

Research Article

Convergence Analysis of the Preconditioned Group Splitting Methods in Boundary Value Problems

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The construction of a specific splitting-type preconditioner in block formulation applied to a class of group relaxation iterative methods derived from the centred and rotated (skewed) finite difference approximations has been shown to improve the convergence rates of these methods. In this paper, we present some theoretical convergence analysis on this preconditioner specifically applied to the linear systems resulted from these group iterative schemes in solving an elliptic boundary value problem. We will theoretically show the relationship between the spectral radiuses of the iteration matrices of the preconditioned methods which affects the rate of convergence of these methods. We will also show that the spectral radius of the preconditioned matrices is smaller than that of their unpreconditioned counterparts if the relaxation parameter is in a certain optimum range. Numerical experiments will also be presented to confirm the agreement between the theoretical and the experimental results.

1. Introduction

Consider the finite difference discretisation schemes for solving the following boundary value problem which is the two-dimensional Poisson equation with Dirichlet boundary conditions:

$$\begin{aligned}u_{xx} + u_{yy} &= f(x, y), & (x, y) \in \Omega, \\u(x, y) &= g(x, y), & (x, y) \in \partial\Omega.\end{aligned}\tag{1.1}$$

Here, Ω is a continuous unit square solution domain with boundary $\partial\Omega$. This equation plays a very important role in the modelers of fluid flow phenomena and heat conduction problems.

Let Ω be discretised uniformly in both x and y directions with a mesh size $h = 1/n$, where n is an integer. The simplest finite difference approximation of the Laplacian is

$$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h^2} - \frac{1}{12}h^2 \left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} \right) - O(h^4) = f_{ij}. \quad (1.2)$$

Here, $u_{ij} = u(x_i, y_j)$. Another approximation to (1.1) can be derived from the *rotated* five-point finite-difference approximation to give [1]

$$\begin{aligned} & \frac{u_{i+1,j+1} + u_{i-1,j-1} + u_{i+1,j-1} + u_{i-1,j+1} - 4u_{ij}}{2h^2} \\ & - h^2 \left[\frac{1}{2} \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{1}{12} \left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} \right) \right] - O(h^4) = f_{ij}. \end{aligned} \quad (1.3)$$

Based on the latter approximation, improved point and group iterative schemes have been developed over the last few years in solving several types of partial differential equations [1–5]. In particular, the Modified Explicit Decoupled Group (MEDG) method [6, 7] was formulated as the latest addition to this family of four-point explicit group methods in solving the Poisson equation. This method has been shown to be the fastest among the existing explicit group methods due to its lesser computational complexity.

Since it is well known that preconditioners play a vital role in accelerating the convergence rates of iterative methods, several preconditioned strategies have been used for improving the convergence rate of the explicit group methods derived from the standard and skewed (rotated) finite difference operators [8–11]. In particular, Saeed and Ali [12–14] presented an $(I + \bar{S})$ -type preconditioning matrix applied to the systems obtained from the four-point Explicit Decoupled Group (EDG) and the Modified Explicit Decoupled Group (MEDG) methods for solving the elliptic partial differential equation, where \bar{S} is obtained by taking the first upper diagonal groups of the iteration matrix of the original system. The numerical experiments performed on these methods were seen to yield very encouraging results. However, no detailed studies of the spectral radius analysis of all these preconditioned systems have been done to confirm the superiority of this preconditioner.

The focus of this study is to establish the convergence properties of the preconditioned systems based on the splitting-type preconditioner $(I + \bar{S})$ for improving the performance and reliability of this family of explicit group methods derived from the rotated finite-difference formula. We will prove that this type of preconditioner applied to the MEDG SOR can minimize the most the spectral radius of the preconditioned matrix provided the relaxation parameter is in a certain optimum range. This paper is organised as follows: in Section 2, we give a presentation of the preconditioner applied to the system resulted from the EDG SOR method. A brief description of the application of the preconditioner in block formulation to the MEDG SOR is given in Section 3. The theoretical convergence analysis of these methods is discussed in Section 4. In Section 5, we give a numerical example to confirm the results obtained in Section 4. Finally, we report a brief conclusion in Section 6.

2. Preconditioned Explicit Decoupled Group SOR (EDG SOR)

For convenience, we will now briefly explain some of the definitions used in this paper.

Definition 2.1 (see [15]). A matrix A of order n has property A if there exists two disjoint subsets S and T of $W = \{1, 2, \dots, n\}$ such that if $i \neq j$ and if either $a_{ij} \neq 0$ and $a_{ji} \neq 0$, then $i \in S$ and $j \in T$ or else $i \in T$ and $j \in S$.

Definition 2.2 (see [3]). An ordered grouping π of $W = \{1, 2, \dots, n\}$ is a subdivision of W into disjoint subsets R_1, R_2, \dots, R_q such that $R_1 + R_2 + \dots + R_q = W$. Given a matrix A and an ordered grouping π , we define the submatrices $A_{m,n}$ for $m, n = 1, 2, \dots, q$ as follows: $A_{m,n}$ is formed from A deleting all rows except those corresponding to R_m and all columns except those corresponding to R_n .

