in (2.72) has $s-1$ zeros for (a,b) near (a_0,b_0). The proof is finished. □

From the proof of Theorems 1.6 and 1.7, we have immediately the following corollaries.

Corollary 2.5. *Let* (1.15), (1.16), *(H_1), and (H_2) hold. If there exist a point* (a_0,b_0) *and an integer* $s \geq 1$*such that*

$$B_j(a_0,b_0) = 0, \quad j = 1,2,\ldots,s, \quad B_{s+1}(a_0,b_0) < 0 \ (\textit{resp.}, > 0), \tag{2.75}$$

then the origin is a stable (resp., unstable) focus of system (1.13) *for* $(a,b) = (a_0,b_0)$ *and there has at most s limit cycles at the origin for all* (a,b) *near* (a_0,b_0).

Corollary 2.6. *Let* (1.15), (1.16), *(H_1), and (H_2) hold. If there exists* $s \geq 1$ *such that*

$$B_j = O(|B_1, B_2, \ldots, B_s|) \tag{2.76}$$

for $j \geq s+1$, *then for any given* $N > 0$ *there exists a neighborhood* U *of the origin such that for all* $|B_j| < N$, $j = 1,\ldots,s$ *system* (1.13) *has at most* $s-1$ *limits cycles in* U.

Further, about system (1.13), we have the below two remarks in order.

Remark 2.7. If the function $h(y)$ in (1.13) has the form $h(y) = h_1 y + O(y^2)$ with $h_1 > 0$, we also can obtain the same conclusions since this system is equivalent to the following:

$$\begin{aligned} \dot{x} &= \frac{h(y)}{h_1} - \frac{F(x)}{h_1}, \\ \dot{y} &= -\frac{g(x)}{h_1}. \end{aligned} \tag{2.77}$$

Remark 2.8. When vector parameter a or b is constant in (1.13), Theorems 1.6 and 1.7 remain true in this case.

3. Applications

In this section, we present some applications of our main results to systems of the form

$$\begin{aligned} \dot{x} &= y - F(x,a), \\ \dot{y} &= -g(x,b), \end{aligned} \tag{3.1}$$

where F, g satisfy (1.15) and (1.16).

Example 3.1. First consider system (3.1) with

$$g(x,b)=\begin{cases}\sum_{j=0}^{k} b_j x^j, & x\ge 0,\\ -1, & x<0,\end{cases}\qquad F(x,a)=\begin{cases}\sum_{j=1}^{n} a_j^+ x^j, & x\ge 0,\\ \sum_{j=1}^{m} a_j^- x^j, & x<0,\end{cases}\tag{3.2}$$

where $b_0>0$ and $a=(a_1^+,\dots,a_n^+,a_1^-,\dots,a_m^-)$, $b=(b_0,b_1,\dots,b_k)$. By Lemma 2.1, it is clear that

$$\alpha(x,b)=-\sum_{j=0}^{k}\frac{b_j}{j+1}x^{j+1},\tag{3.3}$$

for $x\ge 0$ small. For any given $N>0$, denote $L(k,n,m)$ the cyclicity of system (3.1) near the origin for $|a|+|b|\le N$. We study the cyclicity of system (3.1) under (3.2) near the origin with the following three cases.

Case 1. $m=1$, $k,m,n\ge 1$. In this case, we claim that $L(k,n,1)=\max\{k+1,n\}-1$. In fact, by (3.2) and (3.3), we have

$$\begin{aligned}F^-(\alpha(x,b),a)-F^+(x,a)&=-\sum_{j=0}^{k}\frac{a_1^-b_j}{j+1}x^{j+1}-\sum_{j=1}^{n}a_j^+x^j\\&=\sum_{j=1}^{\max\{k+1,n\}}B_j(a,b)x^j\end{aligned}\tag{3.4}$$

for $0\le x\ll 1$. From (3.4) it is easy to see that $F^-(\alpha(x,b),a)\equiv F^+(x,a)$ for $0\le x\ll 1$ if and only if $B_j=0$ for $j=1,\dots,\max\{k+1,n\}$, which shows that $L(k,n,1)\le\max\{n,k+1\}-1$ by Corollary 2.6. Thus we only need to prove that $L(k,n,1)\ge\max\{n,k+1\}-1$. For $n<k+1$, by (3.4), we can obtain

$$B_j(a,b)=\frac{a_1^-b_{j-1}}{j}-a_j^+,\quad j=1,\dots,n,\qquad B_j(a,b)=\frac{a_1^-b_{j-1}}{j},\quad j=n+1,\dots,k+1.\tag{3.5}$$

