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Research Article

An iterative Algorithm for Hemicontractive Mappings in Banach Spaces

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We First introduce a three-step iterative algorithm for approximating the fixed points of the hemicontractive mappings in Banach spaces. Consequently, we prove the strong convergence of the proposed algorithm under some assumptions. Since three-step iterations include Ishikawa iterations as special cases, our result continue to hold for these problems. Our main results can be viewed as an important refinement of the previously known results.

1. Introduction

In recent years, several convergence results have been proved on iterative methods for approximating fixed points of pseudocontractive mappings (see, e.g., [1–10] and references therein). It is worth mentioning that such iterative type methods are known as Mann iterations and Ishikawa iterations. It is clear that a Lipschitz pseudocontractive mapping with a unique fixed point for which the Mann iteration sequence failed to converge, but it does converge for the sequence obtained by the Ishikawa iterations, see [11]. In 2000, Noor [12] suggested and analyzed three-step iterative methods for finding the approximate solution of a continuous mapping in the Hilbert space using the technique of updating the solution. Three-step iterations are also known as Noor iteration. It is well known [13] that three-step iterative schemes include one-step (Mann) and two-step (Ishikawa) iterations as special cases. It raises an interesting question. Is there exist any Lipschitz pseudo-contractive mapping with a unique fixed point for which Ishikawa iteration sequence fail to convergence, but Does convergence for the sequence obtained from Noor iteration? This is an open and challenging problem. To the best of our knowledge, main result of Ishikawa [14], see Theorem IS has never been extended to more general Banach spaces. Motivated and inspired by the recent research activities in this filed, we suggest and analyze a three-step iterative scheme associated with hemi-contractive mappings in Banach spaces. We also prove the strong convergence of the sequence generated by the three-step iterations under mild conditions. Since three-step iterations include Ishikawa iterations as special cases, our results continue to hold for these problems. It is worth mentioning that our results may be considered as very significant, interesting and important extensions of the previously known results concerning pseudo-contractive mappings.

Let E be a real Banach space and E^* be its dual space. The normalized duality mapping from E to 2^{E^*} is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x^*\| = \|x\|\}, \quad \forall x \in E, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

Let C be a nonempty subset of E, a mapping $T : C \to C$ is called pseudo-contractive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2, \tag{1.2}$$

for all $x, y \in C$. Let $F(T) := \{x \in C : Tx = x\}$. A mapping $T : C \to C$ is called hemicontractive if $F(T) \neq \emptyset$ and

$$\langle Tx - x^*, j(x-x^*) \rangle \leq \|x-x^*\|^2, \quad \forall x \in C,\ x^* \in F(T). \tag{1.3}$$

It is easy to see that the class of pseudo-contractive mappings with fixed points is a subclass of the class of hemicontractions. A mapping $T : C \to C$ is called Lipschitzian if there exists a constant $L \geq 0$ such that $\|Tx - Ty\| \leq L\|x-y\|$ for each $x, y \in C$.

In 1974, Ishikawa [14] proved the following result for the pseudo-contractive mappings.

Theorem 1.1 (see [14]). *If C is a compact convex subset of a Hilbert space H, $T : C \to C$ is a Lipschitzian pseudo-contractive mapping. For $x_0 \in C$, define the sequence $\{x_n\}$ iteratively by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 0, \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying the conditions

(i) $0 \leq \alpha_n \leq \beta_n < 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$;

(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then the sequence $\{x_n\}$ defined by (1.4) *converges strongly to a fixed point of T.*

Since its publication in 1974, Theorem IS, as far as we know, has never been extended to more general Banach spaces.

In this paper, we suggest and analyze a three-step iteration below Algorithm 1.2 associated with hemi-contractive mappings having a strong convergence in the setting of Banach spaces under some appropriate conditions.

Algorithm 1.2. For arbitrary $x_1 \in C$, let the sequence $\{x_n\}$ be generated by

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \quad n \geq 1. \end{aligned} \tag{1.5}$$

It is clear that Algorithm 1.2 includes Mann (one-step) and Ishikawa (two-step) iterations as special cases. For some related works, please refere to [15–19].