Definition 2.3 (see [3]). Let π be an ordered grouping with q groups. A matrix A has Property $A^{(\pi)}$ if the $q \times q$ matrix $Z = (z_{r,s})$ defined by $z_{r,s} = \{0 \text{ if } A_{r,s} = 0 \text{ or } 1 \text{ if } A_{r,s} \neq 0\}$ has Property A .

Definition 2.4 (see [15]). A matrix A of order n is consistently ordered if for some t there exist disjoint subsets S_1, S_2, \dots, S_t of $W = \{1, 2, \dots, n\}$ such that $\sum_{k=1}^t S_k = W$ and such that if i and j are associated, then $j \in S_{k+1}$ if $j > i$ and $j \in S_{k-1}$ if $j < i$, where S_k is the subset containing i .

Note that a matrix A is a π -consistently ordered matrix if the matrix Z in Definition 2.3 is consistently ordered.

From the discretisation of the EDG finite-difference formula in solving the Poisson equation, the linear system

$$Au = b \quad (2.1)$$

is obtained with [1]

$$A = \begin{bmatrix} R_0 & R_1 & & & \\ R_2 & R_0 & R_1 & & \\ & R_2 & R_0 & \ddots & \\ & & \ddots & \ddots & R_1 \\ & & & R_2 & R_0 \end{bmatrix}_{(N-1)^2/2 \times (N-1)^2/2}, \quad R_0 = \begin{bmatrix} R_{00} & R_{01} & & & \\ R_{02} & R_{00} & \ddots & & \\ & \ddots & \ddots & R_{01} & \\ & & R_{02} & R_{00} \end{bmatrix}_{(N-1) \times (N-1)}, \quad (2.2)$$

$$R_{00} = \begin{bmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{4} & 1 \end{bmatrix}, \quad R_{01} = \begin{bmatrix} 0 & 0 \\ -\frac{1}{4} & 0 \end{bmatrix}, \quad R_{02} = R_{01}^T,$$

$$R_1 = \begin{bmatrix} R_{01} & R_{01} & & & \\ & R_{01} & \ddots & & \\ & & \ddots & R_{01} & \\ & & & R_{01} & \\ & & & & R_{01} \end{bmatrix}_{(N-1) \times (N-1)}, \quad R_2 = \begin{bmatrix} R_{02} & & & & \\ R_{02} & R_{02} & & & \\ & \ddots & \ddots & & \\ & & R_{02} & R_{02} \end{bmatrix}_{(N-1) \times (N-1)}.$$

Let $A \in \mathcal{Z}_{\pi,p}^{n_i,n_i}$, ($p = N - 1$) be written as $A = D - E - F$, where $D = \text{diag} (A_{11}, A_{22}, \dots, A_{p,p})$, E and F are strict block lower triangular, and strict block upper triangular parts of A . Here, the diagonal entries A_{ii} are nonsingular and $\mathcal{Z}_{\pi,p}^{n_i,n_i}$ denotes the set of all matrices in $\mathcal{Z}_{\pi,p}^{n_i,n_i}$ which is of the form (2.1) relative to some given block partitioning π . The block Jacobi iteration matrix is $B_J = D^{-1}(E+F) = L+U$, where $L = D^{-1}E$, $U = D^{-1}F$, the block Gauss-Seidel iteration matrix is $B_{GS} = (I-L)^{-1}U$, and the Block Successive Over-Relaxation method (BSOR) iteration matrix is

$$B_{\ell w} = (I - wL)^{-1} \{ (1 - w)I + wU \}. \quad (2.3)$$

Since the matrix A of (2.1) is a π -consistently ordered and possesses property $A^{(\pi)}$, therefore, the theory of block SOR is valid for this iterative method [1].

The theoretical optimum relaxation factor ω_p for implementing the group SOR iterative scheme can be computed from the formula:

$$\omega_p = \frac{2}{1 + \sqrt{1 - \rho^2(J)}}, \quad (2.4)$$

where $\rho(J)$ is the spectral radius of the group Jacobian iterative matrix. Yousif and Evans [16] gave a good estimate of the spectral radius for the EDG method:

$$\rho(J) = 1 - \frac{7}{6}\pi^2 h^2. \quad (2.5)$$

In an effort to further accelerate the convergence rates of this method, Saeed and Ali [12] applied a preconditioner P to the linear system (2.1) and transformed it into an equivalent system:

$$PA\bar{u} = P\bar{b} \quad (2.6)$$

with $P = (I + \bar{S})$, where I is the identity matrix which has the same dimension as A while \bar{S} is obtained by taking the first upper diagonal groups of R_0 in the original system above as the following:

$$\bar{S} = \begin{bmatrix} Z_1 & & & \\ & Z_1 & & \\ & & \ddots & \\ & & & Z_1 \end{bmatrix}_{(N-1)^2/2 \times (N-1)^2/2}, \quad Z_1 = \begin{bmatrix} \bar{0} & -R_{01} & & \\ & \bar{0} & \ddots & \\ & & \ddots & -R_{01} \\ & & & \bar{0} \end{bmatrix}_{(N-1) \times (N-1)}. \quad (2.7)$$

Here, $\bar{0}$ is a (2×2) null matrix.