Take $(a,b)=(a_0,\tilde{b}_0)$ such that $a_{j0}^+=(a_{10}^-b_{j-1,0})/j$, $j=1,\dots,n$, $b_{j0}=0$, $j=n,\dots,k$ and $a_{10}^-\ne 0$. Then by (3.5), we have

$$B_j(a_0,b_0)=0,\quad j=1,\dots,k+1,\qquad \operatorname{rank}\frac{\partial(B_1,\dots,B_n,B_{n+1},\dots,B_{k+1})}{\partial(a_1^+,\dots,a_n^+,b_n,\dots,b_k)}\Big|_{(a_0,\tilde{b}_0)}=k+1\tag{3.6}$$

which implies k limit cycles can appear near the origin for (a,b) near $(a_0,\tilde{b}_0)$. For $n\ge k+1$, we can discuss similarly.

Case 2. $k = 2$, $m = 2$, $n \geq 1$. We will see that $L(2,2,1) = 3$, $L(2,2,2) = L(2,2,3) = 4$, $L(2,2,4) = L(2,2,5) = 5$, and $L(2,2,n) = n-1$ for $n \geq 6$.

In fact, combining (3.2) and (3.3) gives that

$$\begin{aligned} &F^-(\alpha(x,b),a) - F^+(x,a) \\ &\quad= a_1^-\left(-b_0x - \frac{b_1}{2}x^2 - \frac{b_2}{3}x^3\right) + a_2^-\left(-b_0x - \frac{b_1}{2}x^2 - \frac{b_2}{3}x^3\right)^2 - \sum_{j=1}^{n} a_j^+ x^j \\ &\quad= \sum_{j=1}^{\max\{6,n\}} B_j(a,b)x^j. \end{aligned} \tag{3.7}$$

As before, it is easy to get that $L(2,2,n) = n-1$ for $n \geq 6$. For $n \leq 5$, we can discuss system (3.1) case by case with respect to n.

For $n = 1$, by (3.7) we can obtain

$$\begin{gathered} B_1(a,b) = -a_1^- b_0 - a_1^+, \qquad B_2(a,b) = -\frac{1}{2}a_1^- b_1 + a_2^- b_0^2, \qquad B_3(a,b) = -\frac{1}{3}a_1^- b_2 + a_2^- b_0 b_1, \\ B_4(a,b) = a_2^-\left(\frac{2}{3}b_0b_2 + \frac{1}{4}b_1^2\right), \qquad B_5(a,b) = \frac{1}{3}a_2^- b_1 b_2, \qquad B_6(a,b) = \frac{1}{9}a_2^- b_2^2. \end{gathered} \tag{3.8}$$

From (3.8), it is easy to see that $B_5 = O(|B_2, B_3, B_4|)$, $B_6 = O(|B_2, B_3, B_4|)$. By Corollary 2.6, one can see that $L(2,2,1) \leq 3$. Further, from (3.8) we can have $a_0 = (a_{10}^+, a_{10}^-, a_{20}^-)$ with

$$a_{10}^+ = -a_{10}^- b_0, \quad a_{20}^- = \frac{b_1}{2b_0^2}a_{10}^-, \quad a_{10}^- \neq 0 \tag{3.9}$$

such that

$$\begin{gathered} B_1(a_0,b) = B_2(a_0,b) = 0, \qquad B_3(a_0,b) = a_{10}^- \Delta_1(b), \\ B_4(a_0,b) = a_{10}^- \Delta_2(b), \\ B_5(a_0,b) = a_{10}^- \Delta_3(b), \qquad B_6(a_0,b) = a_{10}^- \Delta_4(b), \end{gathered} \tag{3.10}$$

where

$$\begin{gathered} \Delta_1(b) = -\frac{b_2}{3} + \frac{b_1^2}{2b_0}, \qquad \Delta_2(b) = \frac{b_1}{b_0^2}\left(\frac{1}{3}b_0b_2 + \frac{1}{8}b_1^2\right), \\ \Delta_3(b) = \frac{b_1^2 b_2}{6b_0^2}, \qquad \Delta_4(b) = \frac{b_1 b_2^2}{18b_0^2}. \end{gathered} \tag{3.11}$$