2. Preliminaries

Let E be a Banach space, the modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : \|x\| = 1, \|y\| = 1, \|x-y\| \geq \epsilon\right\}. \tag{2.1}$$

A Banach space E is called uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$. For $p > 1$, the (generalized) duality mapping $J_p : E \to 2^{E^*}$ is defined as $J_p(x) := \{x^* \in E : \langle x, x^*\rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}$. In particular, $J = J_2$ is the normalized duality mapping on E. It is known that $J_p(x) = \|x\|^{p-2}J(x)$, $x \neq 0$. A Banach space E is called p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$, $0 < \epsilon \leq 2$. It is known (see e.g., [12]) that L_p is

$$\begin{aligned} &2 \text{ uniformly convex}, \quad \text{if } 1 < p \leq 2, \\ &p \text{ uniformly convex}, \quad \text{if } p \geq 2. \end{aligned} \tag{2.2}$$

For proving our main results, we shall need the following lemmas.

Lemma 2.1 (see [20]). *Let $p > 1$ be a given real number. Then the following statements about a Banach space E are equivalent:*

(i) *E is p-uniformly convex;*

(ii) *there is a constant $c_p > 0$ such that for every $x, y \in E$, $j_p(x) \in J_p(x)$, the following inequality holds:*

$$\|x+y\|^p \geq \|x\|^p + p\langle y, j_p(x)\rangle + c_p\|y\|^p. \tag{2.3}$$

Remark 2.2. Replacing x by $(x+y)$, y by $(-y)$ in inequality (2.3) and using the Cauchy-Schwarz inequality, we can obtain

$$\|x+y\|^p \leq \|x\|^p + p\|y\| \cdot \|x+y\|^{p-1}. \tag{2.4}$$

Lemma 2.3 (see [20]). *Let $p > 1$ be a given real number. Let E be a p-uniformly convex Banach space. Then, there exists a constant $d > 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)d\|x-y\|^p, \tag{2.5}$$

for all $\lambda \in [0,1]$ and $x, y \in E$, where $W_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$.

Lemma 2.4 (see [4]). *Let $\{\rho_n\} : \{\sigma_n\}$ be two nonnegative sequences and for all integers $n \ge N_0$ (for some fixed N_0), $\rho_{n+1} \le \rho_n + \sigma_n$.*

(i) *if $\sum_{n=1}^{\infty} \sigma_n < \infty$, then $\lim_{n\to\infty} \rho_n$ exists;*

(ii) *if $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $\{\rho_n\}$ has a sequence converging to zero, then $\lim_{n\to\infty} \rho_n = 0$.*

3. Main Results

In the sequel, c_p and d will denote the constants appearing in inequalities (2.3) and (2.5), respectively. For the rest of this paper, we shall assume that E be a real p-uniformly convex Banach space such that $2^{-(p-2)}dp > (p-1)c_p^{-1}$ and $p \le 1 + c_p$. For L_p spaces with $1 < p \le 2$, the following inequalities hold (see [12, pages 1131-1132]):

$$\begin{gathered}\|x+y\|^2 \ge \|x\|^2 + 2\langle y, J(x)\rangle + c_p\|y\|^2, \\ \|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - W_2(\lambda)(p-1)\|x-y\|^2,\end{gathered} \tag{3.1}$$

for $\lambda \in [0,1]$ and for all $x, y \in E$, where $c_p = [1 + t_p^{(p-1)}][(1+t_p)^{-(p-1)}]$, and for $0 < t_p < 1$, t_p is the unique solution of the equation $g(t) = (p-2)t^{(p-1)} + (p-1)t^{(p-2)} - 1 = 0$.

Remark 3.1. We observe that the function $h : [0,1] \to [0,\infty)$ defined by $h(x) = (1 + x^{p-1})/(1+x)^{p-1}$ is increasing on $[0,1]$ ($h'(x) = (1+x)^{p-2}(p-1)(x^{p-2}-1)/(1+x)^{2p-2} \ge 0$), hence for L_p $(1 < p \le 2)$, we have $c_p \ge 1$ and $d = p-1$. Therefore, the conditions $2^{-(p-2)}dp > (p-1)c_p^{-1}$ and $p \le 1 + c_p$ are satisfied.