The system (2.1) becomes

$$(I + \bar{S})A\bar{u} = (I + \bar{S})\bar{b}. \quad (2.8)$$

Hence, we have the linear system of equations:

$$\overline{A}\underline{u} = \overline{b} \quad (2.9)$$

with

$$\begin{aligned} \overline{A} &= (I + \overline{S})A = I - L - \overline{S}L - (U - \overline{S} + \overline{S}U), \\ \overline{b} &= (I + \overline{S})b. \end{aligned} \quad (2.10)$$

The SOR iteration matrix of this scheme is called the Modified Block Successive Over-Relaxation iteration matrix (MBSOR) and is given by

$$\tilde{B}_{\ell_w} = \left\{ I - w(L + \overline{S}L) \right\}^{-1} \left[(1 - w)I + w(U - \overline{S} + \overline{S}U) \right]. \quad (2.11)$$

The matrix \overline{A} of (2.9) is π -consistently ordered and possesses property $A^{(\pi)}$ [13].

3. Preconditioned-Modified Explicit Decoupled Group SOR (MEDG SOR)

Using the MEDG approximation formula in discretising the Poisson equation, the following system is obtained [6]:

$$A_m \underline{u} = b_m, \quad (3.1)$$

where

$$\begin{aligned} A_m &= \begin{bmatrix} R_{m_0} & R_{m_1} & & & \\ R_{m_2} & R_{m_0} & R_{m_1} & & \\ & R_{m_2} & R_{m_0} & \ddots & \\ & & \ddots & \ddots & R_{m_1} \\ & & & R_{m_2} & R_{m_0} \end{bmatrix}_{(N-2)^2/2 \times (N-2)^2/2}, \\ R_{m_0} &= \begin{bmatrix} R_{m_{00}} & R_{m_{01}} & & \\ R_{m_{02}} & R_{m_{00}} & \ddots & \\ & \ddots & \ddots & R_{m_{01}} \\ & & R_{m_{02}} & R_{m_{00}} \end{bmatrix}_{(N-2) \times (N-2)}, \end{aligned}$$

$$\begin{aligned}
R_{m_{00}} &= \begin{bmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{4} & 1 \end{bmatrix}, & R_{m_{01}} &= \begin{bmatrix} 0 & 0 \\ -\frac{1}{4} & 0 \end{bmatrix}, & R_{m_{02}} &= R_{m_{01}}^T, \\
R_{m_1} &= \begin{bmatrix} R_{m_{01}} & R_{m_{01}} & & \\ & R_{m_{01}} & \ddots & \\ & & \ddots & R_{m_{01}} \\ & & & R_{m_{01}} \end{bmatrix}_{(N-2) \times (N-2)}, \\
R_{m_2} &= \begin{bmatrix} R_{m_{02}} & & & \\ R_{m_{02}} & R_{m_{02}} & & \\ & \ddots & \ddots & \\ & & R_{m_{02}} & R_{m_{02}} \end{bmatrix}_{(N-2) \times (N-2)}.
\end{aligned} \tag{3.2}$$

It is observed that the partitioning of A_m is in the following block form:

$$A_m = \begin{bmatrix} A_{m_{11}} & A_{m_{12}} & & & \\ A_{m_{21}} & A_{m_{22}} & A_{m_{23}} & & \\ & A_{m_{32}} & A_{m_{33}} & \ddots & \\ & & \ddots & \ddots & A_{m_{(p-1)p}} \\ & & & A_{m_{p(p-1)}} & A_{m_{pp}} \end{bmatrix} \tag{3.3}$$

with $p = (N - 2)$, where $A_{m_{ii}} \in \mathbb{C}_{\pi,p}^{n_i, n_i}$, $i = 1, 2, \dots, p$, and $\sum_{i=1}^p n_i = n$. Let $A_m = D_m - E_m - F_m$, where $D_m = \text{diag}(A_{m_{11}}, A_{m_{22}}, \dots, A_{m_{pp}})$ and

$$E_m = E_{m_{ij}} = \begin{cases} -A_{m_{ij}} & \text{for } j < i \\ 0 & \text{for } j \geq i \end{cases} \quad F_m = F_{m_{ij}} = \begin{cases} -A_{m_{ij}} & \text{for } j > i \\ 0 & \text{for } j \leq i \end{cases} \tag{3.4}$$