Take $b = \widetilde{b}_0 = (b_{00}, b_{10}, (3b_{10}^2/2b_{00}))$ with $b_{00} \neq 0$, $b_{10} \neq 0$. Then by (3.11), we have

$$\Delta_1(b_0) = 0, \qquad \Delta_2(b_0) = \frac{5b_{10}^3}{8b_{00}^2} \neq 0. \tag{3.12}$$

Note that

$$\det \frac{\partial(B_1, B_2)}{\partial(a_1^+, a_2^-)} = \begin{vmatrix} -1 & 0 \\ 0 & b_0^2 \end{vmatrix} = -b_0^2 \neq 0, \qquad \frac{\partial \Delta_1}{\partial b_2} = -\frac{1}{3}. \tag{3.13}$$

Then by Theorem 1.8, we know that system (3.1) has cyclicity 3 for (a, b) sufficiently close to $(a_0, \widetilde{b}_0)$. In other words, $L(2,2,1) = 3$. Similarly, we have $L(2,2,2) = L(2,2,3) = 4$, $L(2,2,4) = L(2,2,5) = 5$.

Case 3. $k = m = n$, $n = 1,2,3$. We claim that $L(n,n,n) = 3n - 2$ for $n = 1,2,3$. In fact, for $k = m = n$, we have

$$\begin{aligned} F^-(\alpha(x,b),a) - F^+(x,a) &= \sum_{j=1}^{n} a_j^- \left(-\sum_{i=0}^{n} \frac{b_i}{i+1} x^{i+1}\right)^j - \sum_{j=1}^{n} a_j^+ x^j \\ &= \sum_{j=1}^{n(n+1)} B_j(a,b) x^j. \end{aligned} \tag{3.14}$$

From Cases 1 and 2, it is easy to see that $L(1,1,1) = 1$, $L(2,2,2) = 4$. For $k = m = n = 3$, $B_j(a,b)$ in (3.14) can be written as

$$\begin{aligned} B_1(a,b) &= -a_1^- b_0 - a_1^+, \qquad B_2(a,b) = -\frac{1}{2} a_1^- b_1 + a_2^- b_0^2 - a_2^+, \\ B_3(a,b) &= -\frac{1}{3} a_1^- b_2 + a_2^- b_0 b_1 - a_3^- b_0^3 - a_3^+, \\ B_4(a,b) &= -\frac{1}{4} a_1^- b_3 + a_2^- \left(\frac{2}{3} b_0 b_2 + \frac{1}{4} b_1^2\right) - \frac{3}{2} a_3^- b_0^2 b_1, \\ B_5(a,b) &= a_2^- \left(\frac{1}{2} b_0 b_3 + \frac{1}{3} b_1 b_2\right) - a_3^- \left(b_0^2 b_2 + \frac{3}{4} b_0 b_1^2\right), \\ B_6(a,b) &= a_2^- \left(\frac{1}{4} b_1 b_3 + \frac{1}{9} b_2^2\right) - a_3^- \left(b_0 b_1 b_2 + \frac{3}{4} b_0^2 b_3 + \frac{1}{8} b_1^3\right), \\ B_7(a,b) &= \frac{1}{6} a_2^- b_2 b_3 - a_3^- \left(\frac{3}{4} b_0 b_1 b_3 + \frac{1}{3} b_0 b_2^2 + \frac{1}{4} b_1^2 b_2\right), \\ B_8(a,b) &= \frac{1}{16} a_2^- b_3^2 - a_3^- \left(\frac{1}{2} b_0 b_2 b_3 + \frac{3}{16} b_1^2 b_3 + \frac{1}{6} b_1 b_2^2\right), \end{aligned}$$

$$B_9(a,b) = -a_3^-\left(\frac{1}{4}b_1b_2b_3 + \frac{3}{16}b_0b_3^2 + \frac{1}{27}b_2^3\right), \qquad B_{10}(a,b) = -a_3^-\left(\frac{3}{32}b_1b_3^2 + \frac{1}{12}b_2^2b_3\right),$$
$$B_{11}(a,b) = -\frac{1}{15}a_3^-b_2b_3^2, \qquad B_{12} = -\frac{1}{64}a_3^-b_3^3. \tag{3.15}$$