Lemma 3.2. *Let E be a real p-uniformly convex Banach space, $\emptyset \ne C \subset E$ nonempty closed convex and bounded, and $T : C \to C$ a hemi-contractive mapping with $F(T) \ne \emptyset$. Then, for each $x \in C$ and for each integer $n \ge 1$, the following inequality holds:*

$$c_p\|Tx - x^*\|^p \le (p-1)\|x-x^*\|^p + \|x - Tx\|^p, \quad \forall x^* \in F(T). \tag{3.2}$$

Proof. Replacing x by $(1/2)(x - x^*)$ and y by $(-1/2)(Tx - x^*)$ in inequality (2.3), we can get

$$\begin{aligned}\|x - Tx\|^p &\ge \|x-x^*\|^p - p2^{p-1}\left\langle Tx - x^*, j_p\left(\frac{1}{2}(x-x^*)\right)\right\rangle \\ &\quad + c_p\|Tx-x^*\|^p \\ &\ge \|x-x^*\|^p - p\|x-x^*\|^p + c_p\|Tx-x^*\|^p.\end{aligned} \tag{3.3}$$

Since $j_p((1/2)(x-x^*)) \in J_p((1/2)(x-x^*)) = 2^{-(p-1)}\|x-x^*\|^{(p-2)}J(x-x^*)$ so that

$$c_p\|Tx-x^*\|^p \leq (p-1)\|x-x^*\|^p + \|x-Tx\|^p. \tag{3.4}$$

This completes the proof. □

Remark 3.3. We note that the function $f : [0,\infty) \to (-\infty,+\infty)$ defined by $f(x) = L^p x^p - dp(1-x)2^{-(p-2)} + (p-1)c_p^{-1}$ is strictly increasing on $(0,\infty)$. Hence, it has at most one zero on $(0,\infty)$, provided that $f(0) = (p-1)c_p^{-1} - dp2^{-(p-2)} < 0$. In this case, since $f(1) = L^p + (p-1)c_p^{-1} > 0$, it follows that the zero $t_p \in (0,1)$.

Lemma 3.4. *Let E be a real p-uniformly convex Banach space such that $2^{-(p-2)}dp > (p-1)c_p^{-1}$ and $p \leq 1 + c_p$. Let C be a nonempty closed convex and bounded subset of E, let $T : C \to C$ be a Lipschitz hemi-contractive mapping with Lipschitz constant $L \geq 0$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $[0,1]$ satisfying the following conditions:*

$$\epsilon \leq 1 - dc_p(1-\alpha_n)2^{-(p-2)} \leq \beta_n \leq b, \qquad \sum_{n=1}^{\infty}\gamma_n < \infty. \tag{3.5}$$

for all integers $n \geq 1$, some $\epsilon > 0$ and $b \in (0, t_p)$, where t_p is the unique solution of the equation:

$$L^p x^p - dp(1-x)2^{-(p-2)} + (p-1)c_p^{-1} = 0, \tag{3.6}$$

on $(0,\infty)$.

For arbitrary $x_1 \in C$, let the sequence $\{x_n\}$ be generated by

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \quad n \geq 1. \end{aligned} \tag{3.7}$$

Then, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Proof. We shall use M to denote the possible different constants appearing in the following reasoning.

Let $x^* \in F(T)$. Using inequality (2.5), we have

$$\begin{aligned} \|x_{n+1}-x^*\|^p &= \left\|(1-\alpha_n)(x_n-x^*) + \alpha_n(Ty_n - x^*)\right\|^p \\ &\leq (1-\alpha_n)\|x_n-x^*\|^p + \alpha_n\left\|Ty_n-x^*\right\|^p \\ &\quad - W_p(\alpha_n)d\left\|x_n - Ty_n\right\|^p. \end{aligned} \tag{3.8}$$

From (3.2), we have

$$c_p\|Tx_n - x^*\|^p \le (p-1)\|x_n - x^*\|^p + \|x_n - Tx_n\|^p, \tag{3.9}$$

$$c_p\|Ty_n - x^*\|^p \le (p-1)\|y_n - x^*\|^p + \|y_n - Ty_n\|^p. \tag{3.10}$$

Moreover, we also have

$$\begin{aligned}\|y_n - x^*\|^p &= \|(1-\beta_n)(x_n - x^*) + \beta_n(Tz_n - x^*)\|^p \\ &\le (1-\beta_n)\|x_n - x^*\|^p + \beta_n\|Tz_n - x^*\|^p \\ &\quad - W_p(\beta_n)d\|x_n - Tz_n\|^p,\end{aligned} \tag{3.11}$$

$$\begin{aligned}\|y_n - Ty_n\|^p &= \|(1-\beta_n)(x_n - Ty_n) + \beta_n(Tz_n - Ty_n)\|^p \\ &\le (1-\beta_n)\|x_n - Ty_n\|^p + \beta_n\|Tz_n - Ty_n\|^p \\ &\quad - W_p(\beta_n)d\|x_n - Tz_n\|^p.\end{aligned} \tag{3.12}$$