are block matrices consisting of the block diagonal, strict block lower triangular, and strict block upper triangular parts of A_m . Here, the diagonal entries $A_{m_{ii}}$ are nonsingular. The block Jacobi iteration matrix is $B_J(A_m) = D_m^{-1}(E_m + F_m) = L_m + U_m$, where $L_m = D_m^{-1}E_m$, $U_m = D_m^{-1}F_m$, while the block Gauss-Seidel iteration matrix is $B_{GS}(A_m) = (I_m - L_m)^{-1}U_m$. The Block Successive Over-Relaxation method (BSOR) iteration matrix is, therefore,

$$T_{\ell_w} = (I_m - wL_m)^{-1}\{(1 - w)I_m + wU_m\}. \tag{3.5}$$

Since the matrix A_m of (3.3) is π -consistently ordered and possesses property $A^{(\pi)}$, the theory of block SOR is also valid for this iterative method and, therefore, is convergent [6].

Similarly, the theoretical optimum relaxation factor ω_p for implementing this group SOR-iterative scheme can be obtained from (2.4). In view of the fact that the grid spacing

$h_{\text{MEDG}} = 2h_{\text{EDG}}$, an estimate of the spectral radius of the group Jacobian iterative matrix of the MEDG method may be obtained from (2.5) as

$$\rho(J) = 1 - \frac{14}{3}\pi^2 h^2. \quad (3.6)$$

Good agreement between the theoretical estimates and experimental values of the optimum relaxation parameters was observed in our numerical experiments. Upon applying the left-sided preconditioner $P = (I + \tilde{S})$ to system (3.1), the following system is obtained [13]:

$$PA_m \underline{u} = Pb_m \quad (3.7)$$

with

$$\tilde{S} = \begin{bmatrix} s_1 & & & \\ & s_1 & & \\ & & \ddots & \\ & & & s_1 \end{bmatrix}_{(N-2)^2/2 \times (N-2)^2/2}, \quad s_1 = \begin{bmatrix} \mathcal{Q} & -R_{m01} & & \\ & \mathcal{Q} & \ddots & \\ & & \ddots & -R_{m01} \\ & & & \mathcal{Q} \end{bmatrix}_{(N-2) \times (N-2)}, \quad (3.8)$$

where \mathcal{Q} is a (2×2) null matrix.

The preconditioner $I + \tilde{S}$ is of the following form:

$$I + \tilde{S} = \begin{bmatrix} s_2 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_2 \end{bmatrix}_{(N-2)^2/2 \times (N-2)^2/2}, \quad s_2 = \begin{bmatrix} I_0 & -R_{m01} & & \\ & I_0 & \ddots & \\ & & \ddots & -R_{m01} \\ & & & I_0 \end{bmatrix}_{(N-2) \times (N-2)} \quad (3.9)$$

Here, I_0 is a (2×2) identity matrix and the system (3.7) becomes

$$(I + \tilde{S})A_m \underline{u} = (I + \tilde{S})b_m. \quad (3.10)$$

Hence,

$$\tilde{A}_m \underline{u} = \tilde{b}_m, \quad (3.11)$$

where

$$\begin{aligned} \tilde{A}_m &= (I + \tilde{S})A_m = I_m - L_m - \tilde{S}L_m - (U_m - \tilde{S} + \tilde{S}U_m), \\ \tilde{b}_m &= (I + \tilde{S})b_m. \end{aligned} \quad (3.12)$$

The SOR iteration matrix will result in an Improved Modified Block Successive Over-Relaxation iteration matrix (IMBSOR) and is given by

$$\tilde{T}_{\ell_w} = \left\{ I_m - w(L_m + \tilde{S}L_m) \right\}^{-1} \left[(1-w)I_m + w(U_m - \tilde{S} + \tilde{S}U_m) \right]. \quad (3.13)$$

4. Convergence Properties of the Preconditioned Group Methods

In this section, we will derive several properties related to the convergence of the preconditioned methods discussed in Sections 2 and 3. We will begin with the presentation of several preliminary relevant theorems and lemmas which are needed for the proof of the convergence properties. The spectral radius of a matrix is denoted by $\rho(\cdot)$, which is defined as the largest of the moduli of the eigenvalues of the iteration matrix.

Theorem 4.1 (see [15]). *If $A = M - N$ is a regular splitting of the matrix A and $A^{-1} \geq 0$, then*

$$\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1 + \rho(A^{-1}N)} < 1. \quad (4.1)$$

Thus, an iterative method with coefficient matrix $M^{-1}N$ is convergent for any initial vector $x^{(0)}$.

An accurate analysis of convergence properties of the SOR method is possible if the matrix A is consistently ordered in the following sense (see [17]).