From (3.15), it is easy to prove that $B_i = O(|B_4,\ldots,B_8|)$, $i = 9,\ldots,12$. Then by Corollary 2.6, we have $L(3,3,3) \le 7$. In order to prove $L(3,3,3) = 7$, let for convenience

$$Z(b) = 36b_0^2b_1b_3 - 12b_0b_1^2b_2 - 32b_0^2b_2^2 - 9b_1^4. \tag{3.16}$$

Then we can find $(a,b) = (a_0,\widetilde{b}_0)$ with $Z(\widetilde{b}_0) \neq 0$ and

$$a_{10}^+ = -a_{10}^-b_0, \qquad a_{20}^+ = \frac{1}{2}a_{10}^-b_1 - a_{20}^-b_0^2, \qquad a_{30}^+ = -\frac{1}{3}a_{10}^-b_2 + a_2^-b_0b_1 - a_{30}^-b_0^3,$$
$$a_{10}^- \neq 0, \quad a_{20}^- = \frac{3a_{10}^-b_3\left(4b_0b_2 + 3b_1^2\right)}{Z(b)}, \qquad a_{30}^- = \frac{2a_{10}^-b_3(3b_0b_3 + 2b_1b_2)}{b_0Z(b)} \tag{3.17}$$

for $b = \widetilde{b}_0$ such that

$$B_1\left(a_0,\widetilde{b}_0\right) = \cdots = B_5\left(a_0,\widetilde{b}_0\right) = 0, \qquad B_6\left(a_0,\widetilde{b}_0\right) = a_{10}^-\Delta_1\left(\widetilde{b}_0\right),$$
$$B_7\left(a_0,\widetilde{b}_0\right) = a_{10}^-\Delta_2\left(\widetilde{b}_0\right), \qquad B_8\left(a_0,\widetilde{b}_0\right) = a_{10}^-\Delta_3\left(\widetilde{b}_0\right), \qquad B_9\left(a_0,\widetilde{b}_0\right) = a_{10}^-\Delta_4\left(\widetilde{b}_0\right),$$
$$B_{10}\left(a_0,\widetilde{b}_0\right) = a_{10}^-\Delta_5\left(\widetilde{b}_0\right), \qquad B_{11}\left(a_0,\widetilde{b}_0\right) = a_{10}^-\Delta_6\left(\widetilde{b}_0\right), \qquad B_{12}\left(a_0,\widetilde{b}_0\right) = a_{10}^-\Delta_7\left(\widetilde{b}_0\right), \tag{3.18}$$

where

$$\Delta_1(b) = Z_1\left(-\frac{9}{2}b_0^3b_3^2 - 6b_0^2b_1b_2b_3 + \frac{4}{3}b_0^2b_2^3 + \frac{3}{2}b_0b_1^2b_3 - 3b_0b_1^2b_2^2 - \frac{1}{2}b_1^4b_2\right),$$
$$\Delta_2(b) = Z_1\left(-\frac{9}{2}b_0^2b_1b_3^2 - 3b_0b_1^2b_2b_3^2 - \frac{4}{3}b_0b_1b_2^3 - b_1^3b_2^2\right),$$
$$\Delta_3(b) = Z_1\left(-\frac{9}{4}b_0^2b_2b_3^2 - \frac{9}{16}b_0b_1^2b_3^2 - 3b_0b_1b_2^2b_3 - \frac{3}{4}b_1^2b_2b_3 - \frac{2}{3}b_1^2b_2^3\right),$$
$$\Delta_4(b) = Z_1\left(-\frac{9}{8}b_0^2b_3^2 - \frac{9}{4}b_0b_1b_2b_3^2 - \frac{2}{9}b_0b_2^3b_3 - b_1^2b_2^2b_3 - \frac{4}{27}b_1b_2^4\right), \tag{3.19}$$
$$\Delta_5(b) = Z_1\left(-\frac{9}{16}b_0b_1b_3^2 - \frac{1}{2}b_0b_2^2b_3^2 - \frac{3}{8}b_1^2b_2b_3^2 - \frac{1}{3}b_1b_2^3b_3\right),$$
$$\Delta_6(b) = Z_1\left(-\frac{3}{8}b_2b_3^3 - \frac{1}{4}b_1b_2^2b_3^2\right), \qquad \Delta_7(b) = Z_1\left(-\frac{3}{32}b_0b_3^4 - \frac{1}{16}b_1b_2b_3^2\right),$$

with $Z_1 = b_3/b_0 Z$. Let's choose $\tilde{b}_0 = (b_{00}, 0, (3/2)b_{00}^{1/3}b_{30}^{2/3}, b_{30})$ with $b_{00} > 0$, $b_{30} \neq 0$. Obviously, $Z(\tilde{b}_0) \neq 0$. Then by (3.19), we have

$$\Delta_1\left(\tilde{b}_0\right) = \Delta_2\left(\tilde{b}_0\right) = 0, \qquad \Delta_3\left(\tilde{b}_0\right) = \frac{3b_{30}^{7/3}}{64b_{00}^{4/3}} \neq 0. \tag{3.20}$$