At the same time, applying (2.4), we can obtain the following estimates:

$$\begin{aligned}\|Tz_n - x^*\|^p &= \|Tx_n - x^* + Tz_n - Tx_n\|^p \\ &\le \|Tx_n - x^*\|^p + p\|Tz_n - Tx_n\|\|Tz_n - x^*\|^{p-1} \\ &\le \|Tx_n - x^*\|^p + pL\|z_n - x_n\|\|Tz_n - x^*\|^{p-1} \\ &= \|Tx_n - x^*\|^p + pL\gamma_n\|Tx_n - x_n\|\|Tz_n - x^*\|^{p-1} \\ &\le \|Tx_n - x^*\|^p + M\gamma_n,\end{aligned} \tag{3.13}$$

$$\begin{aligned}\|x_n - Tx_n\|^p &= \|Tz_n - Tx_n + x_n - Tz_n\|^p \\ &\le \|x_n - Tz_n\|^p + p\|Tz_n - Tx_n\|\|x_n - Tx_n\|^{p-1} \\ &\le \|x_n - Tz_n\|^p + M\gamma_n,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\|Tz_n - Ty_n\|^p &= \|Tx_n - Ty_n + Tz_n - Tx_n\|^p \\ &\le \|Tx_n - Ty_n\|^p + p\|Tz_n - Tx_n\|\|Tz_n - Ty_n\|^{p-1} \\ &\le \|Tx_n - Ty_n\|^p + M\gamma_n.\end{aligned} \tag{3.15}$$

Substitute (3.13) and (3.14) into (3.11) to get

$$\begin{aligned}\|y_n - x^*\|^p &\le (1-\beta_n)\|x_n - x^*\|^p + \beta_n\|Tx_n - x^*\|^p \\ &\quad - W_p(\beta_n)d\|x_n - Tx_n\|^p + M\gamma_n,\end{aligned} \tag{3.16}$$

this together with (3.9) implies that

$$\begin{aligned}\left\|y_n - x^*\right\|^p &\le (1-\beta_n)\|x_n - x^*\|^p \\ &\quad + \beta_n c_p^{-1}\{(p-1)\|x_n - x^*\|^p + \|x_n - Tx_n\|^p\} \\ &\quad - W_p(\beta_n) d\|x_n - Tx_n\|^p + M\gamma_n \\ &= \left[1 + \beta_n c_p^{-1}(p-1-c_p)\right]\|x_n - x^*\|^p \\ &\quad + \left[\beta_n c_p^{-1} - W_p(\beta_n) d\right]\|x_n - Tx_n\|^p + M\gamma_n.\end{aligned} \tag{3.17}$$

Set $t_n = \beta_n c_p^{-1}(p-1-c_p)$, $r_n = \beta_n c_p^{-1} - W_p(\beta_n)d$. Then,

$$\left\|y_n - x^*\right\|^p \le (1+t_n)\|x_n - x^*\|^p + r_n\|x_n - Tx_n\|^p + M\gamma_n. \tag{3.18}$$

From (3.12), (3.14), and (3.15), we have

$$\begin{aligned}\left\|y_n - Ty_n\right\|^p &\le (1-\beta_n)\left\|x_n - Ty_n\right\|^p + \beta_n\left\|Tx_n - Ty_n\right\|^p \\ &\quad - W_p(\beta_n) d\|x_n - Tx_n\|^p + M\gamma_n.\end{aligned} \tag{3.19}$$

Substitution of (3.18) and (3.19) into (3.10) yields

$$\begin{aligned}c_p\left\|Ty_n - x^*\right\|^p &\le (p-1)(1+t_n)\|x_n - x^*\|^p + (p-1)r_n\|x_n - Tx_n\|^p \\ &\quad + (1-\beta_n)\left\|x_n - Ty_n\right\|^p + \beta_n\left\|Tx_n - Ty_n\right\|^p \\ &\quad - W_p(\beta_n) d\|x_n - Tx_n\|^p + M\gamma_n \\ &= (p-1)(1+t_n)\|x_n - x^*\|^p + \left[(p-1)r_n - W_p(\beta_n)d\right] \\ &\quad \times \|x_n - Tx_n\|^p + (1-\beta_n)\left\|x_n - Ty_n\right\|^p \\ &\quad + \beta_n\left\|Tx_n - Ty_n\right\|^p + M\gamma_n.\end{aligned} \tag{3.20}$$