Definition 4.2. A matrix A is a generalized (q, r) -consistently ordered matrix (a $\text{GCO}(q, r)$ -matrix) if $\Delta = \det(\alpha^q E + \alpha^{-r} F - kD)$ is independent of α for all $\alpha \neq 0$ and for all k . Here, $D = \text{diag } A$ and E and F are strictly lower and strictly upper triangular matrices, respectively, such that $A = D - E - F$.

Definition 4.3 (see [17]). A matrix A of the form (3.3) is said to be generally consistently ordered (π, q, r) or simply $\text{GCO}(\pi, q, r)$, where q and r are positive integers, if for the partitioning π of A , the diagonal submatrices A_{ii} , $i = 1, 2, \dots, p (\geq 2)$, are nonsingular, and the eigenvalues of

$$B_J(\alpha) = \alpha^r L + \alpha^{-q} U \quad (4.2)$$

are independent of α , for all $\alpha \neq 0$, where L and U are strict block lower and upper triangular parts of A respectively.

For any matrix $C = (c_{ij})$ in $\mathcal{C}_{\pi, p}^{n_i, n_i}$, let $|C|$ denote the block matrix in $\mathcal{C}_{\pi, p}^{n_i, n_i}$ with entries $|c_{ij}|$. Given the matrix

$$B_J = L + U, \quad (4.3)$$

then $\bar{\mu}$ denotes the spectral radius of the matrix:

$$|B_J| = |L + U|, \quad (4.4)$$

so that

$$\bar{\mu} = \rho(|B_J|). \quad (4.5)$$

Lemma 4.4 (see [17]). Let $|B_J|$ of (4.4) be a GCO (q, r) -matrix and $p = q + r$. Then, for any real nonnegative constant α, β , and γ with $\gamma \neq 0$ satisfying: $\alpha^r \beta^q \bar{\mu}^p < \gamma^p$, the matrix $A'' = \gamma I - \alpha|L| - \beta|U|$ is such that $A''^{-1} \geq 0$.

Lemma 4.5 (see [14]). Suppose $A = I - L - U$ is a GCO (π, q, r) , where $-L$ and $-U$ are strictly lower and upper triangular matrices, respectively. Let B_{ℓ_w} be the block iteration matrix of the SOR method given by (2.3). If $0 < w < 2$, then the block SOR method converges, that is, $\rho(B_{\ell_w}) < 1$.

Theorem 4.6 (see [14]). Suppose $A = I - L - U$ is a GCO (π, q, r) , where $-L$ and $-U$ are strictly lower and upper triangular matrices, respectively. Let B_{ℓ_w} and \tilde{B}_{ℓ_w} be the iteration matrices of the SOR method given by (2.3) and (2.11), respectively. If $0 < w < 2$, then

- (i) $\rho(\tilde{B}_{\ell_w}) < \rho(B_{\ell_w})$ if $\rho(B_{\ell_w}) < 1$,
- (ii) $\rho(\tilde{B}_{\ell_w}) = \rho(B_{\ell_w})$ if $\rho(B_{\ell_w}) = 1$,
- (iii) $\rho(\tilde{B}_{\ell_w}) > \rho(B_{\ell_w})$ if $\rho(B_{\ell_w}) > 1$.

Using the results and definitions stated above, we can prove the following lemma and theorems in relation to the spectral radius of the iteration matrices of the preconditioned group methods and their unpreconditioned counterparts.

Lemma 4.7. Suppose $A_m = I_m - L_m - U_m$ is a GCO (π, q, r) , where $-L_m$ and $-U_m$ are strictly lower and upper triangular matrices, respectively. Let T_{ℓ_w} be the block iteration matrix of the SOR method given by (3.5). If $1 \leq w < 2$, then the block SOR method converges, that is, $\rho(T_{\ell_w}) < 1$.

Proof. Let the matrix A_m with partitioning π be given as in (3.3) and let the block SOR iteration matrix T_{ℓ_w} be given as in (3.5).

Set

$$B'_{\ell_w} = (I - |wL_m|)^{-1} \{ |1 - w|I_m + |w||U_m| \}. \quad (4.6)$$

Clearly, we can see that $|T_{\ell_w}| < B'_{\ell_w}$ and hence we can conclude that $\rho(T_{\ell_w}) \leq \rho(B'_{\ell_w})$.

Consider the matrix $A' \in \mathcal{L}_{\pi, p}^{n_i, n_i}$ defined by

$$A' = \overline{M}_m - \overline{N}_m, \quad (4.7)$$

where $\overline{M}_m = I_m - |w||L_m|$ and $\overline{N}_m = |1 - w|I_m + |w||U_m|$. It is easily seen that \overline{M}_m is nonsingular and $B'_{\ell_w} = \overline{M}_m^{-1} \overline{N}_m$. Moreover, since $\overline{M}_m^{-1} \geq 0$ and $\overline{N}_m \geq 0$, $\overline{M}_m - \overline{N}_m$ is a regular splitting of A' (cf. [11]). For w satisfying the condition $1 \leq w < 2$, Lemma 4.4 implies that $A'^{-1} \geq 0$. Therefore, recalling Theorem 4.1 above, we have $\rho(B'_{\ell_w}) < 1$. Hence, $\rho(T_{\ell_w}) < 1$, which completes the proof. \square