Note that

$$\begin{gathered} \det \frac{\partial(B_1, B_2, B_3, B_4, B_5)}{\partial(a_1^+, a_2^+, a_3^+, a_2^-, a_3^-)}\Big|_{(a_0,\tilde{b}_0)} = \frac{9}{8}b_{00}^{8/3}b_{30}^{4/3} \neq 0, \\ \det \frac{\partial(\Delta_1, \Delta_2)}{\partial(b_1, b_2)}\Big|_{\tilde{b}_0} = \frac{81}{2}b_{00}^{14/3}b_{30}^{10/3} \neq 0. \end{gathered} \tag{3.21}$$

Then by Theorem 1.8, it is clear that 7 limit cycles can appear near the origin for (a, b) sufficiently close $(a_0, \tilde{b}_0)$. Then we have $L(3, 3, 3) = 7$.

Example 3.2. Consider system (3.1) with

$$g(x, b) = \begin{cases} x^2, & x \geq 0, \\ -1 + b_1 x + b_2 x^2, & x < 0, \end{cases} \qquad F(x, a) = \begin{cases} \sum_{j=2}^{n} a_j^+ x^j, & x \geq 0, \\ \sum_{j=1}^{m} a_j^- x^j, & x < 0. \end{cases} \tag{3.22}$$

Clearly, the origin is a center or focus by Theorem 1.2. Denote $H(m, n)$ the cyclicity of system (3.1) near the origin. By (3.22), we can obtain

$$G(x, b) = \begin{cases} \dfrac{x^3}{3}, & x \geq 0, \\ -x + \dfrac{b_1}{2}x^2 + \dfrac{b_2}{3}x^3, & x < 0. \end{cases} \tag{3.23}$$

Using Lemma 2.1 and (3.23), for $x \geq 0$ small, we can solve from the equation $G^-(\alpha(x, b), b) = G^+(x, b)$

$$\begin{aligned} \alpha(x, b) = \sum_{j\geq 1}\alpha_j x^{3j} = & -\frac{1}{3}x^3 + \frac{1}{18}b_1 x^6 - \left(\frac{b_1^2}{54} + \frac{b_2}{81}\right)x^9 + \left(\frac{5}{648}b_1^3 + \frac{5}{486}b_1 b_2\right)x^{12} \\ & - \left(\frac{7}{1944}b_1^4 + \frac{7}{972}b_1^2 b_2 + \frac{1}{729}b_2^2\right)x^{15} \\ & + \left(\frac{7}{3888}b_1^5 + \frac{7}{1458}b_1^3 b_2 + \frac{14}{6561}b_1 b_2^2\right)x^{18} + O\left(x^{21}\right). \end{aligned} \tag{3.24}$$

Take $m = 4$, $n = 5$. Combining (3.22) and (3.24), we can obtain

$$F^{-}(\alpha(x,b),a) - F^{+}(x,a) = \sum_{j=1}^{5} B_j(a,b)x^{j+1} + \sum_{j\geq 6} B_j(a,b)x^{3(j-3)}, \tag{3.25}$$