Substitution of this inequality into (3.8) now gives

$$\begin{aligned}\|x_{n+1}-x^*\|^p &\le (1-\alpha_n)\|x_n-x^*\|^p \\ &\quad + \alpha_n c_p^{-1}\{(p-1)(1+t_n)\|x_n-x^*\|^p + [(p-1)r_n - W_p(\beta_n)d] \\ &\qquad \times\|x_n-Tx_n\|^p + (1-\beta_n)\|x_n-Ty_n\|^p + \beta_n\|Tx_n-Ty_n\|^p\} \\ &\quad - W_p(\alpha_n)d\|x_n-Ty_n\|^p + M\gamma_n \\ &= \left[(1-\alpha_n)+\alpha_n c_p^{-1}(p-1)(1+t_n)\right]\|x_n-x^*\|^p \\ &\quad + \left[\alpha_n c_p^{-1}(1-\beta_n) - W_p(\alpha_n)d\right]\|x_n-Ty_n\|^p \\ &\quad + \alpha_n c_p^{-1}[(p-1)r_n - W_p(\beta_n)d]\|x_n-Tx_n\|^p \\ &\quad + \alpha_n\beta_n c_p^{-1}\|Tx_n-Ty_n\|^p + M\gamma_n,\end{aligned} \tag{3.21}$$

that is,

$$\begin{aligned}\|x_{n+1}-x^*\|^p &\le \left\{1+\alpha_n\left[(p-1)c_p^{-1}(1+t_n)-1\right]\right\}\|x_n-x^*\|^p \\ &\quad - \left[W_p(\alpha_n)d - c_p^{-1}\alpha_n(1-\beta_n)\right]\|x_n-Ty_n\|^p \\ &\quad - c_p^{-1}\alpha_n[W_p(\beta_n)d-(p-1)r_n]\|x_n-Tx_n\|^p \\ &\quad + \alpha_n\beta_n c_p^{-1}\|Tx_n-Ty_n\|^p + M\gamma_n.\end{aligned} \tag{3.22}$$

Observe that $c_p^{-1}(p-1)(1+t_n)-1 = c_p^{-2}(p-1-c_p)[(p-1)\beta_n+c_p]$ and that by condition (3.5), since $W_p(\alpha_n) \ge \alpha_n(1-\alpha_n)2^{-(p-2)}$, we get $W_p(\alpha_n)d - c_p^{-1}\alpha_n(1-\beta_n) \ge 0$. so that

$$\begin{aligned}\|x_{n+1}-x^*\|^p &\le \left\{1+\alpha_n c_p^{-2}(p-1-c_p)[(p-1)\beta_n+c_p]\right\}\|x_n-x^*\|^p \\ &\quad - \alpha_n c_p^{-1}[W_p(\beta_n)d-(p-1)r_n]\|x_n-Tx_n\|^p \\ &\quad + \alpha_n\beta_n c_p^{-1}\|Tx_n-Ty_n\|^p + M\gamma_n.\end{aligned} \tag{3.23}$$

Since T is Lipschitzian, we have

$$\begin{aligned}\|Tx_n-Ty_n\|^p &\le L^p\|x_n-y_n\|^p = L^p\|\beta_n(x_n-Tz_n)\|^p \\ &\le L^p\beta_n^p\|x_n-Tz_n\|^p \\ &= L^p\beta_n^p\|x_n-Tx_n+Tx_n-Tz_n\|^p \\ &\le L^p\beta_n^p\left[\|x_n-Tx_n\|^p + p\|Tx_n-Tz_n\|\|x_n-Tz_n\|^{p-1}\right] \\ &\le L^p\beta_n^p\|x_n-Tx_n\|^p + M\gamma_n.\end{aligned} \tag{3.24}$$

By the assumption $p \le 1 + c_p$, hence

$$\|x_{n+1} - x^*\|^p \le \|x_n - x^*\|^p - \alpha_n \beta_n c_p^{-1} \left[dp(1-\beta_n)2^{-(p-2)} - (p-1)c_p^{-1} - \beta_n^p L^p \right] \times \|x_n - Tx_n\|^p + M\gamma_n. \tag{3.25}$$

Since $b \in (0, t_p)$, it follows that $\delta = dp(1-b)2^{-(p-2)} - (p-1)c_p^{-1} - b^p L^p > 0$. We can choose some ϵ such that $\epsilon' = 1-(1-\epsilon)2^{-(p-2)}c_p d > 0$. Then, condition (3.5) implies $\alpha_n \ge \epsilon' > 0$. Furthermore, inequality (3.25) now yields the following estimates:

$$\begin{aligned} \|x_{n+1} - x^*\|^p &\le \|x_n - x^*\|^p - \epsilon\epsilon' c_p^{-1}\delta \|x_n - Tx_n\|^p + M\gamma_n \\ &\le \|x_n - x^*\|^p + M\gamma_n. \end{aligned} \tag{3.26}$$