The result of Lemma 4.7 enables us to prove the following theorem

Theorem 4.8. Suppose $A_m = I_m - L_m - U_m$ is a GCO(π, q, r), where $-L_m$ and $-U_m$ are strictly lower and upper triangular matrices, respectively. Let T_{ℓ_w} and \tilde{T}_{ℓ_w} be the iteration matrices of the SOR method given by (3.5) and (3.13), respectively. If $1 \leq w < 2$, then

- (i) $\rho(\tilde{T}_{\ell_w}) < \rho(T_{\ell_w})$ if $\rho(T_{\ell_w}) < 1$,
- (ii) $\rho(\tilde{T}_{\ell_w}) = \rho(T_{\ell_w})$ if $\rho(T_{\ell_w}) = 1$,
- (iii) $\rho(\tilde{T}_{\ell_w}) > \rho(T_{\ell_w})$ if $\rho(T_{\ell_w}) > 1$.

Proof. From Lemma 4.7 and since the matrix A_m of (3.3) is a GCO(π, q, r) and $T_{\ell_w} = (I - wL_m)^{-1}\{(1-w)I_m + wU_m\}$, there exists a positive vector y such that

$$T_{\ell_w}y = \lambda y, \quad (4.8)$$

where $\lambda = \rho(T_{\ell_w})$ or equivalently

$$[(1-w)I_m + wU_m]y = \lambda(I_m - wL_m)y. \quad (4.9)$$

Also, since

$$\tilde{T}_{\ell_w} = \left\{ I_m - w(L_m + \tilde{S}L_m) \right\}^{-1} \left[(1-w)I_m + w(U_m - \tilde{S} + \tilde{S}U_m) \right], \quad (4.10)$$

we can write

$$\begin{aligned} \tilde{T}_{\ell_w}y - \lambda y &= \left\{ I_m - w(L_m + \tilde{S}L_m) \right\}^{-1} \left[(1-w)I_m + w(U_m - \tilde{S} + \tilde{S}U_m) \right. \\ &\quad \left. - \lambda \left\{ I_m - w(L_m + \tilde{S}L_m) \right\} \right] y. \end{aligned} \quad (4.11)$$

Rearrange (4.11), we can get

$$\begin{aligned} \tilde{T}_{\ell_w}y - \lambda y &= \left\{ I_m - w(L_m + \tilde{S}L_m) \right\}^{-1} \left[(1-w)I_m + w(U_m + \lambda L_m) \right. \\ &\quad \left. - \lambda \left(I_m - w\tilde{S}L_m \right) + w\tilde{S}(U_m + I_m) \right] y. \end{aligned} \quad (4.12)$$

But from (4.9), we have

$$[\lambda wL_m + wU_m]y = [(\lambda - 1 + w)I_m]y. \quad (4.13)$$

Therefore, (4.12) can be written as

$$\tilde{T}_{\ell_w}y - \lambda y = \left\{ I_m - w(L_m + \tilde{S}L_m) \right\}^{-1} \left[w\tilde{S}(I_m + \lambda L_m + U_m) \right] y. \quad (4.14)$$

Hence, for $1 \leq w < 2$ and from [10], we can get

- (i) $\lambda < 1$, then $\tilde{T}_{\ell_w} \mathbf{y} - \lambda \mathbf{y} < 0$ and from Theorem 4.6 we have $\rho(\tilde{T}_{\ell_w}) < \rho(T_{\ell_w})$,
- (ii) $\lambda = 1$, then $\tilde{T}_{\ell_w} \mathbf{y} = \lambda \mathbf{y}$ and from Theorem 4.6 we have $\rho(\tilde{T}_{\ell_w}) = \rho(T_{\ell_w}) = 1$,
- (iii) $\lambda > 1$, then $\tilde{T}_{\ell_w} \mathbf{y} - \lambda \mathbf{y} > 0$ and from Theorem 4.6 we have $\rho(\tilde{T}_{\ell_w}) > \rho(T_{\ell_w})$.

Thus, the proof is complete. \square

Theorem 4.9. Suppose $A = I - L - U$ and $A_m = I_m - L_m - U_m$ are GCO(π, q, r), where $-L$, $-L_m$, $-U$ and $-U_m$ are strictly lower and upper triangular matrices of A and A_m , respectively. Let B_{ℓ_w} , \tilde{B}_{ℓ_w} , T_{ℓ_w} and \tilde{T}_{ℓ_w} be the iteration matrices of the SOR method given by (2.3), (2.11), (3.5), and (3.13), respectively. If $1 \leq w < 2$, then

- (i) $\rho(\tilde{T}_{\ell_w}) < \rho(T_{\ell_w}) < \rho(\tilde{B}_{\ell_w}) < \rho(B_{\ell_w})$ if $\rho(B_{\ell_w}) < 1$,
- (ii) $\rho(\tilde{T}_{\ell_w}) = \rho(T_{\ell_w}) = \rho(\tilde{B}_{\ell_w}) = \rho(B_{\ell_w})$ if $\rho(B_{\ell_w}) = 1$,
- (iii) $\rho(\tilde{T}_{\ell_w}) > \rho(T_{\ell_w}) > \rho(\tilde{B}_{\ell_w}) > \rho(B_{\ell_w})$ if $\rho(B_{\ell_w}) > 1$.