where

$$\begin{aligned}
&B_1(a,b) = -a_2^+, \qquad B_2(a,b) = -a_3^+ - \frac{1}{3}a_1^-, \qquad B_3(a,b) = -a_4^+, \qquad B_4 = -a_5^+,\\
&B_5(a,b) = \frac{1}{18}a_1^- b_1 + \frac{1}{9}a_2^-, \qquad B_6(a,b) = a_1^-\left(-\frac{1}{54}b_1^2 - \frac{1}{81}b_2\right) - \frac{1}{27}a_2^- b_1 - \frac{1}{27}a_3^-,\\
&B_7(a,b) = a_1^-\left(\frac{5}{648}b_1^3 + \frac{5}{486}b_1 b_2\right) + a_2^-\left(\frac{5}{324}b_1^2 + \frac{2}{243}b_2\right) + \frac{1}{54}a_3^- b_1 + \frac{1}{81}a_4^-,\\
&B_8(a,b) = a_1^-\left(-\frac{1}{1944}b_1^4 - \frac{1}{972}b_1^2 b_2 - \frac{1}{729}b_2^2\right) + a_2^-\left(-\frac{7}{972}b_1^3 - \frac{2}{243}b_1 b_2\right)\\
&\qquad - a_3^-\left(\frac{1}{108}b_1^2 + \frac{1}{243}b_2\right) - \frac{2}{243}a_4^- b_1,\\
&B_9(a,b) = a_1^-\left(\frac{7}{3888}b_1^5 + \frac{7}{1458}b_1^3 b_2 + \frac{14}{6561}b_1 b_2^2\right) + a_2^-\left(\frac{1}{648}b_1^4 + \frac{5}{2187}b_1^2 b_2 + \frac{7}{6561}b_2^2\right)\\
&\qquad + a_3^-\left(\frac{7}{1458}b_1^3 + \frac{7}{1458}b_1 b_2\right) + a_4^-\left(\frac{7}{1458}b_1^2 + \frac{4}{2187}b_2\right),
\end{aligned} \tag{3.26}$$

which implies that $B_j = O(|B_1,\ldots,B_9|)$, $j \geq 10$. Then we can obtain $H(4,5) \leq 8$ by Corollary 2.6. Now we prove that $H(4,5) = 8$. Suppose $a = (a_2^+,\ldots,a_5^+,a_1^-,\ldots,a_4^-)$. Then we can find

$$a_0 = (\overline{a}_{20}^+,\ldots,\overline{a}_{50}^+,\overline{a}_{10}^-,\ldots,\overline{a}_{40}^-), \tag{3.27}$$

where

$$a_{20}^+ = a_{40}^+ = a_{50}^+ = a_{40}^- = 0, \qquad a_{30}^+ = -\frac{1}{3}a_{10}^-, \qquad a_{20}^- = -\frac{1}{2}a_{10}^- b_1, \qquad a_{30}^- = -\frac{1}{3}a_{10}^- b_2, \qquad a_{10}^- \neq 0, \tag{3.28}$$

such that

$$\begin{gathered}
B_1(a_0,b) = B_2(a_0,b) = B_3(a_0,b) = B_4(a_0,b) = B_5(a_0,b) = B_6(a_0,b) = B_7(a_0,b) = 0,\\
B_8(a_0,b) = \frac{1}{162}a_1^-\Delta_0, \qquad B_9(a_0,b) = \frac{a_1^-}{162}\Delta_1,
\end{gathered} \tag{3.29}$$

where

$$\Delta_0 = b_1^2\left(\frac{1}{2}b_1^2 + b_2\right), \qquad \Delta_1 = b_1^3\left(\frac{5185}{31096}b_1^2 + \frac{1}{3}b_2\right). \tag{3.30}$$

Solving the equation $\Delta_0 = 0$ gives $b = (b_{10}, -(1/2)b_{10}^2) \equiv b_0$, $b_{10} \neq 0$. Inserting b_0 into (3.30) gives

$$\Delta_1(b_0) = \frac{5185}{186576}b_{10}^5 \neq 0. \tag{3.31}$$

Further, by (3.26) and (3.30), we have

$$\frac{\partial(B_1, B_2, B_3, B_4, B_5, B_6, B_7)}{\partial(a_2^+, a_3^+, a_4^+, a_5^+, a_2^-, a_3^-, a_4^-)} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{b_1}{27} & -\frac{1}{27} & 0 \\ 0 & 0 & 0 & 0 & \frac{5b_1^2}{324} + \frac{2b_2}{243} & \frac{b_1}{54} & \frac{1}{81} \end{pmatrix},$$
$$\frac{\partial\Delta_0}{\partial b_2}(b_0) = b_{10}^2 \neq 0. \tag{3.32}$$

Hence, by the conclusion (i) of Theorem 1.8, it is easy to see that there exists cyclicity 8 when (a, b) near (a_0, b_0), which implies $H(4, 5) = 8$.

Remark 3.3. In a similar way, for $1 \leq m \leq 4$, $2 \leq n \leq 5$, we can obtain $H(m, n) = m + n - 1$ for $n \neq 2$, $H(m, n) = m + n - 2$ for $n = 2$, $m \neq 2$ and $H(2, 2) = 3$.

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