Since $\sum_{n=0}^{\infty} \gamma_n < \infty$, it follows from Lemma 2.3 that $\lim_{n\to\infty} \|x_n - x^*\|^p$ exists. Let $\lim_{n\to\infty} \|x_n - x^*\|^p = r$. Inequality (3.26) also yields

$$0 < \epsilon\epsilon' c_p^{-1}\delta \|x_n - Tx_n\|^p \le \|x_n - x^*\|^p - \|x_{n+1} - x^*\|^p + M\gamma_n \to 0. \tag{3.27}$$

Hence, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. This completes the proof. □

Remark 3.5. The interest and importance of Lemma 3.2 lie in the fact that strong convergence of the sequence $\{x_n\}$ is achieved under certain mild compactness assumptions either on T or on its domain.

Now, we give a strong convergence theorem as follows.

Theorem 3.6. *Let E be a real p-uniformly convex Banach space such that $2^{-(p-2)}dp > (p-1)c_p^{-1}$ and $p \le 1 + c_p$. Let C be a nonempty closed convex and bounded subset of E, $T : C \to C$ be a completely continuous Lipschitz hemi-contractive mapping with Lipschitz constant $L \ge 0$ and $F(T) \ne \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three real sequences in $[0,1]$ satisfying the following conditions:*

$$\epsilon \le 1 - dc_p(1-\alpha_n)2^{-(p-2)} \le \beta_n \le b, \qquad \sum_{n=1}^{\infty} \gamma_n < \infty, \tag{3.28}$$

for all integers $n \ge 1$, some $\epsilon > 0$ and $b \in (0, t_p)$, where t_p is the unique solution of the equation:

$$L^p x^p - dp(1-x)2^{-(p-2)} + (p-1)c_p^{-1} = 0, \tag{3.29}$$

on $(0, \infty)$.

For arbitrary $x_1 \in C$, let the sequence $\{x_n\}$ be defined by (3.7). Then, $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By Lemma 3.2 $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Since T is completely continuous, there exists a subsequence $\{Tx_{n_i}\}$ of $\{Tx_n\}$ such that $Tx_{n_i} \to y^*$. This implies, by Lemma 3.2 that

$$x_{n_i} \longrightarrow y^*. \tag{3.30}$$

By the continuity of T and Lemma 3.2, we obtain $Ty^* = y^*$, that is, y^* is a fixed point of T. Replacing the x^* by y^* in inequality (3.26), we obtain that

$$\begin{aligned} \|x_{n+1} - y^*\|^p &\le \|x_n - y^*\|^p - \epsilon\epsilon' c_p^{-1}\delta\|x_n - Tx_n\|^p \\ &\le \|x_n - y^*\|^p + M\gamma_n. \end{aligned} \tag{3.31}$$

From (3.30), we know that $\{\|x_n - y^*\|\}$ has a sequence converging to zero. We note that the condition $\sum_{n=0}^{\infty}\gamma_n < \infty$. Hence from inequality (3.31) and Lemma 2.3, we can conclude that $\|x_n - y^*\| \to \infty$ as $n \to \infty$, that is, $\{x_n\}$ converges to a fixed point of T. This completes the proof. □

From Theorem 3.6, we can obtain the following result.

Corollary 3.7. *Let E be a real p-uniformly convex Banach space such that $2^{-(p-2)}dp > (p-1)c_p^{-1}$ and $p \le 1 + c_p$. Let C be a nonempty closed convex and bounded subset of E, $T : C \to C$ be a completely continuous Lipschitz hemi-contractive mapping with Lipschitz constant $L \ge 0$ and $F(T) \ne \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $[0,1]$ satisfying the following condition:*

$$\epsilon \le 1 - dc_p(1-\alpha_n)2^{-(p-2)} \le \beta_n \le b, \tag{3.32}$$

for all integers $n \ge 1$, some $\epsilon > 0$ and $b \in (0, t_p)$, where t_p is the unique solution of the equation:

$$L^p x^p - dp(1-x)2^{-(p-2)} + (p-1)c_p^{-1} = 0 \tag{3.33}$$

on $(0,\infty)$.

For arbitrary $x_1 \in C$, let the sequence $\{x_n\}$ be defined by (1.4). Then, $\{x_n\}$ converges strongly to a fixed points of T.

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