Proof. In the same manner of the proof of Theorem 4.8 and since the matrix \bar{A} of (2.9) is a GCO(π, q, r), see [13], and $\tilde{B}_{\ell_w} = \{I - w(L + \bar{S}L)\}^{-1}[(1 - w)I + w(U - \bar{S} + \bar{S}U)]$, there exists a positive vector v such that

$$\tilde{B}_{\ell_w} v = \bar{\lambda} v, \quad (4.15)$$

where

$$\bar{\lambda} = \rho(\tilde{B}_{\ell_w}). \quad (4.16)$$

Equation (4.15) can be written as

$$\left[(1 - w)I + w(U - \bar{S} + \bar{S}U)\right]v = \bar{\lambda} \left\{I - w(L + \bar{S}L)\right\}v. \quad (4.17)$$

Also, since $T_{\ell_w} = (I_m - wL_m)^{-1}\{(1 - w)I_m + wU_m\}$, we can write

$$\begin{aligned} T_{\ell_w} v - \bar{\lambda} v &= (I_m - wL_m)^{-1} \left\{ (1 - w)I_m + wU_m - \bar{\lambda}(I_m - wL_m) \right\} v \\ &= (I_m - wL_m)^{-1} \left\{ (1 - w - \bar{\lambda})I_m + w(\bar{\lambda}L_m + U_m) \right\} v. \end{aligned} \quad (4.18)$$

But, from (4.17) we have

$$\left[w\{\bar{\lambda}L + U + \bar{S}(\bar{\lambda}L + U - I)\}\right]v = \left[(\bar{\lambda} - 1 + w)I\right]v. \quad (4.19)$$

Thus, from (4.19) and since A_m of (3.3) is a GCO(π, q, r) matrix, we can get

$$\left[w\{\bar{\lambda}L_m + U_m + \tilde{S}(\bar{\lambda}L_m + U_m - I_m)\}\right]v = \left[(\bar{\lambda} - 1 + w)I_m\right]v. \quad (4.20)$$

Equation (4.18) can then be written as

$$T_{\ell_w} v - \bar{\lambda} v = \{I_m - wL_m\}^{-1} \left\{ w\tilde{S}(-\bar{\lambda}L_m - U_m - I_m) \right\} v. \quad (4.21)$$

Hence, we can conclude that, for $1 \leq w < 2$, if

- (a) $\bar{\lambda} < 1$, then $T_{\ell_w} v - \bar{\lambda} v < 0$ and from Lemma 4.7 we have $\rho(T_{\ell_w}) < \rho(\tilde{B}_{\ell_w})$,
- (b) $\bar{\lambda} = 1$, then $T_{\ell_w} v = \bar{\lambda} v$ and from Lemma 4.7 we have $\rho(T_{\ell_w}) = \rho(\tilde{B}_{\ell_w}) = 1$,
- (c) $\bar{\lambda} > 1$, then $T_{\ell_w} v - \bar{\lambda} v > 0$ and from Lemma 4.7 we have $\rho(T_{\ell_w}) > \rho(\tilde{B}_{\ell_w})$.

In consequence of the above, for $1 \leq w < 2$ and from Theorems 4.6 and 4.8, we have

- (i) $\rho(\tilde{T}_{\ell_w}) < \rho(T_{\ell_w}) < \rho(\tilde{B}_{\ell_w}) < \rho(B_{\ell_w})$ if $\rho(B_{\ell_w}) < 1$,
- (ii) $\rho(\tilde{T}_{\ell_w}) = \rho(T_{\ell_w}) = \rho(\tilde{B}_{\ell_w}) = \rho(B_{\ell_w})$ if $\rho(B_{\ell_w}) = 1$,
- (iii) $\rho(\tilde{T}_{\ell_w}) > \rho(T_{\ell_w}) > \rho(\tilde{B}_{\ell_w}) > \rho(B_{\ell_w})$ if $\rho(B_{\ell_w}) > 1$,

and the theorem is proved. \square

In view of Theorem 4.9, the superiority of the preconditioned MEDG SOR over the unpreconditioned MEDG SOR, EDG SOR methods and also preconditioned EDG SOR are confirmed for certain relaxation parameters lying in an optimum range.

5. Numerical Experiments and Discussion of Results

To further confirm the results obtained in Theorems 4.8 and 4.9, several experiments were carried out on the following model problem with Dirichlet boundary conditions:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= (x^2 + y^2)e^{xy}, \\ u(x, 0) &= 1, \quad u(0, y) = 1, \quad u(x, 1) = e^x, \quad u(1, y) = e^y. \end{aligned} \quad (5.1)$$

This problem has an exact solution $u(x, y) = e^{xy}$ with the unit square as the solution domain. The values of u were calculated using different mesh sizes, 34, 86, 118, 186, and 222. The tolerance was set to be $\varepsilon = 5 \times 10^{-6}$. The experimental optimum relaxation parameter w was obtained by running the programs repeatedly and choosing the values which gave the fastest rate of convergence. The computer processing unit was Intel(R) Core(TM) 2Duo with memory of 3Gb and the software used to implement and generate the results was Developer C++ Version 4.9.9.2. Tables 1 and 2 display the corresponding number of iterations (k), optimum execution times (t), and the maximum errors (e) for the unpreconditioned and preconditioned methods of EDG SOR and MEDG SOR, respectively.

From the results in Table 1, it is obvious that the original MEDG SOR method is superior to the EDG SOR method in terms of the number of iterations and computing times. The superiority of the preconditioned MEDG SOR over the preconditioned EDG SOR was also depicted in Table 2. The preconditioned EDG SOR was also outperformed by the unpreconditioned MEDG as shown in Figure 1 since the spectral radius of the latter is smaller

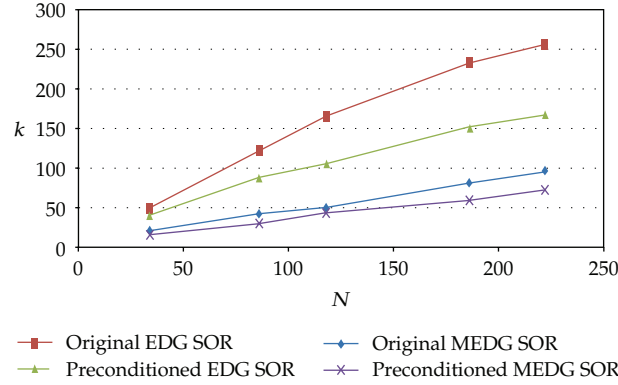


Figure 1: Number of iterations (k) for the four methods for different mesh sizes N .

Table 1: Comparison of performances for the original EDG SOR and MEDG SOR.

N	Unpreconditioned methods							
	EDG SOR				MEDG SOR			
	w	k	t (secs)	e	w	k	t (secs)	e
34	1.753	49	0	$4.11E-06$	1.586	21	0	$5.00E-06$
86	1.894	122	0.047	$4.88E-06$	1.812	42	0.023	$4.57E-06$
118	1.921	166	0.078	$4.89E-06$	1.862	50	0.035	$2.08E-06$
186	1.943	233	0.187	$4.79E-06$	1.914	81	0.064	$4.36E-06$
222	1.948	256	0.327	$4.78E-06$	1.931	96	0.145	$1.84E-06$

than the former as proven in Theorem 4.9. From the numerical results, it is also apparent that the preconditioned MEDG SOR scheme requires the least computing effort amongst the four methods in terms of number of iterations and execution times due to its smallest spectral radius value amongst the four schemes.

Figure 1 shows the number of iterations needed for convergence for the unpreconditioned and preconditioned methods which were shown to be in agreement with the theoretical results obtained in Theorem 4.9.

6. Conclusion

In this paper, we present a theoretical convergence analysis of a specific splitting-type preconditioner in block formulation applied to the linear systems resulted from a class of group iterative schemes specifically the EDG SOR and the MEDG SOR schemes. We have shown that the spectral radius of the iteration matrix of the preconditioned MEDG SOR method is the smallest compared to the unpreconditioned MEDG SOR, EDG SOR, and preconditioned EDG SOR methods provided that the relaxation parameter $\omega \in [1, 2)$. This work confirms the superiority of the preconditioned MEDG SOR method theoretically and experimentally in terms of convergence rates among this class of group iterative methods.

Table 2: Comparison of performances for the Preconditioned EDG SOR and MEDG SOR.

N	Preconditioned methods							
	Preconditioned EDG SOR				Preconditioned MEDG SOR			
	w	k	t (secs)	e	w	k	t (secs)	e
34	1.679	41	0	$3.85E - 06$	1.442	16	0	$3.47E - 06$
86	1.757	88	0.034	$3.87E - 06$	1.5882	30	0.008	$3.09E - 06$
118	1.774	106	0.051	$4.47E - 06$	1.642	43	0.016	$3.55E - 06$
186	1.782	151	0.133	$4.68E - 06$	1.684	59	0.025	$4.24E - 06$
222	1.795	168	0.198	$4.15E - 06$	1.671	72	0.078	$3.99E - 06$

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